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Undergraduates' reasoning while solving integration tasks: discussion of a research framework

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In this paper we investigate the extent to which the research framework on reasoning developed by Lithner (2008) is adequate for characterizing undergraduate students' mathematical reasoning. We conducted a small number of individual task-based think-aloud interviews in which students solved integration tasks. Several examples illustrate how we characterized reasoning types by using the framework. However, we found that some reasoning types were not covered by the framework. We propose to extend the framework by introducing a reasoning type that is mathematically founded but not creative, and as a consequence, may be intertwined with imitative reasoning.

Keywords: Mathematical reasoning, undergraduate students, calculus.

Introduction

Students' mathematical reasoning while solving mathematical tasks is not always as well-founded as it appears, as has already been highlighted by Vinner (1997) in his article on pseudo-conceptual and pseudo-analytical thought processes in mathematics learning. Moreover, undergraduates' reasoning in the domain of calculus is found to be susceptible to ill-founded reasoning (Lithner, 2003). Because of its relevance for mathematics teaching, students' mathematical reasoning demands further investigation. Different frameworks on mathematical reasoning have been reported in literature. Some frameworks focus on argumentation used while solving proving problems (e.g., Blanton & Stylianou, 2014; Stylianides, 2008). Zandieh and Rasmussen (2010) constructed a framework on mathematical reasoning that distinguishes informal and formal reasoning, which seems most suited for investigating students' understanding of abstract concepts. Carlson and Bloom (2005) combined the problem solving phases Orienting, Planning, Executing, and Checking with the use of problem solving attributes Resources, Heuristics, Affect, and Monitoring. Their framework appears capable of identifying students' problem solving activities and the influence of cognitive, affective and/or metacognitive factors. However, this framework does not incorporate the foundation of mathematical reasoning. Lithner (2008) constructed a framework which does incorporate foundations of students' strategic decisions in solving mathematical tasks. This framework is often referred to when characterizing reasoning as either imitative or creative, which are the two main categories in the framework (e.g., Jäder, Sidenvall, & Sumpter, 2016; Jonsson, Norqvist, Liljekvist, & Lithner, 2014). However, the framework offers greater detail, by also defining Memorized Reasoning, and Familiar, Delimiting, and several types of Guided Algorithmic Reasoning (Lithner, 2008). Since this detailed characterization of mathematical reasoning based upon its foundation appears useful for identifying students' reasoning, we selected this framework for our research. Based upon our experiences in applying this research framework, we discuss the framework's possibilities and limitations.

Theoretical framework

Lithner (2008) defines reasoning as:

> the line of thought adopted to produce assertions and reach conclusions in task solving. It is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it (Lithner, 2008, p. 257).

Based on observations of students who solve mathematical tasks, Lithner describes various ways in which students choose a mathematical strategy to solve a task, pointing out students' 'predictive argumentation' (reasoning for choosing a strategy) and 'verificative argumentation' (reflection upon implementation of strategy), where strategy "ranges from local procedures to general approaches" and choice "is seen in a wide sense (choose, recall, construct, discover, guess, etc.)" (Lithner, 2008, p. 257). The resulting framework is visualized in figure 1.

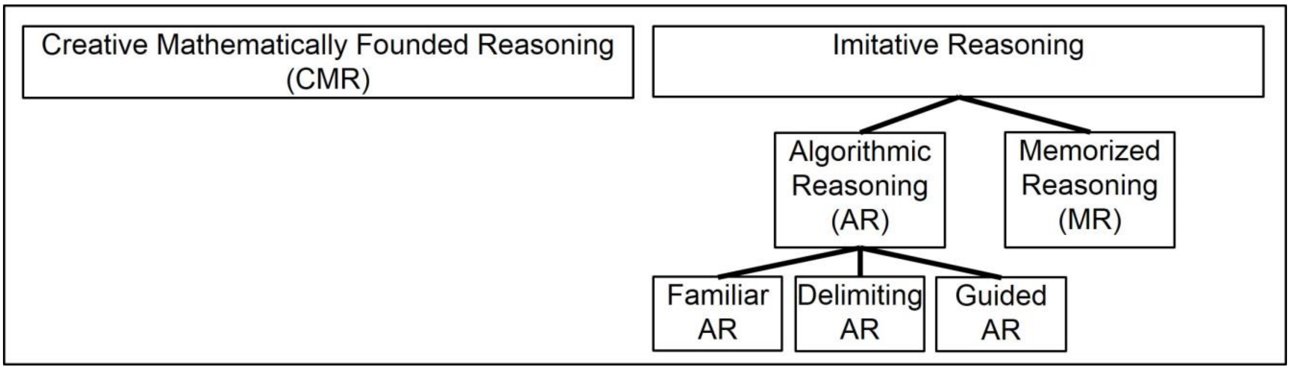

Figure 1: Visualization of reasoning framework as described by Lithner (2008)

The framework distinguishes two main categories, Creative Mathematically founded Reasoning (CMR) and Imitative Reasoning. CMR[1] refers to reasoning that is based on intrinsic mathematical properties, that is novel to the student (the reasoner) and for which the student has arguments (Lithner, 2008). Imitative reasoning is described as reasoning in which an algorithm or answer is recalled in some way. Imitative reasoning is divided into Memorized Reasoning and Algorithmic Reasoning. Memorized Reasoning implies that the student recalls a complete answer, for example a definition or a proof that is learnt by heart. Algorithmic Reasoning occurs when a student recalls an algorithm. Lithner's framework altered over time (see Lithner, 2003, 2004, 2008); in this study we applied the framework as described in Lithner (2008).

The definitions by Lithner (2008) for each of the reasoning types are listed in Table 1. In the definition of Delimiting Algorithmic Reasoning (see Table 1), the term 'set' of algorithms requires some explanation. Lithner (2008) clarifies that if no guidance is available and if the task is unfamiliar to the student, then the student must choose an algorithm from the 'set' of algorithms the student knows, based upon some kind of connection to the task.

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[1] In earlier versions of the framework (e.g., Lithner, 2004), creative mathematically founded reasoning (which was then named Plausible Reasoning) was subdivided in a global and a local subtype, but this distinction has not remained.

Reasoning type	Criteria
Creative Mathematically founded Reasoning	Three criteria: “Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created.” “Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.” “Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning” (Lithner, 2008, p. 266)
Imitative Reasoning	No definition is given. Imitative Reasoning is subdivided into Memorized Reasoning and Algorithmic Reasoning.
Memorized Reasoning	“The strategy choice is founded on recalling a complete answer. The strategy implementation consists only of writing it down.” (Lithner, 2008, p. 258)
Algorithmic Reasoning	“The strategy choice is to recall a solution algorithm. The predictive argumentation may be of different kinds (see below for examples), but there is no need to create a new solution.” “The remaining reasoning parts of the strategy implementation are trivial for the reasoner, only a careless mistake can prevent an answer from being reached.” (Lithner, 2008, p. 259)
Familiar Algorithmic Reasoning	“The reason for the strategy choice is that the task is seen as being of a familiar type that can be solved by a corresponding known algorithm.” “The algorithm is implemented.” (Lithner, 2008, p. 262)
Delimiting Algorithmic Reasoning	“An algorithm is chosen from a set that is delimited by the reasoner through the algorithms’ surface relations to the task. The outcome is not predicted.” “The verificative argumentation is based on surface considerations that are related only to the reasoner’s expectations of the requested answer or solution. If the implementation does not lead to a (to the reasoner) reasonable conclusion it is simply terminated without evaluation and another algorithm may be chosen from the delimited set.” (Lithner, 2008, p. 263)
Guided Algorithmic Reasoning	Text-guided Algorithmic Reasoning: “The strategy choice concerns identifying surface similarities between the task and an example, definition, theorem, rule, or some other situation in a text source.” “The algorithm is implemented without verificative argumentation.” (Lithner, 2008, p. 263) Person-guided Algorithmic Reasoning: “All strategy choices that are problematic for the reasoner are made by a guide, who provides no predictive argumentation.” “The strategy implementation follows the guidance and executes the remaining routine transformations without verificative argumentation.” (Lithner, 2008, p. 264)

Table 1: Definitions of reasoning types, derived from Lithner (2008)

It is important to note that the foundation of Creative Mathematically founded Reasoning is explicitly stated, while this is not the case for Imitative Reasoning: CMR is by definition founded in intrinsic mathematical properties, while the foundation of Imitative Reasoning is not clearly stated. The definitions of various sub-categories of Imitative Reasoning contain criteria like ‘surface relations’, ‘surface considerations’, ‘surface similarities’, ‘no predictive argumentation’, and ‘without verificative argumentation’. This terminology appears to stem from earlier work: Lithner (2004) distinguished mathematically founded reasoning and superficial reasoning, where the latter was based upon surface properties and not upon mathematically relevant properties. Although in Lithner (2008), ‘imitative reasoning’ is not defined as founded in superficial or surface properties, many of the subtypes are (see Table 1). Moreover, all examples and explanations given by Lithner (2008) do refer to situations in which the reasoning is founded in so-called superficial properties and not in intrinsic mathematical properties, which is a criterion for CMR.

Certain studies have already used the framework to characterize students’ reasoning. Boesen, Lithner, and Palm (2010) employed the categories of Memorized Reasoning, Algorithmic Reasoning (without subcategories) and extended the category of Creative Mathematically founded Reasoning by defining the two subtypes Local CMR and Global CMR (in accordance with Lithner (2004)). Sumpter (2013) applied the framework to label episodes of students’ reasoning in a study on the role of beliefs in mathematical reasoning, and showed three examples which were all labeled as Familiar Algorithmic Reasoning. Jäder et al. (2016) used the framework to discern whether students’ reasoning was imitative or creative. We remark that these studies have not used all sub-categories of the framework as described by Lithner (2008). In the case of Sumpter (2013), only one type of reasoning was discussed. Although these studies did make use of the framework, none of them did explicitly reflect upon its applicability. Since we consider the framework a worthwhile addition to literature on mathematical reasoning, we investigate its applicability for characterizing students’ reasoning. The research question we thus aim to answer is: to what extent is the framework by Lithner (2008) adequate to characterize undergraduate students’ mathematical reasoning?

Methodology

The data in this study originates from interviews with three first year mathematics bachelor students, one male and two female, of varying mathematics proficiency levels, determined by previous exam scores. These students are a sub-sample of a group of 12 students participating in a longitudinal study that investigates the development of mathematical reasoning. The students are majors in mathematics at the University of Groningen (the Netherlands) or the KU Leuven (Belgium); universities which offer courses in a wide range of domains at undergraduate and graduate level. The individual task-based think-aloud interviews lasted for approximately 1.5 hours each and are administered by the first author at the end of the students’ first undergraduate year. Students were permitted to use a list with basic calculus formulas, which did not include elaborate integration formulas. The students were asked to explicate their thinking while solving tasks and, after each task, to answer the questions: “How did you come to think of using this strategy?”, “How certain were you that this strategy would help you solve the problem, and why?”, and “Have you seen this type of task before?”. The interviews are video and audio recorded.

We used tasks to create a situation in which the students must choose a suitable strategy from a wide range of possible strategies. We considered integration tasks suitable for this purpose since the students had learnt various mathematical strategies for solving integrals , such as partial integration, substitution, partial fractions, or Euclidean division, in the courses they had taken so far. These considerations led to selection of various tasks, amongst which $\int\sqrt{9-x^2}dx$ and $\int\sqrt{x^2-9}dx$. Both integrals can be simplified through inverse trigonometric substitution, e.g. $x=3\sin(t)$ or $x=3\cos(t)$ to solve the first integral, and $x=3/\sin(t)$ or $x=3\cosh(t)$ to solve the second integral. The students had taken courses in integral calculus in which they solved similar tasks, amongst many other types of tasks. The explicit discussion of these types of integrals had already taken place earlier in the academic year. Based upon teaching experience we expected that these tasks at the time of the interviews would be non-trivial to many students.

While integration tasks may be regarded as tasks that solely require application of procedures, these tasks can arouse various types of mathematical reasoning in students. Considering and selecting suitable procedures is a process in which both Creative Mathematically founded Reasoning as well as Imitative Reasoning can become visible. Familiar Algorithmic Reasoning can be used if the student recognizes the problem type and recalls the corresponding algorithm; Delimiting Algorithmic Reasoning if the student does not recognize the task but recalls various algorithms such as partial integration or substitution of some kind; Creative Mathematically founded Reasoning can be employed if the student is not able to recall a solution strategy but instead constructs a solution or reconstructs a forgotten reasoning sequence, such as drawing a rectangular triangle and deducing a suitable substitution. We did not expect Memorized Reasoning, since the solutions to the tasks are extensive. Guided Algorithmic Reasoning also appeared improbable, since example solutions were unavailable and the interviewer would not offer any hints. The available list with formulas however could serve as inspiration.

Transcripts of the task solutions are split into episodes. An episode begins at the first consideration of a strategy (or a set of strategies) and ends when the strategy is abandoned and a new strategy is about to be considered. Using the framework to characterize parts of a solution is similar to the method of Lithner (2008) and Sumpter (2013). The first author tried to characterize each of the episodes through the definitions given by Lithner (2008). If this was unsuccessful, the difficulties were described. The findings from this analysis were discussed with the other authors until agreement was obtained.

Results

Below we describe several reasoning episodes from our data, which illustrate how we characterized reasoning using the framework and which problems we encountered.

Familiar Algorithmic Reasoning?

Example 1: $\int\sqrt{x^2-9}dx$; student A chose to rewrite the integrand by splitting it into two terms and next integrating them separately: $\sqrt{x^2-9}=\dfrac{x^2-9}{\sqrt{x^2-9}}=\dfrac{x^2}{\sqrt{x^2-9}}+9\dfrac{-1}{\sqrt{x^2-9}}$. The student had applied this strategy earlier in the interview, when (unsuccessfully) solving $\int\sqrt{9-x^2}dx$. The student explained: "The task was similar to the former task, so I figured I could try using the same approach."

We observe that the task is recognized as a familiar type, which made the student decide to apply the same strategy as before. We characterize this reasoning as Familiar Algorithmic Reasoning.

Example 2: $\int\frac{x}{\sqrt{x^2-9}}dx$; this sub-task arose while student A worked on $\int\sqrt{x^2-9}dx$. This sub-task is of a standard form because the derivative of x^2-9 is in the numerator, making $t=x^2-9$ an appropriate substitution. Student A effectively chose this strategy and explained afterwards: "Because I knew that the derivative of this (points at x^2-9 in the denominator), was something like this (points at the numerator) […] So I applied substitution to this (points at x^2-9), because this (points at the numerator) would then be eliminated."

We observe that the student noticed the relevant mathematical characteristics of this task and knew what algorithm would solve tasks of this type. Either the task type was familiar or the student constructed the approach. The ease with which the student came to this conclusion and the fact that this type of task has been practiced extensively make us expect the student to be familiar with the task type: we characterize the reasoning as Familiar Algorithmic Reasoning. However, the student clearly founds the reasoning in intrinsic mathematical properties, which is very different from the reasoning that occurred during Example 1, where the student appears to base the strategy choice solely on familiarity with the task type.

We conclude that Examples 1 and 2 provide two rather distinct reasoning types while both satisfying the criteria for Familiar Algorithmic Reasoning.

Memorized Reasoning or Creative Mathematically founded Reasoning?

Example 3: $\int\frac{1}{\cos\theta}d\theta$; this sub-task arose while student C worked on $\int\sqrt{x^2-9}dx$. The student searched the primitive of $\frac{1}{\cos\theta}$ with respect to θ, which is $\ln|\sec\theta+\tan\theta|$. "It was something like $\ln$ to the power…, $\ln$ of, wait. $\int\frac{1}{\cos\theta}d\theta$. it does not have to be so complicated. There must be something that I overlook. […] eh. Ah, no, wait wait, hey. sec θ times … Secant tangent? There was something about that. $\frac{\sec\theta}{\ln|\sec\theta\tan\theta|}$. I rely on my memory now, because I have solved those integrals. I know it's an integral with secant, with ln. (student calculates the derivative of $\ln|\sec\theta\tan\theta|$, infers it is not correct) What was it like? […] wait, I think I know. $\ln|\sec\theta+\tan\theta|$ is (student calculates derivative) $\frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}$. Then you can cancel this ('$\sec\theta+\tan\theta$' in numerator and denominator) and then you obtain indeed… I knew it was something with $\ln$."

We observe that the student solves the task by making use of answer recall, but also reasons on the intrinsic mathematical properties of the task to be successful. In the framework, the only reasoning type that makes use of recall of an answer is Memorized Reasoning. However, the second criterion of Memorized Reasoning is not fulfilled. The strategy implementation was not just writing down the answer, since the answer was constructed and verified building upon the intrinsic mathematical properties of the components involved in the reasoning. On the other hand, the category of Creative Mathematically founded Reasoning does not reflect the important role of memory in this solution. This example shows hybrid reasoning with elements from Creative Mathematically founded Reasoning and from Memorized Reasoning.

Intrinsic mathematical properties or surface properties?

Example 4: $\int \sqrt{9-x^2}\,dx$; student B rewrote $y=\sqrt{9-x^2}$ to $x^2+y^2=9$ and remarked it is a circle: "That gives a nice circle. Then you have got the radius, a circle with radius 9, radius 3, I mean. It's not transformed, so you get this. Circular coordinates. Let's take a look at circular coordinates […] Then you get $x=r\cos\theta$ and $y=r\sin\theta$. So $r^2\cos^2\theta+r^2\sin^2\theta=r^2$. […] This is of course… This is just $r^2=r^2$ because $\cos^2\theta+\sin^2\theta=$ … r^2." The student stops using this strategy.

We observe that the student considered the circular coordinates (polar coordinates) since the integrand made the student think of a circle. The strategy appears to be selected based upon intrinsic mathematical properties of the task. However, the student employed the circle coordinates in an ineffective way, which shows that the student did not know why the property of the task, that it concerns a circle, implies the use of circle coordinates. The strategy of using circle coordinates are selected only because the task concerned a circle, therefore the foundation for strategy selection should be regarded as based on the task's surface properties. This example raises doubts concerning whether it is always possible to distinguish a surface property from an intrinsic mathematical property.

Conclusions and discussion

The framework by Lithner (2008) provides means to highlight foundations that underlie students' reasoning when solving a mathematical task. However, we faced several difficulties when employing the framework as an analysis instrument. Examples 1 and 2 concern rather distinct reasoning episodes, while both satisfy the definition of Familiar Algorithmic Reasoning. Whether or not the student provides mathematically founded reasons is a relevant characteristic but not included in the definitions. In Examples 2 and 3, the predictive argumentation of a strategy was imitative (based on recall of any kind), but verificative reasoning was founded in intrinsic mathematical properties of the task. These examples reveal that the framework does not cover such 'hybrid' types of reasoning. Example 4 confronted us with the more fundamental issue how to decide whether reasoning is based on 'surface properties' or on 'intrinsic mathematical properties'.

A way to improve the applicability of the framework is to include reasoning types that are mathematically founded as well as make use of some kind of imitative reasoning. This is not the same as local CMR (Lithner, 2004), which is reasoning that is partly Creative Mathematically founded Reasoning while the remainder is Imitative Reasoning. We propose that reasoning can be mathematically founded without being creative, and in addition, that mathematically founded reasoning can be intertwined with imitative reasoning.

Whether a property is an intrinsic mathematical property or a surface property appears to depend on the student's understanding, e.g. of why a certain task property leads to a certain strategy selection. Distinguishing between the use of surface properties and intrinsic mathematical properties therefore requires a more complete picture of the students' reasoning as a whole.

These suggestions are based upon difficulties faced when applying the framework on a small number of reasoning episodes within the domain of integration. To obtain a framework adequate to characterize any type of mathematical reasoning not only requires thorough investigation of specific examples, but also requires investigation of the structure of the framework such that the framework will be decisive for each reasoning episode to be characterized.

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