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Switch observability for switched linear systems

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Abstract

Mode observability of switched systems requires observability of each individual mode. We consider other concepts of observability that do not have this requirement: Switching time observability and switch observability. The latter notion is based on the assumption that at least one switch occurs. These concepts are analyzed and characterized both for homogeneous and inhomogeneous systems.

Keywords: mode detection, observability, switched systems, fault detection

1. Introduction

Mode observability of switched systems is concerned with recovering the initial state as well as the switching signal from the output (and the input) and has been widely studied, see e.g. Babaali and Pappas (2005) for inhomogeneous discrete-time systems, Elhamifar et al. (2009) for homogeneous discrete-time systems, Vidal et al. (2003) for homogeneous systems, and Lou and Si (2009) for inhomogeneous systems. For a recent overview of observability notions see De Santis and Di Benedetto (2016).

Since for mode observable systems it is in particular possible to recover the state for constant switching signals, each mode necessarily has to be observable. In the context of fault-detection (or diagnosis) the different modes of a switched system describe faulty and non-faulty variants of the system and a switch represents a fault. Requiring observability of each mode, in particular of each faulty mode, might be a too strong assumption. Instead of mode observability, it would be sufficient to compute the switching signal and the state if an error occurs. This idea is formalized in the novel notion of switch observability, \((x, \sigma_1)\)-observability for short.

Before characterizing \((x, \sigma_1)\)-observability, we first have to consider the problem of detecting switches (switching time observability or \(t_S\)-observability). This has been done in Babaali and Pappas (2005) in the homogeneous case, but the generalization to inhomogeneous systems is not straightforward as the switch might occur in an interval where the state is zero. This difficulty has been avoided so far, e.g. in Elhamifar et al. (2009) by assuming mode observability. We are able to relax this assumption and to fully characterize \(t_S\)-observability without any additional assumptions.

Similar to the classical observability of linear systems, we derive characterizations of the observability notions based on rank-conditions on the Kalman observability matrices. Our results are summarized in Figure 1 where \(\Theta_i\) and \(\Gamma_i\) are the Kalman observability matrix and Hankel matrix of mode \(i\), respectively. These notions are defined in Section 2 and 3;

\[ \text{rank}(A) \] denotes the rank of \(A\).

The first column in Figure 1 gives the result for the homogeneous case: The strongest notion considered here is \((x, \sigma)\)-observability, which coincides with switching signal observability \((\sigma)\)-observability. It implies \((x, \sigma_1)\)-observability and \(t_S\)-observability. The reverse implications are false in general, we will show this by some examples. For the inhomogeneous case, we consider two different setups. First we restrict our attention to systems with analytic input and with some restriction on the input matrices (assumption \(A_2\)). Then we drop \(A_2\) and require only smooth input. This makes it necessary to consider equivalence classes of switching signals, but gives observability notions with the same characterizations as in the more restrictive setup.

Our main contribution is the concept of (strong) \((x, \sigma_1)\)-observability and its characterization. Also the characterization of strong switching time observability for inhomogeneous systems is new.

2. Homogeneous Systems

2.1. System class and preliminaries

A switching signal is a piecewise constant, right-continuous function \(\sigma : \mathbb{R} \to \mathcal{P} := \{1, \ldots, N\}, N \in \mathbb{N}\), with locally finitely many discontinuities. The discontinuities of \(\sigma\) are also called switching times:

\[ T_\sigma := \{ t_\sigma \in \mathbb{R} | t_\sigma \text{ is a discontinuity of } \sigma \} \]

We assume that all switches occur for \(t > 0\), i.e. \(T_\sigma \subset \mathbb{R}_{>0}\). Consider switched linear systems of the form

\[
\begin{align*}
\dot{x} &= A_\sigma x, & x(0) &= x_0, \\
y &= C_\sigma x
\end{align*}
\]
with switching signal $\sigma$ and $A_i \in \mathbb{R}^{n \times n}$, $C_i \in \mathbb{R}^{p \times n}$ for all $i \in \mathcal{P}$ and denote its solution and output by $x(t, \sigma)$ and $y(t, \sigma)$, respectively.

Furthermore, let $\Theta_i^{[\nu]}$ be the Kalman observability matrix for mode $i$ with $\nu$ row blocks, i.e.,

$$\Theta_i^{[\nu]} = \begin{bmatrix} C_i^T & (C_i A_i) & \cdots & (C_i A_i^{\nu-1})^T \end{bmatrix}^T$$

and let $\Theta_i^{[\infty]}$ be the corresponding infinite Kalman observability matrix. For observability of unswitched systems, it suffices to consider $\nu = n$. In our setting, the required size increases as we have to compare the output from different modes.

For any sufficiently smooth function $y : \mathbb{R} \to \mathbb{R}^p$ denote by $y^{[\nu]} : \mathbb{R} \to \mathbb{R}^p$ the vector of $y$ and its first $\nu - 1$ derivatives and by $y^{[\infty]}$ the (countably) infinite vector of $y$ and its derivatives. The same can be done for piecewise-smooth functions, where $y(t^-)$ and $y(t^+)$ denote the left-hand side and right-hand side limit at $t$, respectively. Then the output $y(t, \sigma)$ of (1) satisfies for all $t \in \mathbb{R}$:

- $y^{[\nu]}(t^+) = \Theta_i^{[\nu]} x_i(t)$, $\nu \in \mathbb{N} \cup \{\infty\}$,
- $y^{[\nu]}(t^-) = \Theta_i^{[\nu]} x_i(t)$, $\nu \in \mathbb{N} \cup \{\infty\}$.

### 2.2. Known results and definitions

**Definition 1.** The switched system (1) is called

- $(x, \sigma)$-observable iff for all $(x_0, \tilde{x}_0) \neq (0, 0)$ the following implication holds:
  $$ (x_0 \neq \tilde{x}_0 \lor \sigma \neq \tilde{\sigma}) \implies y_{(x_0, \sigma)} \neq y_{(\tilde{x}_0, \tilde{\sigma})}, $$
  i.e., if it is possible to determine simultaneously the state and current mode from the output;
  - $\sigma$-observable iff for all $(x_0, \tilde{x}_0) \neq (0, 0)$
    $$ \sigma \neq \tilde{\sigma} \implies y_{(x_0, \sigma)} \neq y_{(\tilde{x}_0, \tilde{\sigma})}. $$
  (3)
  i.e., if it is possible to determine the current mode from the output;
  - $t_\sigma$-observable (or switching time observable) iff for all $x_0 \neq 0$, $\sigma$ nonconstant and all $\tilde{x}_0$, $\tilde{\sigma}$:
    $$ T_\sigma \neq T_{\tilde{\sigma}} \implies y_{(x_0, \sigma)} \neq y_{(\tilde{x}_0, \tilde{\sigma})}, $$
  i.e., if it is possible to determine the switching times from the output.

Clearly, $(x, \sigma)$-observability implies $\sigma$-observability which in turn implies $t_\sigma$-observability. Furthermore, it seems quite obvious that it is much harder to determine both the state and the switching signal compared to just determining the current mode from the output. However, this intuition is wrong:

**Lemma 2.** For the switched system (1) it holds that

$$(x, \sigma)$-observability $\iff$ $\sigma$-observability.

**Proof.** The implication “$\Rightarrow$” is clear. Now let the system be $\sigma$-observable, but not $(x, \sigma)$-observable. This means there exist $(x_0, \tilde{x}_0) \neq (0, 0)$ and $\sigma$, $\tilde{\sigma}$ with

$$(x_0 \neq \tilde{x}_0 \lor \sigma \neq \tilde{\sigma}) \land y_{(x_0, \sigma)} = y_{(\tilde{x}_0, \tilde{\sigma})}. $$

$\sigma \neq \tilde{\sigma}$ would contradict $\sigma$-observability. Hence we have $\sigma = \tilde{\sigma}$ and $x_0 \neq \tilde{x}_0$. This means $y_{(x_0, \sigma)} = y_{(\tilde{x}_0, \sigma)}$ and, by linearity,

$y_{(x_0, \tilde{x}_0, \sigma)} \equiv 0$. This contradicts $\sigma$-observability, as it implies $y_{(x_0, \tilde{x}_0, \sigma)} \equiv 0 \equiv y_{(0, \sigma)}$ for all $\tilde{\sigma}$. $\square$
This relation was already implicitly stated in [7] for discrete-time systems. Note that observability of the (continuous) state in each mode is necessary for \((x, \sigma)-\)observability (just consider the constant switching signals). However, state-observability in each mode is not sufficient for \((x, \sigma)-\)observability (c.f. [11]). A trivial counterexample for the latter is a system for which each mode describes the same observable system.

The next example shows that \(t_3\)-observability is indeed weaker than \((x, \sigma)-\)observability:

**Example 3.** The system (1) with modes \((A_1, C_1) = \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)\), \((A_2, C_2) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)\) is \(t_3\)-observable, but not \((x, \sigma)-\)observable as the individual modes are not observable.

**Remark 4 (Observability and invertibility).** Most observability notions are concerned with the invertibility of certain maps involving the output and it is helpful to compare the different concepts side-by-side in regard of these sought inverse maps, see Table 1. For this comparison we consider a general nonlinear switched systems as in Figure 2.

**Figure 2:** General nonlinear switched system with initial state \(x_0\), input \(u\), switching signal \(\sigma\) and output \(y\).

<table>
<thead>
<tr>
<th>sought map</th>
<th>name, reference footnotes</th>
</tr>
</thead>
<tbody>
<tr>
<td>((y, u, \sigma))</td>
<td>(x_0) observability</td>
</tr>
<tr>
<td>((y, u, u))</td>
<td>((u, \sigma)) invertibility</td>
</tr>
<tr>
<td>((y, u, \sigma))</td>
<td>((x_0, \sigma)) ((x, \sigma)-)observability</td>
</tr>
<tr>
<td>((y, u))</td>
<td>(\sigma) (\sigma)-observability</td>
</tr>
<tr>
<td>((y, u))</td>
<td>((x_0, \sigma)) strong ((x, \sigma)-)observability</td>
</tr>
<tr>
<td>((y, u))</td>
<td>(\sigma) strong (\sigma)-observability</td>
</tr>
</tbody>
</table>

Table 1: Comparison of different observability notions based on the sought inverse maps.

Note that most results on observability of switched systems are only for the linear case (one exception is [16].

We now recall the known characterization for \(t_3\)- and \((x, \sigma)-\)observability in terms of the Kalman observability matrices:

**Lemma 5** ([5] (2003)). System (1) is \(t_3\)-observable if, and only if,

\[
\begin{align*}
\text{rank}(\sigma^{[2n]}_i - \sigma^{[2n]}_j) &= n & \forall i, j \in \mathcal{P} \text{ with } i \neq j.
\end{align*}
\]

It is \((x, \sigma)-\)observable if, and only if,

\[
\begin{align*}
\text{rank}(\sigma^{[2n]}_i - \sigma^{[2n]}_j) &= 2n & \forall i, j \in \mathcal{P} \text{ with } i \neq j.
\end{align*}
\]

The characterization (4) can be nicely interpreted by considering the homogeneous augmented system \(\Sigma_{i,j}^{\text{hom}}\), \(i, j \in \mathcal{P}\):

\[
\Sigma_{i,j}^{\text{hom}}: \begin{bmatrix} \dot{\xi} = A_i \xi \\ y_{\Delta_{ij}} = C_i \xi - C_j \xi \end{bmatrix},
\]

because (4) is equivalent to (classical) observability of \(\Sigma_{i,j}^{\text{hom}}\), indeed \(\sigma^{[v]}_{ij} = [\sigma^{[v]}_i, -\sigma^{[v]}_j]\). This also justifies why it suffices to consider the order \(v = 2n\) in (4).

### 2.3. \(\sigma_1\)-observability

As already mentioned in the introduction assuming observability of each (in particular, each faulty) mode is often too restrictive. Furthermore, the notion of \((x, \sigma)-\)observability (and hence \(\sigma\)-observability) reduces to the ability to determine the current mode of a (locally) unswitched systems. In particular, the event of the switch itself is not utilized for recovering the switching signal. We illustrate this with the following example:

**Example 6.** The system (1) with modes \((A_1, C_1) = (0, 1), (A_2, C_2) = (0, 2)\) is not \((x, \sigma)-\)observable, because both systems produce constant outputs for constant switching signals. However, in the presence of a switch, the output is either halved or doubled, which allows us to determine whether we switched from mode 1 to 2 or vice versa. This observability property is lost if we modify \(C_2\) to \(-1\), because the output then just changes its sign and we are not able to distinguish the two possible mode sequences. However it is still possible to detect the switching time, because of the sign change (which always occurs as long as \(x_0 \neq 0\), which we assumed here).

This motivates us to define the following more suitable observability notion:

**Definition 7.** The system (1) is called \((x, \sigma_1)-\)observable (or switch observable) if (3) holds for all \(x_0 \neq 0\) and all \(\sigma\) with at least one switch, i.e. \(\sigma\) nonconstant, and all \(x_0, \vec{\sigma}\). It is called \(\sigma_1\)-observable if (3) holds for \(x_0, \vec{\sigma}\) as above.

**Lemma 2** holds accordingly and gives

\[
(x, \sigma_1) - \text{observability} \iff \sigma_1 - \text{observability}.
\]

We now present our first main result which characterizes \((x, \sigma_1)-\)observability for homogeneous switched linear systems.
Theorem 8. The system (1) is \((x, \sigma_1)\)-observable if, and only if, for all \(i, j, p, q \in \mathcal{P}\) with \(i \neq j, p \neq q\) and \((i, j) \neq (p, q)\):
\[
\text{rk} \begin{bmatrix} \sigma_{[2n]}^i & \sigma_{[2n]}^j \\ \sigma_{[2n]}^p & \sigma_{[2n]}^q \end{bmatrix} = 2n. \tag{7}
\]
Proof. “⇒”: Assume that (7) does not hold, i.e. there exist \(i, j, p, q\) as above and \((x_1, \bar{x}_1) \neq (0, 0)\) such that
\[
\begin{bmatrix} \sigma_{[2n]}^i & \sigma_{[2n]}^j \\ \sigma_{[2n]}^p & \sigma_{[2n]}^q \end{bmatrix} [x_1, \bar{x}_1] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{8}
\]
Without loss of generality, we can assume \(x_1 \neq 0\). Define \((x_0, \bar{x}_0) := (e^{-A_t \bar{x}_1} x_1, e^{-A_t \bar{x}_1} \bar{x}_1)\) and
\[
\sigma(t) = \begin{cases} i, & t < t_s, \\ j, & t \geq t_s, \end{cases} \quad \tilde{\sigma}(t) = \begin{cases} p, & t < t_s, \\ q, & t \geq t_s. \end{cases} \tag{9}
\]
Then we have \(x_0 \neq 0\) and \(\sigma \neq \tilde{\sigma}\). From (8) we can conclude
\[
y_{[2n]}(t_s^i) \equiv y_{[2n]}(t_s^j) \wedge y_{[2n]}(t_s^p) \equiv y_{[2n]}(t_s^q).
\]
In terms of (5) with initial value \((x_1, \bar{x}_1)\) this is equivalent to \(y_{[2n]}(0) = 0\) and \(y_{[2n]}(0) = 0\). By the classical observability theory, this implies \(y_{[2n]}(\infty) = 0\) and \(y_{[2n]}(\infty) = 0\), i.e. \(\Delta_{\infty} \equiv 0\) and \(\Delta_{\infty} \equiv 0\). We can conclude \(y_{(x_0, \sigma)} \equiv y_{(x_0, \tilde{\sigma})}\).

“⇐”: Using (5), it suffices to show \(\sigma_1\)-observability. (7) implies \(\tau_s\)-observability as for \(p = j \neq i = q\) we have
\[
\text{rk} \begin{bmatrix} \sigma_{[2n]}^i & \sigma_{[2n]}^j \\ \sigma_{[2n]}^p & \sigma_{[2n]}^q \end{bmatrix} = 2n \Rightarrow \text{rk} \begin{bmatrix} \sigma_{[2n]}^i - \sigma_{[2n]}^j \\ \sigma_{[2n]}^p - \sigma_{[2n]}^q \end{bmatrix} = n.
\]
Now let \(x_0, \bar{x}_0, \sigma, \tilde{\sigma}\) be given with \(x_0 \neq 0\), \(\sigma\) nonconstant and \(\sigma \neq \tilde{\sigma}\). It remains to show \(y_{(x_0, \sigma)} \neq y_{(x_0, \tilde{\sigma})}\). For \(T_{\sigma} \neq T_{\tilde{\sigma}}\) this follows directly from \(t_s\)-observability, hence let \(T_{\sigma} = T_{\tilde{\sigma}}\). Then there exists a common switching time \(t_s\) with \(\sigma(t_s^i) \neq \tilde{\sigma}(t_s^j)\) or \(\sigma(t_s^j) \neq \tilde{\sigma}(t_s^i)\). Let \(i, j, p, q\) be as in (7). As \(x_{(x_0, \sigma)}(t_s^i) \neq 0\), (7) implies
\[
y_{[2n]}(t_s^i) \neq y_{[2n]}(t_s^j) \vee y_{[2n]}(t_s^j) \neq y_{[2n]}(t_s^i).
\]
Thus the system is \(\sigma_1\)-observable.

Condition (7) also appears in Johnson et al. (2014) as a characterization of what those authors call ST-observability. The main difference to our approach is that observability of the individual modes \(i, j, p\) is assumed there.

Remark 9. Vidal et al. (2003) chose a different approach for observability of systems with nonconstant switching signals. They required for all \(i \neq j\):
\[
\text{rk} \begin{bmatrix} \sigma_{[2n]}^i & \sigma_{[2n]}^j \\ \sigma_{[2n]}^p & \sigma_{[2n]}^q \end{bmatrix} = \text{rk} \sigma_{[2n]}^i + \text{rk} \sigma_{[2n]}^j,
\]
which guarantees that one can determine the current mode whenever the output is nonzero. Together with \(t_s\)-observability, this gives that mode and state can be determined whenever the switching signal is nonconstant and the initial state is nonzero. This means (10) and \(t_s\)-observability imply \((x, \sigma_1)\)-observability. The reverse is not true, as the first part of Example 6 shows.

Clearly, \((x, \sigma_1)\)-observability works also for systems with more than one switch, but then each switching instant is treated independently of the others (analogously as for \((x, \sigma)\)-observability each mode is treated independently of the others). If we restricted our attention to systems with at least two (or more generally at least \(k\)) switches and defined \((x, \sigma_1)\)-observability accordingly, one would get even weaker conditions than (7). However, these conditions would then depend on the differences of the switching times, i.e. the duration times. It is questionable whether these weaker observability notions are really relevant in praxis and whether the technical effort to find corresponding characterizations is justified.

The results of this sections for homogeneous linear switched systems are summarized in the left column of Figure 1 and Example 6 shows that the converse implications do not hold in general.

3. Inhomogeneous Systems

For unswitched systems or switched systems with known switching signal the system dynamics are known and thus the output’s dependence on the input can be computed a priori; it is therefore common to restrict the analysis to homogeneous systems. For unknown switching signals this reduction to the homogeneous case is not possible, because the effect of the input on the output depends on the switching signal.

There are several ways to generalize the observability notions to inhomogeneous systems, depending on the treatment of the inhomogeneity. We consider strong observability notions, i.e. we require the system to be \(t_s/-\sigma-/\{x, \sigma_1\}\)-observable for all inputs. Other approaches are that one requires the existence of an input that makes the system observable (weak notion) or requires observability for almost all inputs. This generic notion actually coincides with the weak one, see Babaei and Pappas (2005). The literature focuses on the weak or the generic case, see e.g. De Santis and Di Benedetto (2016), Baglietto et al. (2007) and we are not aware of available results for strong observability notions.

We consider the switched system
\[
\dot{x} = A_t x + B_t u, \quad x(0) = x_0, \tag{11a}
\]
\[
y = C_t x + D_t u, \tag{11b}
\]
with matrices \(A_t \in \mathbb{R}^{n \times n}, B_t \in \mathbb{R}^{n \times q}, C_t \in \mathbb{R}^{p \times n}, D_t \in \mathbb{R}^{p \times q}\) for \(t \in \mathcal{P}\). Solutions and outputs are denoted by \(x_{(x_0, \sigma, u)}\) and \(y_{(x_0, \sigma, u)}\), respectively. In order to define suitable observability notions we make the following two assumptions:

\[
\text{rk} \begin{bmatrix} B_j \\ B_j - D_j \end{bmatrix} = 0 \quad \forall i \neq j. \tag{A2}
\]

Definition 10. Consider the switched system (11) satisfying (A2). Then we define (11) to be \(\text{strongly } (x, \sigma_1)/\sigma-/\{x, \sigma_1\}-\text{observable}\) iff the analogous conditions of Definitions 1 and 7 hold for all inputs \(u\) satisfying (A1).
Consider the system (11) with mode 

\[ \begin{cases} x_1 = \sigma_1 x_1 + u_1, \\ x_2 = \sigma_2 x_2 + u_2, \end{cases} \]

Example 12. This example is illustrated in Figure 3. 

The first example shows what can happen when assumption (A1) is not satisfied. 

Example 12. Consider the system (11) with modes 

\[ \begin{align*} 
(A_1, B_1, C_1, D_1) &:= ( [0 0 0], [0 0 0], [0 0 0], [0 0 0] ), \\
(A_2, B_2, C_2, D_2) &:= ( [0 0 0], [0 0 0], [0 0 0], [0 0 0] ), \\
(A_3, B_3, C_3, D_3) &:= ( [0 0 0], [0 0 0], [0 0 0], [0 0 0] ). 
\end{align*} \]

This means assumption (A2) does not hold. Define 

\[ x_0 := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \] 

\[ u(t) := -\frac{1}{t} \cos \left( \frac{\pi}{2} t \right) \]

\[ \sigma(t) := \begin{cases} 1, & t < 1, \\ 2, & 1 \leq t < 2, \\ 3, & 1 \leq t < 2, \\ 1, & t \geq 2. \end{cases} \]

Then \( x_{[\sigma, \sigma, \sigma]}(1) = x_{[\sigma, \sigma, \sigma]}(2) = \begin{bmatrix} 0 \end{bmatrix} \) and thus \( x_{[\sigma, \sigma, \sigma]}(t) = \begin{bmatrix} 0 \end{bmatrix} \) for \( t \in [1, 2] \). Hence the switching signals cannot be distinguished for this particular choice of input. This example is illustrated in Figure 3. 

The second example shows what can happen when assumption (A1) is not satisfied. 

Example 12. Consider the system (11) with mode 

\[ \begin{cases} x_1 = \sigma_1 x_1 + u_1, \\ x_2 = \sigma_2 x_2 + u_2, \end{cases} \]

This means assumption (A2) has no effect on the solution and hence the system cannot be \( t_0 \)-observable or even \( (x, \sigma) \)-observable. Such a \( u \) is clearly non-analytic. In contrast to the previous example, no switch is required to achieve an interval with zero state, see Figure 4. 

For a characterization of strong \((x, \sigma)\)-observability we need to define \( \Gamma^{[\sigma]} \) corresponding to the unswitched inhomogeneous system 

\[ \begin{align*} 
\Sigma : \quad & \dot{\xi} = A_i \xi + B_i u, \\
& y = C_i \xi + D_i u, 
\end{align*} \]

We would like to recall the notion of unknown-input observability for unswitched systems: 

**Definition 13.** The system \( \Sigma \) is unknown-input (ui-) observable iff \( y \equiv 0 \) implies \( x \equiv 0 \) (independently of the input \( u \)). 

A system \( \Sigma \) is ui-observable iff 

\[ \text{rk} \begin{bmatrix} \sigma^{[n]} \Gamma^{[n]} \end{bmatrix} = n + \text{rk} \Gamma^{[n]}, \]

or, equivalently, 

\[ \text{rk} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n + \text{rk} \begin{bmatrix} B \\ D \end{bmatrix} \forall s \in \mathbb{R}, \]

see Kratz (1995) and Hautus (1983), respectively. This means the system is ui-observable iff it has no zeroes (in the sense of Hautus (1983)). 

Applying this characterization on the augmented system \( \Sigma_{ij}, i, j \in \mathcal{N} \) : 

\[ \begin{align*} 
\dot{x}_{ij} : \quad & \dot{x}_{ij} = A_{ij} x_{ij} + B_{ij} u, \\
y_{ij} : \quad & y_{ij} = C_{ij} x_{ij} + D_{ij} u, 
\end{align*} \]

we can conclude that \( \Sigma_{ij} \) is ui-observable if and only if 

\[ \text{rk} \begin{bmatrix} \sigma^{[2n]}_{i} & \sigma^{[2n]}_{j} \\ \Gamma^{[2n]}_{i} & \Gamma^{[2n]}_{j} \end{bmatrix} = 2n + \text{rk} \begin{bmatrix} \sigma^{[2n]}_{i} & \sigma^{[2n]}_{j} \\ \Gamma^{[2n]}_{i} & \Gamma^{[2n]}_{j} \end{bmatrix}. \]
If (12) holds for all \( i \neq j \), one can determine mode and state of the system as long as the state is nonzero. This has already been shown by [Lou and Si (2009)]. By requiring (A1), (A2) and \( x_0 \neq 0 \) we can guarantee that on any interval the state is not constantly zero or the mode can be uniquely determined by the direct feedthrough. Hence we have:

**Lemma 14 (cf. Lou and Si (2009)).** System (11) satisfying (A1) and (A2) is strongly \((x, \sigma)\)-observable if and only if (12) holds for all \( i, j \in \mathcal{P}, i \neq j \).

For the characterization of \( t_S\)-observability, the following notion will be essential:

**Definition 15 (Trentelman et al. (2001)).** The set of controllable weakly unobservable states of the system \( \Sigma \) is

\[
\mathcal{R}(\Sigma) := \left\{ x_0 \in \mathbb{R}^n \mid \exists u(\cdot) \text{ smooth, } T > 0 : y(x_0,u) \equiv 0 \text{ and } x(x_0,u)(T) = 0 \right\}.
\]

Note that one obtains the same set if we restrict the inputs to be analytic. Furthermore, \( \mathcal{R}(\Sigma) = \{0\} \) if, and only if,

\[
\text{rk} \begin{bmatrix} A-sI & B \\ C & D \end{bmatrix} = n + \text{rk} \begin{bmatrix} B \\ D \end{bmatrix}, \quad \text{for all but finitely many } s \in \mathbb{R},
\]

see [Trentelman et al. (2001)].

**Lemma 16.** Let (11) satisfy (A1), (A2) and

\[
\mathcal{R}(\Sigma_{ij}) = \{0\} \quad \text{for all } i \neq j.
\]

Let \((x_0, \tilde{x}_0) \neq (0,0), u \text{ and } \sigma, \tilde{\sigma} \text{ be given with } \sigma(T^+) \neq \tilde{\sigma}(T^+) \text{ and } x(x_{ij},\sigma)(T) = x(\tilde{x}_{ij},\tilde{\sigma})(T) = 0 \text{ for some } T > 0. \text{ Then } y(x_{ij},\sigma,u) \neq y(\tilde{x}_{ij},\tilde{\sigma},u).

**Proof.** As a nonzero state is steered to a zero state, the input \( u \) cannot be zero. Using (A1), this means that \( u \) is nonzero on any interval.

Let \( \mathcal{J} := [T, T + \varepsilon], \varepsilon > 0, \) be an interval with \( \sigma \) and \( \tilde{\sigma} \) constant. Set \( i := \sigma(T^+) \) and \( j := \tilde{\sigma}(T^+) \). If \( B_i u = B_j u \equiv 0 \) on \( \mathcal{J} \), (A2) implies \( D_i u \neq D_j u \) on \( \mathcal{J} \), hence \( y(x_{ij},\sigma,u) \neq y(\tilde{x}_{ij},\tilde{\sigma},u) \).

Thus let \( B_i u \neq 0 \) or \( B_j u \neq 0 \) on \( \mathcal{J} \). This means that for some \( \tilde{t} \in \mathcal{J} \) we have \((x_{ij},\tilde{x}_{ij}) : (x_{ij},\sigma)(\tilde{t}), x(\tilde{x}_{ij},\tilde{\sigma})(\tilde{t})\neq (0,0), y(x_{ij},\sigma,u) \equiv y(\tilde{x}_{ij},\tilde{\sigma},u) \) on \( \mathcal{J} \) would imply \((x_{ij},\tilde{x}_{ij}) \in \mathcal{R}(\Sigma_{ij}) \), hence the outputs have to be different. \( \square \)

**Lemma 17.** Consider the switched system (11) satisfying (A1) and (A2). Then (11) is strongly \( t_S \)-observable if, and only if (13) holds and, for all \( i \neq j \),

\[
\text{rk} \begin{bmatrix} \sigma_i^{(2n)} - \sigma_j^{(2n)} \\ r_i^{(2n)} - r_j^{(2n)} \end{bmatrix} = n + \text{rk} \begin{bmatrix} r_i^{(2n)} - r_j^{(2n)} \end{bmatrix}. \tag{14}
\]

**Proof.** Necessity of (13): Assume there exists \( [\sigma_i, e_i] \in \mathcal{R}(\Sigma_{ij}) \setminus \{0\} \). This means there exists an analytic input \( u \) and a time \( t_S \) such that

\[
y(x_{ij},u) \equiv y(\tilde{x}_{ij},\tilde{u})(t_S) = x(\tilde{x}_{ij},\tilde{u})(t_S) = 0. \tag{15}
\]

Both \( y(x_{ij},u) \) and \( y(\tilde{x}_{ij},\tilde{u}) \) are analytic. Define \( \sigma \equiv i \) and

\[
\tilde{\sigma}(t) = \begin{cases} i, & t < t_S, \\
\tilde{j}, & t \geq t_S. \end{cases}
\]

Then \( y(x_{ij},\sigma,u) \) and \( y(\tilde{x}_{ij},\tilde{\sigma},u) \) coincide on \((\infty, t_S)\) by definition and on \([t_S, \infty)\) by (15). Hence for this specific initial value and input it is not possible to detect a switch from mode \( i \) to mode \( \tilde{j} \) at time \( t_S \).

Assume that (14) does not hold for some \( i \neq j \), i.e. there exist some \( x_1 \neq 0 \) and \( U \) with \( \sigma_i^{(2n)} x_1 + r_i^{(2n)} U = \sigma_j^{(2n)} x_1 + r_j^{(2n)} U \). In particular, (12) does not hold (as the nonzero vector \( [x_1^T - x_0^T \ U^T] \) lies in the kernel of the matrix on the left hand side). Hence by Lemma 14 there exists some input \( \tilde{u} \) with \( y(x_{ij},u) \equiv y(\tilde{x}_{ij},\tilde{u}) \). Now let \( t_S > 0, u(\cdot) := \tilde{u}(\cdot - t_S), \sigma = i, \tilde{\sigma} \) as in (16) \([\tilde{A}2]\) and \( x_0 \) such that \( x(x_{ij},\sigma)(t_S) = x_1 \). By construction of \( \sigma \) and \( \tilde{\sigma}, y(x_0,\sigma,u) \) and \( y(\tilde{x},\tilde{\sigma},u) \) coincide on \((\infty, t_S)\). Due to \( y(x_{ij},u) \equiv y(\tilde{x}_{ij},\tilde{u}) \), they also coincide on \([t_S, \infty)\). Hence the system is not strongly \( t_S \)-observable.

To show sufficiency of (13) and (14) for strong \( t_S \)-observability, consider \( x_0 \neq 0, u \) and \( \sigma \) with switching time \( t_S \). Let \( \tilde{x}_0 \) and \( \tilde{\sigma} \) be given with \( \tilde{x}_0 \neq \tilde{x}_0 \). As we want to show that the outputs of these solutions differ in an neighborhood of \( t_S \), it suffices to consider \( T = \{t_S\} \) and \( \tilde{T} \) constant. This means that \( y(x_{ij},\sigma,u) \) is analytic. Equation (14) gives that for \( x(x_{ij},\sigma)(t_S) \neq 0 \) we have \( y(x_{ij},\sigma)(t_S^+) \neq y(x_{ij},\sigma)(t_S^-) \), hence \( y(x_{ij},\sigma,u) \neq y(\tilde{x}_{ij},\tilde{\sigma},u) \). Now let \( x(x_{ij},\sigma,u)(t_S) = 0 \), then \( y(x_{ij},\sigma,u) \equiv y(\tilde{x}_{ij},\tilde{\sigma},u) \) would imply \( y(x_{ij},u) \) is analytic, i.e. that it coincides with \( y(\tilde{x}_{ij},\tilde{\sigma},u) \) for \( \sigma(t) = \tilde{\sigma}(t) \) \( \forall t \). Now Lemma 16 gives a contradiction to \( y(x_{ij},\sigma,u) \equiv y(\tilde{x}_{ij},\tilde{\sigma},u) \). \( \square \)

**Remark 18.** Regarding (13) we observe the following:

(i) In [Elhamifar et al. (2009)] strong \( t_S \)-observability is characterized for discrete time switched systems in terms of (14), but condition (13) does not occur. The reason is due to stronger assumption made in [Elhamifar et al. (2009)] which are specific to the discrete time set up; in particular, they require that each individual mode is observable.

(ii) The conditions (13) and (14) of strong \( t_S \)-observability are indeed not related. Consider for example the system given by

\[
(A_1, B_1, C_1, D_1) = (0, 1, 2, 0),
(A_2, B_2, C_2, D_2) = (0, 2, 1, 0),
\]

which satisfies (14) but not (13). On the other hand (13) holds for any system with \( B_i = 0 \) for all \( i \in \mathcal{P} \), hence it does not imply (14) in general.

(iii) (13) does not imply \( \mathcal{R}(\Sigma_i) = \{0\} \) for the individual modes. As an example, consider the system (11) with modes

\[
(A_1, B_1, C_1, D_1) = (0, 1, 0, 0),
(A_2, B_2, C_2, D_2) = (0, 1, 1, 0).
\]

It is strongly \( t_S \)-observable, in particular, \( \mathcal{R}(\Sigma_{ij}) = \{0\} \). However, for the first mode we have \( \mathcal{R}(\Sigma_i) = \mathbb{R} \).
(iv) (13) and (14) are indeed weaker than (12): The example from (iii) is strongly $t_y$-observable, but not strongly $(x, \sigma)$-observable as $\sigma_1 = 0$.

**Theorem 19.** The switched system (11) satisfying (A1) and (A2) is strongly $(x, \sigma)$-observable if and only if it satisfies (13) and, for all $i, j, p, q \in \mathcal{P}$ with $i \neq j, p \neq q$ and $(i, j) \neq (p, q)$

\[
\mathbf{rk}\left[\begin{array}{ccc}
\mathbf{O}^{[4n]}_p & \mathbf{O}^{[4n]}_q \\
\mathbf{I}^{[4n]}_p & -\mathbf{I}^{[4n]}_q
\end{array}\right] = 2n + \mathbf{rk}\left[\begin{array}{cc}
\mathbf{I}^{[4n]}_p & -\mathbf{I}^{[4n]}_q \\
\mathbf{I}^{[4n]}_q & -\mathbf{I}^{[4n]}_p
\end{array}\right].
\]

(17)

Here the order of the observability matrix is doubled with respect to the previous results. If we only considered $v = 2n$, a vector $U$ as in the proof of Lemma 17 might be related to different inputs $u$ and $\bar{u}$ on the pre-switch interval and post-switch interval.

Again, the statement can be related to ui-observability of an augmented system: (17) is a necessary – but not sufficient – condition for ui-observability of the system $\Sigma_{i,j,p,q}$ defined by

\[
\begin{align*}
A_{i,j,p,q} &= \begin{bmatrix} A_{i,p} & 0 \\ 0 & A_{j,q} \end{bmatrix}, & B_{i,j,p,q} &= \begin{bmatrix} B_{i,p} \\ B_{j,q} \end{bmatrix}, \\
C_{i,j,p,q} &= \begin{bmatrix} C_{i,p} & 0 \\ 0 & C_{j,q} \end{bmatrix}, & D_{i,j,p,q} &= \begin{bmatrix} D_{i,p} \\ D_{j,q} \end{bmatrix}.
\end{align*}
\]

**Proof of Theorem 19.** “(13) and (17) ⇒ strong $t_y$-observability”: From (17) with $p = j, q = i$ and $i \neq j$, we can conclude (14). Then the claim follows by Lemma 17.

“Strong $(x, \sigma)$-observability ⇒ (13)” follows by Lemma 17 as strong $t_y$-observability is necessary for strong $(x, \sigma_1)$-observability.

“Strong $(x, \sigma_1)$-observability ⇒ (17)”: Assume that (17) does not hold for some $i, j, p, q$, i.e. there exist $(x_1, \check{x}_1) \neq (0, 0)$ and $U$ such that

\[
\begin{bmatrix}
\mathbf{O}^{[4n]}_p & \mathbf{O}^{[4n]}_q \\
\mathbf{I}^{[4n]}_p & -\mathbf{I}^{[4n]}_q
\end{bmatrix} = \begin{bmatrix} \mathbf{1} \end{bmatrix}.
\]

We get that $\Sigma_{i,j,p,q}$ is not strongly observable, i.e. for the initial value $\eta_1 := \begin{bmatrix} x_1^T & \check{x}_1^T \end{bmatrix}$ and some $\bar{u}$ with $\bar{u}^{(4n)}(0) = U$ we have $\eta_{i,j,p,q} = 0$, i.e. $\eta_{i,j,p,q} \equiv \eta_{(i,j,p,q)}$ and $\gamma_{x(i,j,p,q)} \equiv \gamma_{(x(i,j,p,q), \bar{u})}$. Define $\sigma$ and $\check{\sigma}$ as in (9) for some $t_y > 0$ and let $u(t) := \bar{u}(t - t_y)$. Let $x_0$ and $\check{x}_0$ be such that $\gamma_{x_0(\sigma)}(t_y^+) = x_1$ and $\gamma_{x_0(\check{\sigma})}(t_y^+) = \check{x}_1$. Then we get $\gamma_{x_0, \sigma}(t_y^+) = \gamma_{x_0, \check{\sigma}}(t_y^+)$, i.e. (11) is not strongly $(x, \sigma_1)$-observable.

(13) and (17) ⇒ strong $(x, \sigma)$-observability**: Let $x_0, \check{x}_0, \sigma, \check{\sigma}$ and $u$ be given with $x_0 \neq 0, \sigma$ nonconstant and $\sigma \neq \check{\sigma}$. We want to show that this implies $\gamma_{x_0(\sigma)}(t_y) \neq \gamma_{x_0(\check{\sigma})}(t_y)$. Assume $T_y = T_{(i,j,p,q)}$ as otherwise $t_y$-observability – which we have by the first step – would yield $\gamma_{x_0, \sigma}(t_y) \neq \gamma_{x_0, \check{\sigma}}(t_y)$. Then there exists a common switching time $t_y$ with $\sigma(t_y^+) \neq \check{\sigma}(t_y^+)$ or $\sigma(t_y^+) \neq \check{\sigma}(t_y^+)$. Define $x_1 := \gamma_{x_0, \sigma}(t_y^+)$ and $\check{x}_1 := \gamma_{x_0, \check{\sigma}}(t_y^+)$. Condition (17) implies that only for $(x_1, \check{x}_1) = (0, 0)$ we can have

\[
\gamma_{x_0(\sigma)}(t_y) = \gamma_{x_0(\sigma)}(t_y^+) \land \gamma_{x_0(\check{\sigma})}(t_y) = \gamma_{x_0(\check{\sigma})}(t_y^+).
\]

However, in this case Lemma 16 already implies $\gamma_{x_0(\sigma)}(t_y^+) \neq \gamma_{x_0(\check{\sigma})}(t_y^+)$. As in Lemma 2, we have equivalence of strong $(x, \sigma)$- and strong $(x, \sigma_1)$-observability.

4. Equivalent switching signals

In the previous section we have highlighted the problem that the switching signal cannot be determined when state and input are identically zero on an interval. This problem was avoided by making the assumptions (A1) and (A2). We can consider smooth instead of analytic input and can drop (A2) if we consider equivalence classes of switching signals:

**Definition 20.** For given $x_0 \in \mathbb{R}^n$ and $u : \mathbb{R} \rightarrow \mathbb{R}^n$ the switching signals $\sigma$ and $\check{\sigma}$ are equivalent for the switched system (11), denoted by $\sigma \sim_{M} \check{\sigma}$, iff $x_{(x_0, \sigma,u)} \equiv x_{(x_0, \check{\sigma}, u)}$, $\gamma_{x_0, \sigma}(t_y) \equiv \gamma_{x_0, \check{\sigma}}(t_y)$ and $\sigma = \check{\sigma}$, except on intervals $I$ with $(x_{(x_0, \sigma,u)})(t_y) = 0$. The corresponding equivalence class is denoted by

\[
\begin{bmatrix} \sigma(x_0,u) \end{bmatrix} := \{ \check{\sigma} \mid \sigma \sim_{M} \check{\sigma} \},
\]

and the essential switching times are given by

\[
T_{\sigma(x_0,u)} := \bigcap_{\check{\sigma} \sim_{M} \sigma} T_{\check{\sigma}}.
\]

A similar equivalence has been considered in [Kaba, 2014] in the context of invertibility of switched systems.

For $u$ analytic, $(x_0, u) \neq (0, 0)$ and systems satisfying (A2) we have $[\sigma(x_0,u)] = \{ \sigma \}$, i.e. trivial equivalence classes.

Adaption of Definition 10 to equivalence classes of switching signals gives:

**Definition 21.** The system (11) is called

- strongly $(x, \sigma)$-observable iff for all smooth $u$ and all $x_0, \check{x}_0, \sigma, \check{\sigma}$ the following implication holds:

\[
(x_0, \sigma(u)) \neq (\check{x}_0, \check{\sigma}(u)) \Rightarrow \gamma_{x_0, \sigma}(t_y) \neq \gamma_{x_0, \check{\sigma}}(t_y); \quad (18)
\]

- strongly $(x, \sigma)$-observable iff (19) holds for all smooth $u$ and all $x_0, \check{x}_0, \sigma, \check{\sigma}$ with

\[
1 \leq \min \left\{ |T_y| \mid \sigma \sim_{M} \check{\sigma} \right\}.
\]

- strongly $(t_y)$-observable iff for all smooth $u$ and all $x_0, \check{x}_0, \sigma, \check{\sigma}$ the following implication holds:

\[
T_{(x_0(u))} \neq T_{(\check{x}_0(u))} \Rightarrow \gamma_{x_0, \sigma}(t_y) \neq \gamma_{x_0, \check{\sigma}}(t_y);
\]

One can also define strong $[\sigma]$- and strong $[\sigma_1]$-observability. Lemma 2 holds accordingly. While the setup is more general, the same characterizations hold:

**Theorem 22.** The system (11) is strongly $(t_y)$./(x, [σ])/-/(x, [σ_1])-observable if and only if the conditions (13) + (14), (13) + (17), (12) are satisfied, respectively (c.f. Figure 1).
5. Conclusion

Switching time observability and switch observability were introduced and characterized by rank-conditions. The relation of these notions is illustrated in Figure 1. A possible future research topic is the extension to the case of switched differential-algebraic equations (DAEs); we already have obtained some preliminary results in Küsters et al. (2017b,a). Based on the notion of strong \((x,\sigma_1)-\)observability, another future research topic is the construction of an observer; some preliminary results have been presented in Küsters et al. (2017c).

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References


Derzsi, D., Deboort, M. (Eds.), Hybrid Dynamical Systems. Springer-Verlag, pp. 205–240.


