Constructing tensegrity frameworks and related applications in multi-agent formation control
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Document Version
Publisher's PDF, also known as Version of record

Publication date: 2018

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

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Chapter 3

Merging rigid tensegrity frameworks

This chapter is to analyze the existence of a strictly proper self-stress for the merged tensegrity framework obtained by connecting two separate ones. It discusses what kind of condition allows the combined framework to be in equilibrium with the new stress without altering any existing member’s type, viz. cable or strut, in the previously existing frameworks. In addition, this chapter also studies the problem of merging two rigid tensegrity frameworks. It is shown that there exists an insertion of four new members with which the combined tensegrity framework is still rigid. By injecting distance perturbations into the combined framework, we propose a method with which the type of the fourth member can be determined.

3.1 Introduction

Rigidity graph theory serves as a fundamental mathematical tool to solve a wide range of problems in different fields, such as formation control of teams of mobile robots [6, 15, 35, 54, 64, 75, 98], localization of sensor networks [41, 110], molecular structural analysis in bio-chemistry [61, 128] and construction of stable frameworks in [107]. Of particular theoretical and practical interest are tensegrity frameworks, which are able to support large loads because of the use of cables and struts in comparison with bar frameworks. This property has been well employed in the design and control of tensegrity robots, see e.g. [94, 105].

In many, if not most, applications, the framework is expected to be rigid. This means the formation shape of the framework can be maintained as long as the distance constraints associated with all the edges are maintained, i.e. for a bar, an exact distance is maintained, for a cable, an upper bound is maintained, and for a strut, a lower bound is maintained. As stated in Chapter 2, global rigidity and universal rigidity can be accordingly defined if rigidity still holds in the whole given space and any higher dimensional space, respectively.

From an engineering point of view, a framework may be required to be augmented by adding one or more vertices, or even merging or becoming connected with another framework. More precisely, by merging we mean, given two frameworks, the operations of superimposing some of their vertices and adding additional
members joining a vertex pair with the vertices drawn from the two different frameworks. Normally, rigidity of frameworks is aimed to be preserved after adding vertices or merging.

In the plane, it is well known that the Henneberg construction (HC) [120] is an efficient technique to grow minimally rigid graphs. Recall that a rigid graph is said to be minimally rigid if no single edge can be removed without losing rigidity. The constructions of [120] propose two techniques, termed vertex addition and edge splitting. Due originally to Henneberg [58], a minimally rigid framework (in an ambient two or three-dimensional space) can acquire an additional vertex (in the process that additional members are introduced). Henneberg also proposed a merging procedure for two (minimally) rigid graphs in an ambient two-dimensional space, whereby three members (bars in a normal framework) were inserted to link the two frameworks.

However, the HC did not impose any geometric constraints, such as the length and angle, on the newly added edges, which might result in contradictions with the practical use. To solve this problem, a new construction method using Delaunay triangulation is proposed such that the angle measurements and the number of links can be optimized in [39]. As a follow-up, [40] addressed several topics relevant to rigid frameworks, including minimal cover problem, splitting and merging problem, and closing ranks problem. By minimal cover problem, we mean finding a new set of edges to be inserted into a given framework, such that the resulting framework is minimally rigid. The merging problem was regarded as one special case of minimal cover problem therein, and a type of new strategy based on reduction procedure was designed. Later, the conditions on how to generate globally rigid frameworks through merging in two- and three-dimensional space are provided, respectively. To fully cover all the possible cases of merging frameworks, where it is permitted to have one or more of the vertices of one merging framework made coincident with the same number of the other framework, three principles to conduct optimal merging of minimally or globally rigid frameworks were proposed in [133] for $\mathbb{R}^2$ and $\mathbb{R}^3$ frameworks. The merging is said to be optimal if the number of newly added member for a given number of shared vertices is minimized. At the same time, merging of multiple (more than two) rigid frameworks was investigated in [134], where each framework is regarded as a meta-vertex, and thus the problem can be solved from the meta-meta-formation prospective. In this way, the proposed strategies in [133] can be extended to the case involving multiple frameworks. Relying on HC operations, [136] investigated optimal growing of rigid frameworks in the sense of $H_2$ performance. Motivated by the implications of rigid networks in formation control and localizability, [18] identified the conditions for rigidity-preserving splitting as opposed to merging, under which the corresponding algorithms to perform the partition were also proposed therein. In addition to these work on growing rigid frameworks with undirected underlying graphs, some
efforts were also made based on directed graphs [53, 57]. In parallel to rigidity of undirected graphs, persistence was introduced for directed graphs in [57], where the conditions to ensure persistence after merging was given in both two- and three-dimensional space. Recently in [53], the dynamic merging problem under switching topology has been solved under the condition that each follower is jointly reachable from a leader over any time interval of certain length.

Even though some strategies have been proposed for growing rigid bar frameworks, to the best of our knowledge, the systematic analysis on how to create rigid tensegrity frameworks by merging has been rarely reported due to the complexity caused by the inequality constraints (2.4). One relevant work was presented in [119], where it has been shown that labeled 1-extension operation on a rigid tensegrity graph does not change its rigidity. A Tensegrity graph \( \mathcal{G}(V, C, S) \) is obtained through replacing the edges of a graph \( \mathcal{G}(V, E) \) by cables and struts. Denote by \( C \) and \( S \) the cable and strut set, respectively. The labeled 1-extension is defined in such a way that the original member \((u, w)\) is removed and a new vertex \(v\) is added together with three new members \((v, u), (u, w)\) and \((v, t)\) under the constraint that the type of \((u, w)\) is the same as at least one of \((v, u)\) and \((v, w)\).

Motivated by this circumstance, in this chapter, we will first prove that by merging two isolated infinitesimally rigid tensegrity framework with four members, there exist a proper self-stress such that the resulting tensegrity framework is still infinitesimally rigid and the type of the members can be preserved. We then explore the existence of the assignment of cables and struts to the four linking members when merging two rigid tensegrity frameworks, under which the combined tensegrity framework is rigid. In this chapter, all of the results are constrained to \( \mathbb{R}^2 \) unless otherwise indicated.

The rest of this chapter is organized as follows. In Section 3.2, we prove that the infinitesimal rigidity can be preserved by linking two originally infinitesimally rigid tensegrity frameworks with four appropriate members. In addition to infinitesimal rigid tensegrity frameworks, we also considered the problem of merging rigid tensegrity frameworks in Section 3.3. Concluding remarks are given in Section 3.4.

### 3.2 Merging infinitesimally rigid tensegrity frameworks

In this section, we investigate whether infinitesimal rigidity can be maintained after merging two pre-existing infinitesimally rigid tensegrity frameworks by inserting four members of the different type.

To be specific, the two given planar separate tensegrity frameworks \( \mathcal{T}_A \) and \( \mathcal{T}_B \), shown in Fig. 3.1, are infinitesimally rigid with underlying graphs \( \mathcal{G}_A(V_A, E_A) \) and \( \mathcal{G}_B(V_B, E_B) \), respectively. It is assumed that the underlying graphs \( \mathcal{G}_A \) and \( \mathcal{G}_B \) respectively consist of \( n_A \) and \( n_B \) nodes, which are linked via \( m_A \) and \( m_B \) members,
i.e., $|\mathcal{V}_A| = n_A$, $|\mathcal{V}_B| = n_B$, $|\mathcal{E}_A| = m_A$, and $|\mathcal{E}_B| = m_B$. For infinitesimally rigid tensegrity frameworks in $\mathbb{R}^2$, it follows from Lemma 2.8 that

$$m_A \geq 2n_A - 2 \quad \text{and} \quad m_B \geq 2n_B - 2. \quad (3.1)$$

Note that to obtain a rigid tensegrity framework from interconnecting two separate rigid tensegrity frameworks, it is necessary to have at least four connecting members. In addition, the absolute position (centroid location and orientation) of the frameworks $T_A$ and $T_B$ is irrelevant (though after insertion of the connections some freedom is of course lost).

In the sequel, denote by $\omega_i^A, i = 1, \ldots, m_A$, the original stress of member $i$ in $T_A$, and denote by $\hat{\omega}_i^A, i = 1, \ldots, n_A$, the new stress after inserting the four members $(A_i, B_i), i = 1, \ldots, 4$. Analogously, we can define the stress $\omega_i^B$ and $\hat{\omega}_i^B, i = 1, \ldots, m_B$ associated with tensegrity framework $T_B$. Those four members are labeled as member $\overline{1}, i = 1, \ldots, 4$, shown in Fig. 3.1. All the points of $A_i$ are distinct and so are all the points of $B_i$. It is also assumed that at least three of the members, without loss of generality $(A_i, B_i), i = 1, 2, 3$, are not concurrent or parallel. If all four members are concurrent or parallel, it is not hard to see that there is an infinitesimal displacement at right angles to each of the $A_i$ exists for which the corresponding infinitesimal length changes are zero, i.e. infinitesimal rigidity cannot hold.

**Theorem 3.1.** Consider two infinitesimally rigid tensegrity frameworks $T_A$ and $T_B$ in $\mathbb{R}^2$ and assume they are connected by 4 new members that are not concurrent or parallel. Then there exist an assignment of cables and struts to the new members such that the combined tensegrity framework is infinitesimally rigid as well. Furthermore,
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the sign of the new proper self-stress associated with pre-existing members can be maintained, namely, the members’ type, viz. cable or strut, can be preserved in both $\mathcal{T}_A$ and $\mathcal{T}_B$.

Proof. We first prove that there exists a new strictly proper self-stress such that the signs of the stress associated with the previously existing members do not change. Then by invoking Lemma 2.7, the infinitesimal rigidity of the combined tensegrity framework can be ensured.

For an infinitesimally rigid tensegrity framework $\mathcal{T}_A$, the self-stress satisfies

$$R_A^T(q)\omega^A = 0,$$  \hspace{1cm} (3.2)

where $R_A(q) \in \mathbb{R}^{m_A \times 2n_A}$ is the rigidity matrix of $\mathcal{T}_A$, and $\omega^A = [\omega_1^A, \cdots, \omega_{m_A}^A]^T \in \mathbb{R}^{m_A}$ is the stress with respect to $\mathcal{T}_A$. With the four members $(A_i, !'!B_i), i = 1, \cdots, 4$, joining the two separate tensegrity frameworks, the combined self-stress satisfies

$$R_A^T(q)\hat{\omega}^A + r_A^T w^N = 0,$$  \hspace{1cm} (3.3)

where $w^N$ is the stress associated with new edges in the form of $w^N = [\omega_1, \cdots, \omega_4]^T \in \mathbb{R}^4$. The matrix $r_A \in \mathbb{R}^{4 \times 2n_A}$ is defined by

$$r_A = \begin{bmatrix} 0 & \cdots & 0 & (q_{A1} - q_{B1})^T & 0 & 0 & 0 \\ 0 & \cdots & 0 & (q_{A2} - q_{B2})^T & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & (q_{A3} - q_{B3})^T & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & (q_{A4} - q_{B4})^T \end{bmatrix}.$$  \hspace{1cm} (3.4)

Suppose the types of pre-existing members are maintained after interconnection. A sufficient condition meeting this requirement is given by

$$\hat{\omega}^A = \omega^A + \Psi^A \text{sign} (\omega^A),$$  \hspace{1cm} (3.5)

where $\Psi^A = \text{diag}(\psi_1^A, \cdots, \psi_{m_A}^A) \in \mathbb{R}^{m_A \times m_A}$ is a diagonal matrix with $\psi_i^A > 0, i = 1, \cdots, m_A$, an arbitrary scalar. $\text{sign} (\omega^A) = [\text{sign}(\omega_1^A), \cdots, \text{sign}(\omega_{m_A}^A)]^T \in \mathbb{R}^{m_A}$. The setting $\hat{\omega}^A$ as in (3.5) ensures that $\text{sign}(\hat{\omega}^A) = \text{sign}(\omega^A)$. Note that (3.5) can be equivalently written as

$$\hat{\omega}_i^A = \begin{cases} \omega_i^A + \psi_i^A, & \text{if } \omega_i^A > 0, \\ \omega_i^A - \psi_i^A, & \text{if } \omega_i^A < 0. \end{cases}$$  \hspace{1cm} (3.6)
Substituting (3.5) into (3.3), we get
\[ R_A^T(q) (\omega^A + \Psi^A \text{sign} (\omega^A)) + r_A^T w^N = 0. \] (3.7)

In light of (3.2), we have
\[ R_A^T(q) \left( \Psi^A \text{sign}(\omega^A) \right) + r_A^T w^N = 0. \] (3.8)

Analogously, for tensegrity framework \( T_B \), we also have
\[ R_B^T(q) \left( \Psi^B \text{sign}(\omega^B) \right) + r_B^T w^N = 0, \] (3.9)

where variables \( R_B^T(q), \Psi^B, \omega^B \) and \( r_B \) are defined in the same way as the corresponding variables associated with \( T_A \). Denote by \( \psi^A \in \mathbb{R}^{m_A} \) the vector consisting of the diagonal entries of \( \Psi^A \), i.e., \( \psi^A = [\psi^A_1, \ldots, \psi^A_{m_A}]^T \). Note that for diagonal matrices \( \Psi^A \), there holds
\[ \Psi^A \text{sign}(\omega^A) = \text{diag}(\text{sign}(\omega^A)) \psi^A, \] (3.10)

and
\[ \Psi^B \text{sign}(\omega^B) = \text{diag}(\text{sign}(\omega^B)) \psi^B. \] (3.11)

Then, in view of (3.8)-(3.11), we have
\[ \begin{bmatrix} R_A^T(q) \text{diag}(\text{sign}(\omega^A)) & 0 \\ 0 & R_B^T(q) \text{diag}(\text{sign}(\omega^B)) \end{bmatrix} \begin{bmatrix} \psi^A \\ \psi^B \end{bmatrix} + r^T \omega^N = 0, \] (3.12)

where \( r = [r_A, r_B] \in \mathbb{R}^{4 \times 2(n_A + n_B)} \). Summarizing, to show the existence of a strictly proper self-stress associated with the combined tensegrity framework is equivalent to showing that the linear algebraic equation (3.12) has nontrivial solutions to the unknowns \( \psi^A, \psi^B \) and \( \omega^N \) subject to \( \psi^A > 0 \) and \( \psi^B > 0 \). [Here by denoting \( x > 0, x \in \mathbb{R}^n \), we mean that all of its entries are positive.]

This equation can be further written as
\[ \begin{bmatrix} R_A^T(q) \text{diag}(\text{sign}(\omega^A)) \\ 0 \end{bmatrix} \begin{bmatrix} \psi^A \\ \psi^B \end{bmatrix} + \begin{bmatrix} \psi^A \\ \psi^B \\ \omega^N \end{bmatrix} = 0. \] (3.13)

Note that the rigidity matrix of the combined tensegrity framework is in the form of
\[ R(q) = \begin{bmatrix} R_A(q) & 0 \\ 0 & R_B(q) \end{bmatrix}. \] (3.14)
Therefore, (3.13) can be transformed to

\[
R_{\top}(q) \begin{bmatrix}
\text{diag}(\text{sign}(\omega^A)) & 0 & 0 \\
0 & \text{diag}(\text{sign}(\omega^B)) & 0 \\
0 & 0 & I_4
\end{bmatrix} \begin{bmatrix}
\psi^A \\
\psi^B \\
\omega^N
\end{bmatrix} = 0.
\]

From the definition of \( R(q) \) in (3.14), it is easy to see that the dimension of \( R(q) \) is \((m_A + m_B + 4) \times 2(n_A + n_B)\), and thus \( R_{\top}(q) \in \mathbb{R}^{2(n_A+n_B) \times (m_A+m_B+4)} \). This rigidity matrix \( R(q) \) corresponds to the underlying bar framework as well. Given two infinitesimally rigid bar frameworks in the plane, it has been established by Henneberg [58] that if one joins the two frameworks by bars \((A_i, B_i), i = 1, 2, 3\), that are neither concurrent (when prolonged if necessary) nor parallel, then the framework with the three bars inserted is also infinitesimally rigid. Therefore, the combined new bar framework is infinitesimally rigid. Thus, there holds

\[
\text{rank}(R(q)) = \text{rank}(R_{\top}(q)) = \text{rank}(R_{\top}(q)D) = 2(n_A + n_B) - 3,
\]

which implies

\[
\text{dim}(\text{null}(R_{\top}(q))) = (m_A + m_B + 4) - 2(n_A + n_B) + 3
\]

\[
\geq (2n_A - 2 + 2n_B - 2 + 4) - 2(n_A + n_B) + 3
\]

\[
= 3
\]

Hence there always exist nontrivial solutions with respect to \([\psi^A, \psi^B, \omega^N]_{\top}\) of equation (3.15), disregarding for the moment sign constraints. We now choose one set of specified solutions with nonzero entries as \(\eta^A, \eta^B\) and \(\eta^N\), satisfying

\[
R_{\top}(q) \begin{bmatrix}
\text{diag}(\text{sign}(\omega^A)) & 0 & 0 \\
0 & \text{diag}(\text{sign}(\omega^B)) & 0 \\
0 & 0 & I_4
\end{bmatrix} \begin{bmatrix}
\eta^A \\
\eta^B \\
\eta^N
\end{bmatrix} = 0,
\]

where one or more entries of \(\eta^A\) and \(\eta^B\) might be negative. Hence to show the existence of a strictly proper self-stress, we still need to find the positive solutions with respect to \(\psi^A\) and \(\psi^B\) satisfying (3.15).

Before moving on, note that (3.18) can be equivalently rewritten as

\[
\begin{bmatrix}
R_{A\top}(q)\text{diag}(\text{sign}(\omega^A)) & 0 \\
0 & R_{B\top}(q)\text{diag}(\text{sign}(\omega^B))
\end{bmatrix} \begin{bmatrix}
\eta^A \\
\eta^B
\end{bmatrix} + r_{\top}\eta^N = 0.
\]
Now, we consider the homogeneous part of (3.12), i.e.,
\[
\begin{bmatrix}
R_A^\top(q) & 0 \\
0 & R_B^\top(q)
\end{bmatrix}
\begin{bmatrix}
\text{diag}(\text{sign}(\omega^A)) & 0 \\
0 & \text{diag}(\text{sign}(\omega^B))
\end{bmatrix}
\begin{bmatrix}
\psi^A \\
\psi^B
\end{bmatrix}
= 0. \quad (3.20)
\]

Then any solution of (3.20), denoted by \(\xi^A\) and \(\xi^B\), satisfies
\[
\begin{bmatrix}
R_A^\top(q)\text{diag}(\text{sign}(\omega^A)) & 0 \\
0 & R_B^\top(q)\text{diag}(\text{sign}(\omega^B))
\end{bmatrix}
\begin{bmatrix}
\xi^A + \eta^A \\
\xi^B + \eta^B
\end{bmatrix}
+ r^\top \eta^N = 0.
\] (3.21)

Therefore, \([\xi^A + \eta^A, \xi^B + \eta^B, \eta^N]^\top\) can be regarded as one set of solutions to the unknowns \([\psi^A, \psi^B, \omega^N]^\top\) in (3.12). Here note that \(\xi^A\) and \(\xi^B\) are general solutions to the homogeneous linear algebraic equation (3.20) rather than specific vectors. However, to satisfy the constraints that \(\psi^A > 0\) and \(\psi^B > 0\) in (3.12) under the chosen \(\omega^N = \eta^N\), we need to find at least one set of solutions among \(\xi^A\) and \(\xi^B\), denoted by \(\zeta^A\) and \(\zeta^B\), such that \(\zeta^A + \eta^A > 0\) and \(\zeta^B + \eta^B > 0\).

From (3.2), it is evident that
\[
\begin{cases}
\text{span}(\omega^A) \subseteq \text{null}(R_A^\top(q)), \\
\text{span}(\omega^B) \subseteq \text{null}(R_B^\top(q)).
\end{cases} \quad (3.22)
\]

Note that the dimension of \(R_A^\top(q)\) is \(2n_A \times m_A\), and \(\text{rank}(R_A^\top(q)) = 2n_A - 3\) for a infinitesimally rigid framework \(T_A\). We denote by \(\text{col}(X)\) and \(\text{null}(X)\) the column space and the null space of a matrix \(X\). Since
\[
\dim(\text{col}(R_A^\top(q))) + \dim(\text{null}(R_A^\top(q))) = m_A,
\]
and
\[
\text{rank}(R_A^\top(q)) = \dim(\text{col}(R_A^\top(q))),
\]
we have
\[
\dim(\text{null}(R_A^\top(q))) = m_A - 2n_A + 3. \quad (3.23)
\]

In view of (3.1), i.e., \(m_A \geq 2n_A - 2\), (3.23) satisfies
\[
\dim(\text{null}(R_A^\top(q))) \geq 1. \quad (3.24)
\]

By taking (3.22) into account, (3.24) implies that though \(\text{span}(\omega^A)\) lies in the null space of \(R_A^\top(q)\), it might not fully span the null space. Only when the equality sign holds in (3.24), \(\text{span}(\omega^A) = \text{null}(R_A^\top(q))\). For framework \(T_B\), we have the similar
results. So the specific solution $\zeta^A$ and $\zeta^B$ to (3.20) can be set to satisfy

$$\begin{cases} 
\text{diag}(\text{sign}(\omega^A))\zeta^A \in \text{span}(\omega^A), \\
\text{diag}(\text{sign}(\omega^B))\zeta^B \in \text{span}(\omega^B).
\end{cases} \tag{3.25}$$

To be specific, $\zeta^A$ and $\zeta^B$ can be chosen as

$$\zeta^A = k^A|\omega^A| \quad \text{and} \quad \zeta^B = k^B|\omega^B|, \tag{3.26}$$

where $k^A$ and $k^B$ are positive scalars. This equation can be written in the component-wise form as

$$\begin{bmatrix} \zeta_1^A \\ \vdots \\ \zeta_{m_A}^A \end{bmatrix} = k^A \begin{bmatrix} |\omega_1^A| \\ \vdots \\ |\omega_{m_A}^A| \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \zeta_1^B \\ \vdots \\ \zeta_{m_B}^B \end{bmatrix} = k^B \begin{bmatrix} |\omega_1^B| \\ \vdots \\ |\omega_{m_B}^B| \end{bmatrix}. \tag{3.27}$$

Recall the requirement that $\zeta^A + \eta^A > 0$, which is equivalent to

$$\begin{cases} 
\zeta_1^A + \eta_1^A > 0, \\
\vdots \\
\zeta_{m_A}^A + \eta_{m_A}^A > 0.
\end{cases} \tag{3.28}$$

Substituting (3.27) into (3.28), it yields

$$\begin{cases} 
k^A|\omega_1^A| + \eta_1^A > 0, \\
\vdots \\
k^A|\omega_{m_A}^A| + \eta_{m_A}^A > 0.
\end{cases} \tag{3.29}$$

Then, the parameter $k^A$ can be derived from (3.29) that

$$k^A > \max \left\{ -\frac{\eta_1^A}{|\omega_1^A|}, \ldots, -\frac{\eta_{m_A}^A}{|\omega_{m_A}^A|} \right\}. \tag{3.30}$$

Analogously, in terms of parameter $k^B$, we can also have

$$k^B > \max \left\{ -\frac{\eta_1^B}{|\omega_1^B|}, \ldots, -\frac{\eta_{m_B}^B}{|\omega_{m_B}^B|} \right\}. \tag{3.31}$$

Therefore, the existence of a strictly proper self-stress associated with the combined framework is proved. In addition, the type of pre-existing members can be maintained. Then it follows from Lemma 2.7 that the overall tensegrity frame-
work after interconnection is also infinitesimally rigid. This completes the proof of Theorem 3.1.

3.3 Merging rigid tensegrity frameworks

In this section we explain how to choose the type, viz. strut or cable, of the four elements in a tensegrity framework generated by joining two separate tensegrity frameworks.

More precisely, we shall consider two tensegrity frameworks, call them $T_A$ and $T_B$, with four identified points in each framework, viz, $A_1, \ldots, A_4, B_1, \ldots, B_4$. We assume that these are in general position and that connections are made, with either a strut or a cable, between each $A_i$ and the corresponding $B_i$, $i = 1, \ldots, 4$.

3.3.1 General approach to the problem

We first assume that the two separate tensegrity frameworks are infinitesimally rigid. In addition, there are (for the moment) bars between three points $A_1, A_2, A_3$ say of framework $T_A$ and the corresponding three points $B_1, B_2, B_3$ of framework $T_B$. We can measure the distance between $A_4$ and $B_4$ but there is no bar.

It was established by Henneberg [58] that if one extends the lines joining $A_i, B_i$, $i = 1, 2, 3$ and they are not concurrent or parallel, then the single framework with the three bars inserted to join the separate frameworks $T_A$ and $T_B$ is also infinitesimally rigid (the converse also holds).

This means that there is a finite number of noncongruent frameworks (defined up to inessential translation, reflection and rotation) realizing the associated set of lengths, with the frameworks $T_A$ and $T_B$ being held invariant apart from possible translation, reflection or rotation. Consequently, there is a finite number of possibility for the squared distance $\bar{d}_4 := ||q_{A4} - q_{B4}||^2$, one value of squared distance being associated with each of the noncongruent frameworks. (Of course, in some special cases, the distance for two noncongruent frameworks might coincide, but in general this cannot be expected).

Now if the squared distances $\bar{d}_i := ||q_{Ai} - q_{Bi}||^2$ for $i = 1, 2, 3$ are varied by an infinitesimally small amount to $d_i = \bar{d}_i + \delta d_i$, by for example holding framework $A$ fixed and rotating and/or translating framework $B$ infinitesimally, then there will be necessarily be a corresponding infinitesimal change replacing $\bar{d}_4$ by $d_4 = \bar{d}_4 + \delta d_4$.

It is evident that having fixed a particular framework from the finite set realizing the joining of $T_A$ and $T_B$ using bars of squared lengths $\bar{d}_1, \bar{d}_2, \bar{d}_3$ we can find a

\[^1\text{The majority of this section is taken from one of Prof. Brian D. O. Anderson’s unpublished technical reports. For completeness of the strategy for growing locally rigid tensegrity frameworks, we put it in this thesis.}\]
smooth function \( f : \mathbb{R}^3 \to \mathbb{R} \) for which
\[
f(d_1, d_2, d_3) = d_4
\] (3.32)
with in particular
\[
f(\bar{d}_1, \bar{d}_2, \bar{d}_3) = \bar{d}_4.
\] (3.33)
Moreover, to first order,
\[
\delta d_4 = \frac{\partial f}{\partial d_1} \delta d_1 + \frac{\partial f}{\partial d_2} \delta d_2 + \frac{\partial f}{\partial d_3} \delta d_3.
\] (3.34)

It is assumed that for generic \( \bar{d}_1, \bar{d}_2, \bar{d}_3 \), the function \( f \) will not have a critical point, i.e. there will not hold \( \frac{\partial f}{\partial d_i} = 0, i = 1, 2, 3 \) at \( \bar{d}_1, \bar{d}_2, \bar{d}_3 \). Now we make a change of viewpoint. We suppose that the three bars linking \( A_i \) to \( B_i \) for \( i = 1, 2, 3 \) are replaced by either a strut or a cable, and a strut or cable is placed between \( A_4 \) and \( B_4 \). We claim the following theorem.

**Theorem 3.2.** Consider the arrangement described above. There exists a choice of strut or cable for each of the four linkages between \( q_{A_i} \) and \( q_{B_i} \) to ensure rigidity.

**Proof.** When a desired distance is \( \bar{d}_i \), a strut of this length allows a positive value of \( \delta d_i \) and a chain a negative value. For \( i = 1, 2, 3 \), choose link \( i \) to be a strut or a cable according as \( \frac{\partial f}{\partial d_i} \) is positive or negative. For \( i = 4 \), choose a cable.

Observe that the choices for \( i = 1, 2, 3 \) ensure that any allowed changes of length in three links will always cause the right side in (3.34) to be positive. The choice for \( i = 4 \) however means that any allowed change in \( d_4 \) must be negative. This means there is no set of nonzero changes \( \delta d_i \) consistent with the assignation of struts and cables. Equivalently, the framework is rigid. This proves the theorem.

Obviously, the same conclusion follows if we reverse the assignment of cables and struts in the proof.

While the above theorem asserts the existence of a mixture of struts and cables assuring rigidity, the actual determination of the link type appears to involve knowledge of the functions \( f_i \), or their derivatives. The functions themselves in general may be very difficult to find. The derivatives on the other hand are easier to find, and we now show how this can be done using the rigidity matrix.

### 3.3.2 Determining \( \delta d_4 \) using the rigidity matrix

Suppose that the frameworks \( T_A \) and \( T_B \) referred to above are described by stacked vectors of vertex positions \( q_A, q_B \) and stacked vectors of squared edge lengths realized in the frameworks using tensegrity elements given by \( \bar{d}_A, \bar{d}_B \). In obvious
notation, the rigidity matrix of the overall framework with four tensegrity elements connecting the two frameworks is given by

\[
R = \begin{bmatrix}
R_A & 0 \\
0 & R_B \\
r_1^\top \\
r_2^\top \\
r_3^\top \\
r_4^\top 
\end{bmatrix}.
\] (3.35)

Now let us consider infinitesimal perturbations of the squared distances \(\bar{d}_1, \bar{d}_2, \bar{d}_3\) by amounts \(\delta d_1, \delta d_2\) and \(\delta d_3\). There will be a consequential perturbation in \(d_4\) of \(\delta d_4\), and consequential perturbations \(\delta q_A, \delta q_B\) of the vertex positions \(q_A, q_B\). The fact that \(R\) is the Jacobian of the mapping from vertex positions to squared distances means we can write

\[
\begin{bmatrix}
0 \\
\vdots \\
0 \\
\delta d_1 \\
\delta d_2 \\
\delta d_e \\
\delta d_4 
\end{bmatrix} = R \begin{bmatrix}
\delta q_A \\
\delta q_B
\end{bmatrix}.
\] (3.36)

Note that for given \(\delta d_1, \delta d_2\) and \(\delta d_3\), the perturbations \(\delta q_A, \delta q_B\) are not unique; the adjustment of any vertex perturbation vector by the same infinitesimal translation and/or rotation will leave the right side of the above equation invariant.

In pursuit of our ultimate goal of explaining how the derivatives \(\frac{\partial d_4}{\partial d_i}, i = 1, 2, 3\) can be computed using the rigidity matrix, it is convenient to eliminate this nonuniqueness. This is done as follows. Suppose that a new framework is formed by making four changes:

1. Bars replace all tensegrity elements within framework \(\mathcal{T}_A\) and framework \(\mathcal{T}_B\).
2. Bars are removed from framework \(\mathcal{T}_A\) and framework \(\mathcal{T}_B\) to make the frameworks both minimally rigid.
3. The tensegrity elements of squared lengths \(\bar{d}_1, \bar{d}_2, \bar{d}_3\) and \(\bar{d}_4\) are replaced by bars.
4. One vertex of framework \(\mathcal{T}_A\) is pinned at the origin and a neighbor vertex in the framework is pinned on the \(x\) axis.

The resulting framework is pinned and is a rigid joint-bar framework. It is not minimally rigid but has the property that removal of the bar corresponding
to length $\bar{d}_4$ would make it minimally rigid. Call the associated rigidity matrix $\hat{R}$; this is a ‘reduced’ rigidity matrix, of size $(2n - 2) \times (2n - 3)$ where $n$ is the total vertex count for frameworks $T_A$ and $T_B$. The reduced rigidity matrix is obtained from $R$ defined in (3.35) through deletion of rows within the blocks $[R_A \ 0]$ and $[0 \ R_B]$, corresponding to step 2 of the reduction procedure and deletion of three columns of $R$, corresponding to step 4. The rank of $\hat{R}$ is the same as the rank of $R$, viz $2n - 3$. Let $\hat{r}^T$ denote the last row of $\hat{R}$, and note that it is obtainable from $r^T$ through deletion of three entries. It is also a linear combination of the rows first $2n - 3$ rows of $\hat{R}$.

Let us also define one further matrix, $\hat{\hat{R}}$, which we term a doubly reduced rigidity matrix, obtained from $\hat{R}$ by deleting its last row.

Define $\delta q_{red}^A$ to be the vector obtained from $\delta q_A$ by deletion of those entries of $\delta q_A$ associated with the pinning process of step 4 above. Define $\delta \hat{q}_{red}^A$ and $\delta \hat{q}_B$ to be the infinitesimal perturbations in vertex positions (disregarding pinned coordinates) given infinitesimal perturbations $\delta d_1, \delta d_2, \delta d_3$ for the transformed framework.

Then these perturbations still satisfy (3.36) but they are now unique. To see this, observe that for the transformed framework, there will hold (with $0_{m_A}, 0_{m_B}$ denoting vectors of zeros with $2n_A - 3, 2n_B - 3$ entries, $n_A, n_B$ denoting the number of nodes in frameworks $T_A, T_B$)

$$
\begin{bmatrix}
0_{m_A} \\
0_{m_B} \\
\delta d_1 \\
\delta d_2 \\
\delta d_3
\end{bmatrix}
= \hat{R}
\begin{bmatrix}
\delta \hat{q}_{red}^A \\
\delta \hat{q}_B
\end{bmatrix},
$$

(3.37)

and

$$
\begin{bmatrix}
0_{m_A} \\
0_{m_B} \\
\delta d_1 \\
\delta d_2 \\
\delta d_3 \\
\delta d_4
\end{bmatrix}
= \hat{R}
\begin{bmatrix}
\delta \hat{q}_{red}^A \\
\delta \hat{q}_B
\end{bmatrix}
= \hat{R}
\hat{r}^T
\begin{bmatrix}
\delta \hat{q}_{red}^A \\
\delta \hat{q}_B
\end{bmatrix}.
$$

(3.38)

Now suppose that infinitesimal values of $\delta d_1, \delta d_2$ and $\delta d_3$ are specified. It follows from the invertibility of $\hat{R}$ that $\delta \hat{q}_{red}^A, \delta \hat{q}_B$ are expressible uniquely in terms
of $\delta d_1, \delta d_2, \delta d_3$ and entries of the inverse of $\hat{R}$, i.e.

$$
\begin{bmatrix}
\hat{\delta q}_A \\
\delta q_B
\end{bmatrix} = \hat{R}^{-1}
\begin{bmatrix}
0_{mA} \\
0_{mB} \\
\delta d_1 \\
\delta d_2 \\
\delta d_3
\end{bmatrix}.
$$

(3.39)

Then from (3.38) we have simply

$$
\delta d_4 = \hat{r}_4^\top \hat{R}^{-1}
\begin{bmatrix}
0_{mA} \\
0_{mB} \\
\delta d_1 \\
\delta d_2 \\
\delta d_3
\end{bmatrix},
$$

(3.40)

and now we see how the partial derivatives of the function $\bar{d}_4 = f(\bar{d}_1, \bar{d}_2, \bar{d}_3)$ are identified in terms of entries of the rigidity matrix. The derivatives are in fact the last three entries of the row vector $\hat{r}_4^\top \hat{R}^{-1}$.

### 3.4 Concluding remarks

In this chapter, we considered two scenarios of merging rigid tensegrity frameworks. First, we have shown that by interconnecting infinitesimally rigid tensegrity frameworks with four new members, there exists a distribution of cables and struts to the new members such that the merged tensegrity framework is still infinitesimally rigid. Furthermore, we also proved that the infinitesimal rigidity of the combined tensegrity framework can be obtained without changing the type of pre-existing members. In the case of merging rigid tensegrity frameworks, it has been shown that the rigidity can also be preserved by properly choosing the joining members. To efficiently determine the type of the fourth member once others are fixed, one method has been proposed by invoking the rigidity matrix.