Constructing tensegrity frameworks and related applications in multi-agent formation control
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Appendix A

Lemma on the rank of the matrix $\hat{\Omega}$ in (4.12)

Lemma A.1. Consider the matrix $\hat{\Omega} \in \mathbb{R}^{(n+1)\times(n+1)}$ defined in (4.12), where $\Omega \in \mathbb{R}^{n \times n}$ and $\Omega_u \in \mathbb{R}^{4 \times 4}$ are the stress matrices associated with super stable tensegrity frameworks with three common vertices. Then

$$\text{rank}(\hat{\Omega}) = n - 2. \quad (A.1)$$

Proof. We first consider the solution to the following equations

\begin{align*}
\Omega_a x &= 0, \quad (A.2a) \\
\Omega_b y &= 0, \quad (A.2b)
\end{align*}

where $x, y \in \mathbb{R}^{n+1}$. In view of (4.12), (A.2a) can be equivalently written as

$$
\begin{pmatrix}
\Omega & 0_{n\times1} \\
0_{1\times n} & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 
\begin{pmatrix}
0_{n\times1} \\
0
\end{pmatrix},
$$

(A.3)

where $x_1 \in \mathbb{R}^{n\times1}$ and $x_2 \in \mathbb{R}$. After simple calculation, (A.3) can be reduced to

\begin{align*}
\begin{cases}
\Omega x_1 = 0, \\
0x_2 = 0.
\end{cases}
\end{align*}

(A.4)

Since $null(\Omega) = \text{span}(q^T, 1_n)$, the solution space of (A.4) (equivalently, (A.2a)) is as follows

\begin{align*}
\mathcal{S}_a &= \text{span} \left( \begin{pmatrix} q_1 \\ p_1^a \end{pmatrix}, \begin{pmatrix} q_2 \\ p_2^a \end{pmatrix}, \begin{pmatrix} 1_n \\ c_a \end{pmatrix} \right) \\
\stackrel{\Delta}{=} & \text{span} (s_1^a, s_2^a, s_3^a),
\end{align*}

(A.5)

where $q_1 = [q_{11}, \ldots, q_{n1}]^T \in \mathbb{R}^n$ with $q_{11}$ being the first component of $q_i$, $i = 1, \ldots, n$, and $q_2$ is defined analogously. $p_1^a, p_2^a$ and $c_a$ are any arbitrary scalars.
A. Lemma on the rank of the matrix $\hat{\Omega}$ in (4.12)

Similarly, the solution space of (A.2b) is given by

$$S_b = \text{span} \begin{pmatrix}
  p_{11}^b & \vdots & p_{12}^b \\
  \vdots & \ddots & \vdots \\
  p_{(n-3)1}^b & \vdots & p_{(n-3)2}^b \\
  q_{(n-2)1}^b & \vdots & q_{(n-2)2}^b \\
  \vdots & \ddots & \vdots \\
  q_{(n+1)1} & \vdots & q_{(n+1)2}
\end{pmatrix},$$

$$\Delta = \text{span} \left( s_1^b, s_2^b, s_3^b \right)$$

where $p_{ij}^b, i = 1, \ldots, n - 3, j = 1, 2,$ denote the $j$th component of an arbitrary real vector $p_i^b \in \mathbb{R}^2$, and $c_{bi}, i = 1, \ldots, n - 3,$ are arbitrary scalars. In view of Lemma 7.2, we know

$$\text{null}(\hat{\Omega}) = S_a \cap S_b.$$  \hspace{1cm} (A.7)

To determine the non-trivial form of $S_a \cap S_b$, let

$$\alpha_1 s_1^a + \alpha_2 s_2^a + \alpha_3 s_3^a = \beta_1 s_1^b + \beta_2 s_2^b + \beta_3 s_3^b,$$ \hspace{1cm} (A.8)

where $\alpha_i$ and $\beta_i, i = 1, 2, 3,$ are scalars, at least one of which is nonzero. Note that $S_a$ and $S_b$ share the same entries as follows

$$s_c = \begin{pmatrix}
  q_{(n-2)1} \\
  q_{(n-1)1} \\
  q_{n1}
\end{pmatrix}, \begin{pmatrix}
  q_{(n-2)2} \\
  q_{(n-1)2} \\
  q_{n2}
\end{pmatrix}, \begin{pmatrix}
  1 \\
  1 \\
  1
\end{pmatrix}.$$ \hspace{1cm} (A.9)

Combining (A.8) and (A.9), one has

$$(\alpha_1 - \beta_1) \begin{pmatrix}
  q_{(n-2)1} \\
  q_{(n-1)1} \\
  q_{n1}
\end{pmatrix} + (\alpha_2 - \beta_2) \begin{pmatrix}
  q_{(n-2)2} \\
  q_{(n-1)2} \\
  q_{n2}
\end{pmatrix} + (\alpha_3 - \beta_3) \begin{pmatrix}
  1 \\
  1 \\
  1
\end{pmatrix} = 0,$$ \hspace{1cm} (A.10)

which can be equivalently written as

$$\begin{pmatrix}
  q_{(n-2)1} & q_{(n-2)2} & 1 \\
  q_{(n-1)1} & q_{(n-1)2} & 1 \\
  q_{n1} & q_{n2} & 1
\end{pmatrix} \begin{pmatrix}
  \alpha_1 - \beta_1 \\
  \alpha_2 - \beta_2 \\
  \alpha_3 - \beta_3
\end{pmatrix} = 0.$$ \hspace{1cm} (A.11)

Recalling that vertices $i, j$ and $k$ are not collinear, it is equivalent to say that they
are in general positions in the plane, which implies

\[
\text{rank } \begin{pmatrix}
q_{(n-2)1} & q_{(n-2)2} & 1 \\
q_{(n-1)1} & q_{(n-2)1} & 1 \\
q_{n1} & q_{n2} & 1
\end{pmatrix} = 3. \tag{A.12}
\]

Then in view of (A.11), the parameters \(\alpha_i\) and \(\beta_i\), \(i = 1, 2, 3\), in (A.8) satisfy

\[
\begin{cases}
\alpha_1 = \beta_1, \\
\alpha_2 = \beta_2, \\
\alpha_3 = \beta_3.
\end{cases} \tag{A.13}
\]

From the fact that \(\hat{\Omega}\) is a stress matrix associated with configuration \(\bar{q}\), we know

\[(\bar{q}_1, \bar{q}_2, 1_{n+1}) \subseteq \text{null}(\hat{\Omega}), \tag{A.14}\]

where \(\bar{q}_1 = [q_1^T, q_{(n+1)1}]^T\), and \(\bar{q}_2\) is defined analogously. Since \(\text{rank}(\bar{q}_1, \bar{q}_2, 1_{n+1}) = 3\), we have

\[
\text{rank}(\hat{\Omega}) \leq n - 2. \tag{A.15}
\]

Then, to prove \(\text{rank}(\hat{\Omega}) = n - 2\), we need to show that any other vector \(v \in \text{null}(\hat{\Omega})\) can be represented as a linear combination of vectors \(\bar{q}_1, \bar{q}_2,\) and \(1_{n+1}\), namely, there exist scalars \(\gamma_1, \gamma_2,\) and \(\gamma_3\), such that

\[
v = \gamma_1 \bar{q}_1 + \gamma_2 \bar{q}_2 + \gamma_3 1_{n+1}, \quad \forall v \in \text{null}(\hat{\Omega}), \tag{A.16}\]

where at least one of \(\gamma_i, i = 1, 2, 3\), is nonzero. In light of Lemma 7.2, one has

\[
v \in \text{null}(\hat{\Omega}) \iff v \in S_a \text{ and } v \in S_b, \tag{A.17}\]

which implies

\[
v = \alpha_1 s_1^a + \alpha_2 s_2^a + \alpha_3 s_3^a = \beta_1 s_1^b + \beta_2 s_2^b + \beta_3 s_3^b. \tag{A.18}\]

It follows from (A.13) that

\[
\begin{pmatrix}
v \\
v
\end{pmatrix} = \alpha_1 \begin{pmatrix}
s_1^a \\
1
\end{pmatrix} + \alpha_2 \begin{pmatrix}
s_2^a \\
1
\end{pmatrix} + \alpha_3 \begin{pmatrix}
s_3^a \\
1
\end{pmatrix}. \tag{A.19}
\]

Picking out respectively the first \(n\) entries of \(s_i^a\) and the last entry of \(s_i^b, i = 1, 2, 3\), we get

\[
v = \alpha_1 \begin{pmatrix}
q_{(n+1)1} \\
q_{(n+1)2}
\end{pmatrix} + \alpha_2 \begin{pmatrix}
q_{(n+1)2} \\
1
\end{pmatrix} + \alpha_3 \begin{pmatrix}
1_n \\
1
\end{pmatrix}, \tag{A.20}\]
equivalently,

\[ v = \alpha_1 \bar{q}_1 + \alpha_2 \bar{q}_2 + \alpha_3 1_{n+1}. \]  \hspace{2cm} (A.21)

Therefore, there exist scalars \( \gamma_i, i = 1, 2, 3 \), such that any vector \( v \in \text{null}(\hat{\Omega}) \) can be written as a linear combination of \( \bar{q}_1, \bar{q}_2, \) and \( 1_{n+1} \). This completes the proof. \( \square \)