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Remarks on the additional symmetries and W-constraints in the generalized KdV hierarchy

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Additional symmetries of the p -reduced KP hierarchy are generated by the Lax operator L and another M , satisfying $\text{res}(M^n L^{m+n/p}) = 0$ for $1 \leq n \leq p-1$ and $m \geq -1$ with the condition that $\partial L / \partial t_{kp} = 0$, $k = 1, 2, \dots$. We show explicitly that the generators of these additional symmetries satisfy a closed and consistent W-algebra only when we impose the extra condition that $\partial M / \partial t_{kp} = 0$.

It is a remarkable observation that the integrable models share an interesting relationship with the matrix model formulation of 2D quantum gravity [1–5], 2D topological gravity [6–8] and the intersection theory on the moduli space of Riemann surfaces [9–11]. In the matrix model formulation of 2D gravity, one employs the method of orthogonal polynomials and makes use of the operators Q and P which correspond to the insertions of the spectral parameter and a derivative with respect to this in the matrix integral [12,13]. Since these two operators are conjugate to each other, they satisfy the so-called “string equation” [13] $[P, Q] = 1$. As Douglas argued, these operators can be realized in terms of some finite order differential operators and can be recognized as the Lax-pair of an associated integrable model [14–16].

It is also well known that the formulation of the hermitian one- [17] (two-) [18] matrix model in terms of the continuum Schwinger–Dyson equations gives rise to a semi-infinite Virasoro ($W^{(3)}$) constraints on the square root of the partition function. The square root of the partition function is recognized as the τ -function of the KdV (Boussinesq) hierarchy with the flow parameters being identified with the coupling constants of the scaling operators. One of the Virasoro constraints arising in this way is also

seen to be equivalent with the integrated form of Douglas’ string equation. Further, the conjecture [17] that all other Virasoro and W-constraints can be obtained from the string equation itself is also proved [19–21] in different indirect ways. In fact, a lot of effort has been put in understanding the origin and the geometry of the string equation [22,23]. It has been observed that the solution space of the string equation has an interesting connection with the Sato grassmannian [24,25].

The origin of the Virasoro and W-constraints can also be attributed to the existence of additional symmetries of the integrable hierarchy as propounded in ref. [26]. For the p -reduced KP hierarchy [27,28], the Lax operator is given by

$$L = \partial^p + u_{p-2}(x, t) \partial^{p-2} + \dots + u_0(x, t), \quad (1)$$

where $\partial \equiv \partial / \partial x$ and $u_0, \dots, u_{p-2}$ are coefficient functions of the Lax operator depending on the infinite number of evolution parameters $(x, t_2, t_3, \dots) \equiv (x, t)$. The coefficient functions satisfy the following evolution equations:

$$\frac{\partial L}{\partial t_n} = [L_+^{n/p}, L], \quad n = 1, 2, 3, \dots, \quad (2)$$

where $L_+^{n/p}$ denotes the non-negative differential part of the n th power of the formal pseudo-differential operator $L^{1/p}$. For $n = 1$, the above equation gives a consistency relation if we identify t_1 with x . It is clear

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that for $n=0(\bmod p)$, the commutator in (2) vanishes and thus

$$\frac{\partial L}{\partial t_{kp}} = 0, \quad k=1, 2, 3, \dots \quad (3)$$

The additional symmetries [29] of L mean that L does not depend on some additional symmetry parameters $t_{mp+n,n}$. In other words,

$$\frac{\partial L}{\partial t_{mp+n,n}} \equiv [L, (M^n L^{m+n/p})_-] = 0 \quad (4)$$

for $m \geq -1$, $1 \leq n \leq p-1$ and these flows commute with the usual p -reduced KP hierarchy flows given in (2). But they do not commute among themselves [29]. Here, the operator M is defined as

$$M = \frac{1}{p} K \left(\sum_{n=1}^{\infty} n t_n \partial^{n-1} \right) K^{-1}, \quad (5)$$

where K is the pseudo-differential operator,

$$K = 1 + \sum_{i=1}^{\infty} a_i(x, t) \partial^{-i}, \quad (6)$$

and known as the Zakharov-Shabat dressing operator. It satisfies the relation $L = K \partial^p K^{-1}$ which fixes the coefficients $a_i(x, t)$ in terms of u_i and their derivatives.

It has been shown in refs. [20,21] that eqs. (3), (4) along with the definition of M as in (5) give rise to an infinite set of constraints on the τ -function of the p -reduced KP hierarchy which are claimed to be the semi-infinite Virasoro and W -constraints appearing in the matrix models, as discussed earlier. However, that is not quite true, as we explicitly show for the case of $p=3$ that the constraints generated in this way do not satisfy a closed and consistent $W^{(3)}$ algebra [30]. Further, we show that the situation can be salvaged if we impose the extra condition that M satisfies

$$\frac{\partial M}{\partial t_{kp}} = 0, \quad k=1, 2, 3, \dots \quad (7)$$

Only then, the generated constraints satisfy a consistent W -algebra and also they exactly match with the matrix model results in ref. [17,18]. We would also like to point out here that it is not necessary to impose this extra condition in order to obtain the right Virasoro constraints.

We now specialize to the case of $p=3$, i.e., we are dealing with the 3-reduced KP (Boussinesq) hierarchy. The Lax operator for this hierarchy is

$$L = \partial^3 + 4u\partial + (2u' + v), \quad (8)$$

where $u' \equiv \partial u / \partial x$ and the particular form of the Lax operator is chosen such that it transforms covariantly under the conformal transformation [31]. The Douglas string equation, for the two-matrix model, interpolating between various critical points is given as

$$\sum_{k=1}^{\infty} \sum_{i=1,2} (k + \frac{1}{3}i) t_{3k+i} [L, L_+^{(3(k-1)+i)/3}] = 1, \quad (9)$$

where we have introduced an infinite number of evolution parameters t_{3k+i} proportional to $-1/(k + \frac{1}{3}i)$ and we recall that L satisfies $\partial L / \partial t_{3k} = 0$. In this case the operator M has the form

$$M = \frac{1}{3} K \left(\sum_{n=1}^{\infty} n t_n \partial^{n-1} \right) K^{-1}. \quad (10)$$

The operators L and M satisfy the following commutation relations:

$$[L^{1/3}, M] = \frac{1}{3}, \quad (11a)$$

$$[L, M L^{-2/3}] = 1, \quad (11b)$$

$$[M, L^{-2/3}] = \frac{2}{3} L^{-1}. \quad (11c)$$

By explicitly writing $M L^{-2/3}$ with the form of M as given in (10), it is a simple exercise to check that the string equation (9) is equivalent to

$$[L, (M L^{-2/3})_+] = 1. \quad (12)$$

Also, from (11), (12) and using the residual symmetry of the Zakharov-Shabat dressing operator K [21] it can be seen that

$$(M L^{-2/3})_- = \frac{1}{3} L^{-1}. \quad (13)$$

Eq. (13), in turn, implies that for $n \geq 0$

$$(M L^{n+1/3})_- = ((M L^{-2/3})_- L^{n+1})_- = 0. \quad (14)$$

In terms of the additional symmetries, (13) and (14) simply mean that the Lax operator is independent of the symmetry parameters $t_{3n+1,1}$ for $n \geq -1$. As a consequence of (13), it is straightforward to check

that M and L obey the following relations [20]:

$$(M^2 L^{-4/3} - \frac{4}{3} M L^{-5/3} + \frac{4}{9} L^{-2})_- = 0, \quad (15a)$$

and for $n \geq -1$

$$(M^2 L^{n+1/3})_- = 0. \quad (15b)$$

Again, eq. (15) implies that L does not depend on the additional symmetry parameters $t_{-4,2}$, $t_{-5,1}$ and $t_{3n+1,2}$ for $n \geq -1$. It is clear that (13) is consistent with (12). In fact, one can work out the residue (the coefficient of the ∂^{-1} term) of (13) explicitly and demonstrate that it is equivalent to (9).

By making use of the vertex operator techniques of the KP hierarchy [27,28], it is shown in refs. [20,21] that the residues of (13) and (14) reduce to the following differential operator constraints on the τ -function of the Boussinesq hierarchy:

$$L_n \tau = 0, \quad n \geq -1, \quad (16)$$

where the operators L_n (with $t_2 \equiv y$) are defined as

$$L_{-1} = \sum_{\substack{k=1 \\ i=1,2}}^{\infty} (k + \frac{1}{3}i) t_{3k+i} \frac{\partial}{\partial t_{3(k-1)+i}} + \frac{2}{3} xy, \quad (17a)$$

$$L_0 = \sum_{\substack{k=0 \\ i=1,2}}^{\infty} (k + \frac{1}{3}i) t_{3k+i} \frac{\partial}{\partial t_{3k+i}} + \frac{1}{9}, \quad (17b)$$

and for $n \geq 1$

$$L_n = \sum_{\substack{k=0 \\ i=1,2}}^{\infty} (k + \frac{1}{3}i) t_{3k+i} \frac{\partial}{\partial t_{3k+i}} + \frac{1}{6} \sum_{\substack{i+j=3n \\ i,j \neq 0 \pmod{3}}} \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j}. \quad (17c)$$

It can be checked that for $n \geq -1$, the operators L_n , as given above, satisfy the centerless Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n}$. It is worth noting that in the vertex operator techniques, the question how (13) and (14) give rise to the above constraints is not at all transparent. However, by just using the Lax operator approach, one can show in a more direct way that this is indeed true. The details of this approach will be reported elsewhere [32]. Also, note that the constant term $\frac{1}{9}$ in the definition of L_0 , which originates as an integration constant, is fixed so, such that L_n satisfy the Virasoro algebra. This point is not mentioned in ref. [20].

In a similar way, one can work out the consequences of (15a) and (15b) by considering the residue term. After an integration with respect to x and multiplication by τ , eq. (15) reduces to the following differential constraints on τ [20,21]:

$$W_n^{(3)} \tau = 0 \quad (18)$$

for $n \geq -2$ and where $W_n^{(3)}$ is defined as

$$W_n^{(3)} = \frac{1}{27} \sum_{i+j+k=-3n} ijk t_i t_j t_k + \frac{1}{9} \sum_{i+j-k=-3n} i j t_i t_j \frac{\partial}{\partial t_k} + \frac{1}{9} \sum_{i-j-k=-3n} i t_i \frac{\partial^2}{\partial t_j \partial t_k} + \frac{1}{27} \sum_{i+j+k=3n} \frac{\partial^3}{\partial t_i \partial t_j \partial t_k}. \quad (19)$$

It is claimed in ref. [21] that the differential operators L_n for $n \geq -1$ and $W_m^{(3)}$ for $m \geq -2$ as defined in (17) and (19) respectively, form a closed $W^{(3)}$ algebra. At this stage, the following fact should be noted. Since the τ -function of the Boussinesq hierarchy satisfies [27]

$$\frac{\partial}{\partial x} \frac{\partial}{\partial t_n} \log \tau = \text{res}(L^{n/3}), \quad (20)$$

and the right-hand side of this vanishes when $n=3k$ for any $k=1, 2, \dots$, so the τ -function would not depend on these coordinates. Therefore, it is clear that the non-zero contributions to $W_n^{(3)}$ comes from the first term and only from $k \neq 0 \pmod{3}$, $j, k \neq 0 \pmod{3}$ and $i, j, k \neq 0 \pmod{3}$ respectively for the last three terms of (19). Keeping this fact in mind, we can compute the commutators between L_n and W_m . We find, for example, that

$$[L_{-1}, W_{-2}^{(3)}] = -W_{-2}^{(3)} - \frac{24}{27} t_6, \quad (21)$$

where $W_{-2}^{(3)}$ has the form as given in (19). Therefore, the form of this $W_{-2}^{(3)}$ is inconsistent with the algebra

$$[L_m, W_n^{(3)}] = (2m-n) W_{m+n}^{(3)}. \quad (22)$$

Let us also think of modifying $W_{-2}^{(3)}$ by adding to it a term $-\frac{24}{27} t_6$. This is possible because $W_{-2}^{(3)}$ has the same scaling dimension as t_6 and the expression for $W_{-2}^{(3)}$ is obtained from (15a) after an integration with respect to x . So the t_6 term can appear as an integration constant. We mention here that $W_{-2}^{(3)}$ contains a

term proportional to t_2^3 which in fact appears as an integration constant. Thus we write

$$\tilde{W}_{-2}^{(3)} = W_{-2}^{(3)} - \frac{24}{27} t_6. \quad (23)$$

However, it does not serve our purpose since for example, we find

$$[L_0, \tilde{W}_{-2}^{(3)}] = 2\tilde{W}_{-2}^{(3)} + \frac{48}{27} t_6, \quad (24)$$

which is again inconsistent with (22). Thus, from the above analysis, it is clear that $W_n^{(3)}$ cannot be a function of t_{3k} for $k=1, 2, \dots$. Therefore, in all the terms in (19), i, j, k should not take values $0 \pmod{3}$. Note that this is consistent with the matrix model result [17,18]. Upon imposing this extra condition, eq. (19) takes the form

$$\begin{aligned} W_n^{(3)} = & \frac{1}{27} \sum_{\substack{i+j+k=-3n \\ i,j,k \neq 0 \pmod{3}}} ijkt_i t_j t_k + \frac{1}{9} \sum_{\substack{i+j-k=-3n \\ i,j,k \neq 0 \pmod{3}}} ij t_i t_j \frac{\partial}{\partial t_k} \\ & + \frac{1}{9} \sum_{\substack{i-j-k=-3n \\ i,j,k \neq 0 \pmod{3}}} i t_i \frac{\partial^2}{\partial t_j \partial t_k} \\ & + \frac{1}{27} \sum_{\substack{i+j+k=3n \\ i,j,k \neq 0 \pmod{3}}} \frac{\partial^3}{\partial t_i \partial t_j \partial t_k}. \end{aligned} \quad (25)$$

We would like to point out that this restriction has been correctly put in ref. [20] though it does not follow from eqs. (3)–(5), (15). We can now check that (17) and (25) do indeed satisfy a consistent $W^{(3)}$ algebra:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n}, \\ [L_m, W_n] &= (2m-n)W_{m+n}, \\ [W_m, W_n] &= (m-n) \left[\frac{3}{2}(m^2 + 4mn + n^2) \right. \\ &\quad \left. + \frac{27}{2}(m+n) + 21 \right] L_{m+n} - 9(m-n)U_{m+n} \\ &\quad - \frac{1}{10} \delta_{m,-n} m(m^2-1)(m^2-4), \end{aligned} \quad (26)$$

where we have defined

$$U_m = \sum_{k \leq -2} L_k L_{m-k} + \sum_{k \geq -1} L_{m-k} L_k.$$

Let us note that the extra conditions that $i, j, k \neq 0 \pmod{3}$ in (25) cannot be obtained if they are obtained from (15) with the definition of M as given in (10). In order to obtain (25) as a direct consequence

of the additional symmetries, we can modify the operator M itself and take instead

$$M = \frac{1}{3} K \left(\sum_{\substack{n=1 \\ n \neq 0 \pmod{3}}}^{\infty} n t_n \partial^{n-1} \right) K^{-1}. \quad (27)$$

With the above definition, we would of course have

$$\frac{\partial M}{\partial t_{3k}} = 0, \quad k=1, 2, 3, \dots, \quad (28)$$

and eqs. (11) are still valid. Therefore, the string equation (12) and the consequences of it namely (13)–(15) also remain true. But since M does not depend on the parameters t_{3k} , these parameters will never appear either in the Virasoro or in the $W_n^{(3)}$ constraints and we have the required form of these constraints. Note that even if we do not modify the operator M to the above form, one still obtains the correct form for the Virasoro constraints but not for the W -constraints as we saw. This feature remains valid for the higher KdV hierarchy also. Thus, for the p th KdV hierarchy, we need to take the operator M as

$$M = \frac{1}{p} K \left(\sum_{\substack{n=1 \\ n \neq 0 \pmod{p}}}^{\infty} n t_n \partial^{n-1} \right) K^{-1} \quad (29)$$

in order to obtain the right $W^{(p)}$ constraints.

In conclusion, we have shown that the origin of the Virasoro and W -constraints in the p -reduced KP hierarchy can be attributed to the additional symmetries associated with such an integrable system. The generators of these additional symmetries have to be modified with appropriate changes in the operator M appearing in the generators to obtain a closed and consistent $W^{(p)}$ algebra.

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References

- [1] D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127; Nucl. Phys. B 340 (1990) 333.

- [2] M. Douglas and S. Shenker, Nucl. Phys. B 335 (1990) 635.
- [3] E. Brézin and V. Kazakov, Phys. Lett. B 236 (1990) 144.
- [4] E.J. Martinec, University of Chicago preprint EFI-90-67 (1990).
- [5] L. Alvarez-Gaumé, CERN preprint CERN-TH-6123/91 (1991).
- [6] R. Dijkgraaf and E. Witten, Nucl. Phys. B 342 (1990) 486.
- [7] E. Verlinde and H. Verlinde, Nucl. Phys. B 348 (1991) 457.
- [8] J. Distler, Nucl. Phys. B 342 (1990) 523.
- [9] E. Witten, Surv. Diff. Geom. 1 (1991) 243; preprint IASSNS-HEP-91/74.
- [10] R. Myers and V. Periwal, Nucl. Phys. B 333 (1990) 536.
- [11] M. Kontsevich, Max-Planck Institute preprints MPI/91-47, MPI/91-77.
- [12] S. Chadha, G. Mahoux and M.L. Mehta, J. Phys. A 14 (1981) 579.
- [13] M.R. Douglas, Phys. Lett. B 238 (1990) 176.
- [14] P.D. Lax, Comm. Pure Appl. Math. 21 (1968) 467; 28 (1975) 141.
- [15] L.D. Fadeev and L.A. Takhtajan, Hamiltonian methods in the theory of solitons (Springer, Berlin, 1987).
- [16] A. Das, Integrable models (World Scientific, Singapore, 1989).
- [17] M. Fukuma, H. Kawai and R. Nakayama, Intern. J. Mod. Phys. A 6 (1991) 1385.
- [18] E. Gava and K.S. Narain, Phys. Lett. B 263 (1991) 213.
- [19] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B 348 (1991) 435.
- [20] J. Goeree, Nucl. Phys. B 358 (1991) 737.
- [21] M. Adler and P. Van-Moerbeke, Brandeis preprint (1992).
- [22] G. Moore, Commun. Math. Phys. 133 (1990) 261.
- [23] V. Kac and A. Schwartz, Phys. Lett. B 257 (1991) 329.
- [24] M. Fukuma, H. Kawai and R. Nakayama, Commun. Math. Phys. 143 (1992) 371.
- [25] K.N. Anagnostopoulos, M.J. Bowick and A. Schwarz, Syracuse preprint SU-4238-497 (1991).
- [26] A.Y. Orlov and E.I. Shulman, Lett. Math. Phys. 12 (1986) 171.
- [27] E. Date, M. Kashiwara, M. Jimbo and T. Miwa in: Nonlinear integrable systems, eds. M. Jimbo and T. Miwa (World Scientific, Singapore, 1983).
- [28] G. Segal and G. Wilson, Publ. Math. IHES 63 (1985) 1.
- [29] L.A. Dickey, University of Oklahoma preprint (1992).
- [30] A.B. Zamolodchikov, Theor. Math. Phys. 65 (1985) 1205.
- [31] A. Das and S. Roy, Intern. J. Mod. Phys. A 6 (1991) 1429.
- [32] S. Panda and S. Roy, in preparation.