Constructing tensegrity frameworks and related applications in multi-agent formation control
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Chapter 2

Theoretical preliminaries

In this chapter, we first introduce some general notation and definitions that will be used throughout the thesis. To make this thesis self-contained and its definitions consistent, we then mainly follow [25] to review some basic concepts on tensegrity frameworks. Among those, the key concepts from graph rigidity theory associated with tensegrity frameworks and bar frameworks will be highlighted respectively. This lays theoretical foundations in stability analysis of distributed formation control, which in turn serves as one application of tensegrity frameworks in this thesis.

2.1 Notations

In this section, we introduce some standard notations. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. $\mathbb{R}^{m \times n}$ is used to represent the set of real matrices with dimension $m \times n$. Denote by $I_n$ the identity matrix with dimension $n$. We use $1_n$ and $0_n$ to denote the $n$-dimensional column vector with all ones and zeros, respectively. The subscripts will be omitted when there is no confusion in the context.

For a given matrix $X \in \mathbb{R}^{m \times n}$, $X^\top$ denotes its transpose. The rank, column space (i.e., image) and null space of a matrix $X$ are represented by $\text{rank}(X)$, $\text{col}(X)$ and $\text{null}(X)$, respectively. Let $\det(X)$ denote the determinant of a real square matrix $X$. For $x \in \mathbb{R}$, $\text{sign}(x)$ is the signum function. For a vector $x$, $\text{sign}(x)$ is defined in a component-wise manner. For a vector $x = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$, $\|x\|$ represents the Euclidean norm of $x$, and $\text{diag}(x) = \text{diag}(x_1, \cdots, x_n)$ is a diagonal matrix with the vector $x$ on its diagonal. For a matrix $V = [v_1, \cdots, v_n] \in \mathbb{R}^{m \times n}$, we use $\text{span}(V)$ to denote the linear span of the elements $\{v_1, \cdots, v_n\}$. For a set $S$, $|S|$ denotes the cardinality of $S$.

Given two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$, the Kronecker product $A \otimes B$ is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}_{mp \times nq}.$$
2. Theoretical preliminaries

2.2 Preliminary on frameworks and rigidity theory

2.2.1 Graph theory

A graph comprises a set of vertices and edges, in which the edges specify how the vertices are connected. We assume that the graph studied in this thesis is finite and simple, i.e., without loops or multiple edges. Let $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ be, respectively, the vertex set and the edge set of a graph $\mathcal{G}$ representing the neighboring relationships between $n$ vertices. The graph $\mathcal{G}$ is defined as the pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. A graph is said to be directed if the pairs of nodes are ordered, namely, a directed edge $(i, j)$ means that the information flows along the direction from $i$ to $j$, but not necessarily vice versa. In contrast, a graph is undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$ [101]. In this thesis, we mainly focus on undirected graphs, where vertices $i$ and $j$ are neighbors if and only if there exists an edge $(i, j)$. The set of vertices that are adjacent to $i$ is denoted by $\mathcal{N}_i = \{j | (i, j) \in \mathcal{E}\}$. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ associated with the graph $\mathcal{G}$ is defined in such a way that $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. For an undirected graph, $A$ is a symmetric matrix.

Define the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ by

$$l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}, \quad l_{ij} = -a_{ij}, \quad i \neq j. \quad (2.1)$$

It can be checked that the row sums of the Laplacian matrix equal zero, which implies that $1_n$ is always an eigenvector associated with the zero eigenvalue. This property plays a key role in controller design for achieving consensus of multi-agent systems.

Under the assumption that we have assigned an arbitrary orientation to $\mathcal{G}$, the incidence matrix $H = [h_{ij}] \in \mathbb{R}^{n \times |\mathcal{E}|}$ encoding the relationships between edges and nodes is defined by

$$h_{ij} = \begin{cases} 1, & \text{if node } i \text{ is the head of edge } j, \\ -1, & \text{if node } i \text{ is the tail of edge } j, \\ 0, & \text{otherwise,} \end{cases}$$

where $i$ and $j$ are the indices running over the node and edge sets, respectively. With incidence matrix, the Laplacian matrix can be shown to be equal to

$$L = HH^\top.$$

From this property, we can observe that Laplacian matrix is always symmetric and
positive semidefinite [83].

**Lemma 2.1.** [101, Lemma 2.10] Suppose that \( z = [z_1^T, \cdots, z_n^T]^T \) with \( z_i \in \mathbb{R}^d \) and \( L \) defined in (2.1). Then the following statements are equivalent.

(a) \( L \) has a simple zero eigenvalue with an associated eigenvector \( 1_n \) and all other eigenvalues are positive,

(b) The undirected graph of \( L \) is connected,

(c) \( (L \otimes I_d)z = 0 \) implies that \( z_1 = \cdots = z_n \),

(d) Consensus is reached asymptotically for the system \( \dot{z} = -(L \otimes I_d)z \),

(e) The rank of \( L \) is \( n - 1 \).

**Remark 2.2.** The graph in Lemma 2.1 is assumed to be undirected, which is a special case of the directed graph discussed in [101, Lemma 2.10].

### 2.2.2 Frameworks and rigidity

A **configuration** is a finite collection of \( n \) labeled points in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \), denoted by \( q = [q_1, \cdots, q_n] \). We say a configuration \( q \) is **generic** if the elements of \( q \) are algebraically independent over the rational numbers, namely, there is no non-zero polynomial with rational coefficients that vanishes at the elements of \( q \) [23].

Also, to avoid certain special cases, for a framework in a Euclidean \( d \)-dimensional space, an assumption is often made that the framework is at a general position, i.e., no \( k \) points of \( q_1, \cdots, q_n \) lie in a \((k-1)\)-dimensional affine space for \( 1 \leq k \leq d \). We introduce here a class of transformation of \( q \), called **affine transformation**, which is determined by a matrix \( M \in \mathbb{R}^{d \times d} \) and a vector \( b \in \mathbb{R}^d \). Then given a configuration \( q \), an affine image is given by

\[
\mathcal{A}(q) \doteq \{ p = [p_1, \cdots, p_n]| p_i = Mq_i + b, M \in \mathbb{R}^{d \times d} \text{ and } b \in \mathbb{R}^d, i = 1, \cdots, n \},
\]

or equivalently

\[
\mathcal{A}(q) \doteq \{ p = (I_n \otimes M)q + 1_n \otimes b|M \in \mathbb{R}^{d \times d} \text{ and } b \in \mathbb{R}^d \}.
\]

A graph \( G \) together with its configuration \( q \) in \( \mathbb{R}^d \) is called a **framework**, denoted by \((G, q)\). The edges of the underlying graph \( G \) in \((G, q)\) are called **members**. If a member has fixed length constraint, we call this a **bar**. A framework is said to be a **bar framework** if all its members are bars. In the rest of this section, we will use ‘framework’ for ‘bar framework’ unless otherwise stated.
Graph rigidity theory is for identifying whether partial edge lengths can determine the graph shape uniquely up to translations, rotations, and reflections. Some basic concepts are given as follows.

Given a framework \((G, q)\) in \(\mathbb{R}^d\), if there exists another framework \((G, p)\) in \(\mathbb{R}^d\) such that \(\|p_i - p_j\| = \|q_i - q_j\|, \forall (i, j) \in E\), then we say that \((G, p)\) is equivalent to \((G, q)\). Furthermore, they are congruent if \(\|p_i - p_j\| = \|q_i - q_j\|, \forall i, j \in V\). With these concepts, we say that a framework \((G, q)\) in \(\mathbb{R}^d\) is

- **(locally) rigid**, if all frameworks \((G, p)\) in \(\mathbb{R}^d\) equivalent to \((G, q)\) and sufficiently close to \((G, q)\) are congruent to \((G, q)\);
- **globally rigid**, if all frameworks \((G, p)\) in \(\mathbb{R}^d\) equivalent to \((G, q)\) are congruent to \((G, q)\);
- **universally rigid**, if all frameworks \((G, p)\) in any \(\mathbb{R}^d \supset \mathbb{R}^d\) equivalent to \((G, q)\) are congruent to \((G, q)\).

In addition to the intuitive geometric definitions of rigidity, we also use rigidity matrix to justify the rigidity property of a framework by checking its rank. Before introducing the definition, we need the distance function given by

\[ f_G(q_1, \cdots, q_n) = \frac{1}{2} \left[ \cdots, \|q_i - q_j\|^2, \cdots \right]^T, \]

where \((i, j) \in E\).

**Definition 2.3.** [9] A framework \((G, q)\) is rigid in \(\mathbb{R}^d\) if there exist a neighborhood \(P\) of \(q\) such that \(f_G^{-1}(f_G(q)) \cap P = f_K^{-1}(f_K(q)) \cap P\), where \(K\) is the complete graph with the same vertex set \(V\) of \(G\).

For a rigid framework, it means if one node moves, the rest also moves as a whole in order to satisfy the distance constraints. One illustrative example is shown in Fig. 2.1. The rectangle framework presented in Fig. 2.1(a) is not rigid since the top two nodes can smoothly move in the horizontal direction without breaking other distance constraints. This framework becomes rigid if we insert a crossing bar between nodes 1 and 3 (or 2 and 4) shown in Fig. 2.1(b).

To characterize the rigidity of a framework, another useful tool is the rigidity matrix \(R(q) \in \mathbb{R}^{\left|E\right| \times nd}\), which is defined by

\[ R(q) = \frac{\partial f_G(q)}{\partial q}. \] (2.2)

As an example, if we assign the orders of members in Fig. 2.1(b) shown as the
2.3. Tensegrity frameworks and rigidity

circled numbers, the rigidity matrix of the framework can be written as

\[
R(q) = \begin{bmatrix}
(q_1 - q_2)^\top & (q_2 - q_1)^\top & 0 & 0 \\
0 & (q_2 - q_3)^\top & (q_3 - q_2)^\top & 0 \\
0 & 0 & (q_3 - q_4)^\top & (q_4 - q_3)^\top \\
(q_1 - q_4)^\top & 0 & 0 & (q_4 - q_1)^\top \\
(q_1 - q_3)^\top & 0 & (q_3 - q_1)^\top & 0
\end{bmatrix}.
\]

Before moving forward, we introduce some concepts related to infinitesimal rigidity. Given a framework \((G, q)\), \(\dot{q} = [\dot{q}_1, \cdots, \dot{q}_n]\) is called an infinitesimal motion if for each edge \((i, j)\), there holds

\[
(q_i - q_j)^\top (\dot{q}_i - \dot{q}_j) = 0.
\]

It is easy to check that rotations, translations, and their combinations always satisfy the above equation. These motions are said to be trivial. Then we say that a framework is infinitesimally rigid if the infinitesimal motions are trivial. This can also be validated through the following lemma.

**Lemma 2.4.** [55] A framework \((G, q)\) is infinitesimally rigid in a \(d\)-dimensional space if

\[
\text{rank}(R(q)) = nd - d(d + 1)/2.
\]

In general, infinitesimal rigidity implies rigidity, but the converse is not true. Infinitesimal rigidity only allows the motions as combinations of translation and rotation. It has been discussed that the framework in Fig. 2.1(a) is not rigid, and not infinitesimally rigid. One set of non-trivial infinitesimal motions is presented in Fig. 2.1(c). The framework in Fig. 2.1(b) is not only rigid, but also infinitesimally rigid, which can be verified by the fact that \(\text{rank}(R) = 5\).

**Definition 2.5.** [5] A framework is minimally rigid if it is rigid and no edge can be removed without losing rigidity.

To be specific, a rigid framework \((G, q)\) with \(n\) vertices in 2D or 3D is minimally rigid, if it has exactly \(2n - 3\) or \(3n - 6\) edges, respectively. It can be checked that the framework shown in Fig. 2.1(b) is infinitesimally minimally rigid.

### 2.3 Tensegrity frameworks and rigidity

A tensegrity framework \((G, q)\) is obtained by embedding an undirected graph \(G\) in \(\mathbb{R}^d\) and replacing the edges of \(G\) by three types of members: cables, struts and bars, where cables and struts can only carry tensions and compressions respectively, while bars can carry both tensions and compressions. For the physical interpretation,
we know that the member \((i, j)\) carries different internal forces \(f_{ij}\) depending on its rest length \(l_{ij}\), current length \(\|r_{ij}\|\) and stiffness \(k_{ij}\). Their relationship can be formulated by

\[
\begin{cases}
  k_{ij}(\|r_{ij}\| - l_{ij}), & \text{if } (i, j) \text{ is a cable and } \|r_{ij}\| > l_{ij}, \\
  -k_{ij}(l_{ij} - \|r_{ij}\|), & \text{if } (i, j) \text{ is a strut and } \|r_{ij}\| < l_{ij}, \\
  k_{ij}(\|r_{ij}\| - l_{ij}), & \text{if } (i, j) \text{ is a bar}, \\
  0, & \text{otherwise}.
\end{cases}
\]  

(2.3)

As one can see from (2.3), cables and struts sustain positive and negative internal forces, respectively. However, bars can carry both positive and negative forces, which implies that bars can act as springs generating both attractive and repulsive forces. We illustrate these properties graphically in Fig. 2.2.

For a tensegrity framework \((G, q)\) in \(\mathbb{R}^d\) with the fixed configuration \(q\), we are interested in its associated configurations \(p\) that satisfy the following tensegrity
2.3. Tensegrity frameworks and rigidity

![Diagram of forces and lengths](image)

Figure 2.2: Relationships between internal forces and lengths with respect to different type of members.

\[
\begin{align*}
\|p_i - p_j\| &\leq \|q_i - q_j\|, & \text{when } (i, j) \text{ is a cable}, \\
\|p_i - p_j\| &\geq \|q_i - q_j\|, & \text{when } (i, j) \text{ is a strut}, \\
\|p_i - p_j\| & = \|q_i - q_j\|, & \text{when } (i, j) \text{ is a bar}.
\end{align*}
\]
We say that the tensegrity framework \((\mathcal{G}, q)\) whose shape is determined by the configuration \(q\) is \textit{rigid} if any other configuration \(p\) is congruent to \(q\) whenever \(p\) is sufficiently close to \(q\) and satisfies the tensegrity constraints (2.4); furthermore, if the congruent relationship between \(p\) and \(q\) holds for all \(p\) in \(\mathbb{R}^d\), then we say \((\mathcal{G}, q)\) is \textit{globally rigid}; and if this congruent relationship still holds for all \(p\) living in any higher-dimensional space than \(\mathbb{R}^d\), we say \((\mathcal{G}, q)\) is \textit{universally rigid} [21, 104]. For the rest of this thesis, we only consider cable-strut tensegrity frameworks unless otherwise stated.

To distinguish different members in a tensegrity framework \((\mathcal{G}, q)\), we employ the concept of \textit{stress}. For each member \((i, j)\) of \((\mathcal{G}, q)\), we assign a scalar \(\omega_{ij} = \omega_{ji}\), and use \(\omega \in \mathbb{R}^{|E|}\) to denote the vector \(\omega = (\cdots, \omega_{ij}, \cdots)^T\). Then \(\omega\) is called a \textit{stress} of \((\mathcal{G}, q)\); if further, each \(\omega_{ij}\) satisfies \(\omega_{ij} \geq 0\) whenever \((i, j)\) is a cable and \(\omega_{ij} \leq 0\) whenever \((i, j)\) is a strut, then \(\omega\) is said to be a \textit{proper stress}. Note that for a stress to be \textit{proper}, there is no restriction on a bar. In physics, \(\omega_{ij}\) is interpreted as the axial force per unit length along the member \((i, j)\). It is called \textit{strict} if the stress in each cable and strut is nonzero. We say that \(\omega\) is a \textit{self-stress} for the configuration \(p\) in \(\mathbb{R}^d\) of the framework \((\mathcal{G}, p)\) if for each node \(i\), there holds

\[
\sum_{j \in N_i} \omega_{ij}(q_j - q_i) = 0. \tag{2.5}
\]

We also call the stress \textit{an equilibrium stress} if equation (2.5) holds.

Note that for an affine transformation, we have

\[
\sum_{j \in N_i} \omega_{ij}(Mq_i - Mq_j) = M \sum_{j \in N_i} \omega_{ij}(q_i - q_j) = 0, \quad \forall i,
\]

which implies that the affine transformations donot change the equilibrium stress. The corresponding \textit{stress matrix} \(\Omega = [\Omega_{ij}] \in \mathbb{R}^{n \times n}\) is defined by

\[
\Omega_{ij} = \begin{cases}
-\omega_{ij}, & i \neq j, \\
\sum_{j \in N_i} \omega_{ij}, & i = j.
\end{cases} \tag{2.6}
\]

For the rigidity of tensegrity frameworks, we introduce the following lemmas.

\textbf{Lemma 2.6.} [25, Theorem 4.3.1] \textit{If a tensegrity} \((\mathcal{G}, p)\) \textit{is infinitesimally rigid, then it is rigid.}

\textbf{Lemma 2.7.} [104] \textit{Let} \((\mathcal{G}, p)\) \textit{be a tensegrity framework in} \(\mathbb{R}^d\), \textit{and} \((\bar{\mathcal{G}}, p)\) \textit{the corresponding bar framework, where all the members of the tensegrity framework have been replaced by bars. Then} \((\mathcal{G}, p)\) \textit{is infinitesimally rigid (and equivalently statically rigid) if and only if the following two conditions are satisfied:}

1. \((\bar{\mathcal{G}}, p)\) \textit{is infinitesimally rigid in} \(\mathbb{R}^d\), \textit{and}
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2). there is a proper self-stress $\omega$ for $(G, p)$, where for each cable and strut $(i, j)$ of $G$, $\omega_{ij} \neq 0$.

**Lemma 2.8.** [25, Corollary 4.8.2] If a tensegrity $(G, p)$, with $n$ nodes in $\mathbb{R}^d$, $m$ members and at least one strut or cable, is infinitesimally rigid, then $m \geq nd - d(d + 1)/2 + 1$.

To define the universal rigidity for a class of tensegrity frameworks, we present the following lemma.

**Lemma 2.9.** [1] Let $(G, q)$ be a generic tensegrity framework on $n$ vertices in $\mathbb{R}^d$, $d \leq n - 2$. Then $(G, q)$ is universally rigid if and only if there exists a positive semi-definite stress matrix $\Omega$ such that its rank is $n - d - 1$.

Next we present conditions to guarantee super-stability of a tensegrity framework. We first introduce some basic concepts.

**Definition 2.10.** [23] If $\omega$ is a proper equilibrium stress for the tensegrity framework $(G, q)$, then the relative position $q_i - q_j$ is called a stressed direction if $\omega_{ij} \neq 0$.

**Definition 2.11.** [43] We say a function $A : \mathbb{R}^m \to \mathbb{R}^n$ is affine if there is a linear function $L : \mathbb{R}^m \to \mathbb{R}^n$ and a vector $b \in \mathbb{R}^n$ such that

$$A(x) = L(x) + b$$ (2.7)

for all $x$ in $\mathbb{R}^m$.

**Definition 2.12.** [24] A flex of a framework $(G, q)$ is a continuous motion $q(s), 0 \leq s \leq 1$, $q(0) = q$, where $q(s)$ is equivalent to $q$. It is nontrivial if $q(s)$ is not congruent to $q$ for all $s > 0$. If $q(s) = A(s)p(0)$, where $A(s)$ is an affine function of Euclidean space, then we say $q(s)$ is an affine flex.

**Definition 2.13.** Given a collection of vectors $V = \{v_i\}_{i \in \mathbb{N}}$, an affine span of $V$ is defined as the collection of all finite linear combinations, i.e.,

$$\sum_{1}^{k} a_i v_{s_i}, \quad k \in \mathbb{N}^+, \quad v_{s_i} \in V,$$

where $a_i$ are all scalars satisfying $\sum a_i = 1$.

**Lemma 2.14.** [25] Let $(G, q)$ be a tensegrity framework whose affine span of $q$ is $\mathbb{R}^d$, with an equilibrium stress $\omega$ and stress matrix $\Omega$. Suppose further that

1. $\Omega$ is positive semi-definite,

---

$^1\mathbb{N}$ denotes the set of natural numbers.

$^2\mathbb{N}^+$ denotes the set of positive natural numbers.
2. The rank of $\Omega$ is $n - d - 1$,

3. and $(G, q)$ has no affine flex in $\mathbb{R}^d$,
	hen $(G, q)$ is super stable.

**Remark 2.15.** Lemma 2.14 is known as the fundamental theorem for super-stability. When $\omega$ is a proper equilibrium stress for $(G, q)$, a stressed direction is the relative position of two connected nodes $i$ and $j$ with $\omega_{ij} \neq 0$, i.e., $q_i - q_j$. Note that condition (3) of Lemma 2.14 can be replaced by “the framework $(G, q)$ is rigid in $\mathbb{R}^d$" [23], and “the configuration $q$ is in general position” [24]. We also have the conclusion that super stable tensegrity frameworks are universally rigid, but not vice versa.

Another lemma describing translation movements of a generic configuration is introduced as follows.

**Lemma 2.16.** [78] Suppose $q = [q_1^T, \ldots, q_n^T]^T$ is a generic configuration in $\mathbb{R}^d$. A configuration $p \in A(q)$ is a translation of $q$ if and only if there exist at least $d$ pairs of vertices such that the dimension of the convex hull of that $d$ pairs of vertices is $d$ and

$$p_k - p_j = q_k - q_j.$$

The following lemma will be used in the sequel in different places for the discussion of positive semi-definite stress matrices.

**Lemma 2.17.** Given positive semi-definite matrices $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{n \times n}$, let $Z = X + Y$. Then for any nonzero vector $\xi \in \mathbb{R}^n$, $\xi \in \text{null}(Z)$ if and only if $\xi \in \text{null}(X)$ and $\xi \in \text{null}(Y)$. 