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Atkinson, David

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NONLOCALITY IS A NONSEQUITUR

David Atkinson
Institute for Theoretical Physics, Groningen

Abstract
Nonlocality in quantum mechanics does not follow from nonseparability, nor does classical stochastic independence imply physical independence. In this paper an explicit proof of a Bell inequality is recalled, and an analysis of the Aspect experiment in terms of noncontextual, but indefinite weights, or improper probabilities, is given.

1. Introduction

In quantum mechanics one finds that the properties of many-particle states are not in general reducible to the conjunction of the properties of the separate one-particle states, even when these states are spatially separated from one another. This has led some people to posit a certain holism in the quantum mechanical world-view and thence to adduce, if not a Newtonian *action-at-a-distance* in nature, at least an anaemic *passion-at-a-distance,* which is however impotent to hone a tool that implements superluminal information transfer. But this semantic turn is specious. There is no reason to believe that a measurement on one arm of a Bell-Aspect apparatus causes a physical collapse on the other. There are correlations, quantum correlations to be sure, but nothing more.

At the beginning of the 21st century, we are familiar with the idea that Euclid's axioms of geometry do not in general apply to the physical world — when a gravitational 'field' is present, Einstein has shown us how to use non-Euclidean geometry. Does the success of quantum mechanics similarly imply that classical logic and classical probability theory do not apply to the physical world? There is no such unanimity as in the case of geometry. Bas van Fraassen [Fraassen 1991] states categorically:

> The new phenomena do not force violations of classical probability theory or logic.

On the other hand, Kümmerer and Maassen [Kümmerer 1996] discuss

> ... polarization experiments which show the need to extend classical probability theory.

This claim is explicitly denied by Gill [Gill 1996], who takes these authors to task:

> ... though quantum reality is strange, classical probability [is] ... perfectly adequate to describe it.

In fact the dissension is not as serious as it seems. A distinction can be made between what is *required* on the one hand and what is *convenient* on the other, as in the case of geometry and relativity. No departure from the axioms of Euclid is required by the fact of gravitation. It is *possible* to describe the whole content of Einstein's theory within the framework of Euclidean geometry; but it is not very *convenient* to do so, since then light does not always propagate in free space along a geodesic, and planets appear to be acted upon by 'occult' gravitational forces. For that matter, the Copernican heliocentric theory is also not *required* by the facts of planetary motions. A geocentric, geostationary coordinate system may be used, and planetary and solar motions with respect to such axes may be expanded in Fourier-Ptolemy series of epicycles. We shall argue that, in a similar way, it is useful to introduce nonclassical probability in the discussion of quantum mechanics, even though it is not logically necessary to do so.

There are at least three distinct ways in which one can depart from Kolmogorov's canon in probability theory:

1. By allowing that $P(A\cup B) \neq P(A) + P(B)$ although $A\cap B = \emptyset$.
2. By allowing $P(A\cap B) \neq P(A)P(B)$ although A and B are *physically* independent.
3. By allowing $P < 0$.

Both Dirac and Feynman proclaimed the most striking and fundamental feature of quantum mechanics to be precisely the first option. The second option was discussed in [Atkinson 1998], while in this paper we shall concentrate upon the third possibility. The obvious objection to negative probabilities is of course that they cannot represent, or serve as predictions for, relative frequencies of events. However, if it can be arranged that such a negative probability always comes together with, and is added to, positive probabilities, and moreover in such a way that the final predictions for relative frequencies are always positive, then the only remaining objection would seem to be merely linguistic. In this paper we shall show that the Bell-Aspect scenario can indeed be interpreted in terms of negative probabilities (or indefinite weights, if one baulks at the word).

After giving the axioms and definitions of classical probability theory, we shall recall two proofs of the Bell inequality [Fine 1982], one based on the requirement of separability and the other on the assumed existence of noncontextual counterfactual conditional probabilities or weights (which however are required to be positive). Since the Bell inequality is known to be experimentally violated [Aspect 1982], it follows that these weights are *neither* separable *nor* noncontextual, unless indeed they are allowed to have indefinite sign.

In previous work, we have considered nonseparability in connection with ideas of physical independence [Atkinson 1998], this being a special case of the theory dependence of probability itself [Atkinson 1999]. In this paper we give the most general noncontextual, conditional probabilities for the Aspect experiment, going in fact beyond quantum mechanics in this respect. Full mathematical details may be found in [Atkinson 2001].

2. Kolmogorov's Axioms

The axiomatic approach to probability was formulated by A.N. Kolmogorov in 1933 in a book published in German, a Russian translation appearing three years later. We quote from the second edition of Morrison's English translation [Kolmogorov 1956] verbatim:

Let E be a collection of elements ξ, η, ζ, … which we shall call *elementary events,* and F a set of subsets of E ; the elements of the set F will be called *random events.*

I. F is a field of sets.
II. F contains the set E.
III. To each set A in F is assigned a non-negative real number P(A). This number P(A) is called the probability of the event A.
IV. P(E) equals 1.

V. If A and B have no element in common, then $P(A \cup B) = P(A) + P(B)$.

It is not necessary to postulate $P(\emptyset) = 0$, nor $P(A) \leq 1$ for any $A \in F$, for these statements are implied by the above axioms. If E is an infinite collection of elements, then one normally restricts F to be such that it is closed under countable unions of sets, and one replaces axiom V. by

V′. If $\{A_n\}$ is a set of pairwise disjoint sets in F, then $P(\cup_n A_n) = \Sigma_n P(A_n)$.

This is the condition of σ-additivity. To the above axioms are added, as *definitions,* the notions of stochastic independence and of conditional probability:

VI. The necessary *and sufficient* condition that A and B be stochastically independent events is $P(A \cap B) = P(A)P(B)$.

VII. The conditional probability of event A, given event B, is defined by $P(A|\, B) = P(A \cap B) / P(B)$, on condition that $P(B) \neq 0$.

Note that stochastic independence is not always equivalent to physical independence, and that it is quite different from disjointness, for which axiom V applies. Moreover, if A and B are stochastically independent, $P(A|B) = P(A)$.

3. Separability and Bell's Inequality

Suppose that two photons are created in an angular momentum zero state, as in the experiments of Aspect et al. [Aspect 1982] One photon falls on a polarizer at location A, behind which there is a detector, and the other photon falls on a similar polarizer at another location, B, also with a detector behind it. It is supposed that the axis of the polarizer at A is parallel to the vector **a**, and that of the polarizer at B is parallel to the vector **b**. Let $P(\mathbf{a})$ be the probability that the first photon is transmitted by the polarizer at A, so that it is counted by the detector. Otherwise the photon is absorbed by the polarizer and is thus not counted, the probability of this being $1 - P(\mathbf{a})$. Similarly, $P(\mathbf{b})$ and $1 - P(\mathbf{b})$ are the probabilities of transmission or absorption by the polarizer at B. These probabilities can be estimated by running the experiment many times and counting relative frequencies. The prediction of quantum mechanics is

$$P(\mathbf{a}) = ½ = P(\mathbf{b}).$$

Let $P(\mathbf{a}, \mathbf{b})$ be the joint probability of transmission of the photons at both A and B, with polarizer settings **a** and **b** respectively. In the notation of the previous section, this is written $P(A \cap B)$, where $A(\mathbf{a})$ and $B(\mathbf{b})$ are the events corresponding to registering transmission at A with setting **a** and at B with setting **b**. The prediction of quantum mechanics is

$$P(\mathbf{a}, \mathbf{b}) = ½ \cos 2\theta ,$$

where θ is the angle between the vectors **a** and **b**.

For the first derivation of the Bell inequality, it is supposed that this joint probability can be written in the form

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda\, \rho(\lambda)\, P(\mathbf{a} \mid \lambda)\, P(\mathbf{b} \mid \lambda)$$

which may be called the assumption of separability, with a hidden variable, λ. Here $P(\mathbf{a} \mid \lambda)$ is the conditional probability density for transmission at A, given that the setting at A is **a,** and the conditioning is with respect to λ. The unconditional probability for transmission at A, with setting **a**, can be written

$P(\mathbf{a}) = \int d\lambda\, \rho(\lambda)\, P(\mathbf{a} \mid \lambda)$

and similarly for P(**b**). It is required that the weight function, ρ , be non-negative and normalized:

$\int d\lambda\, \rho(\lambda) = 1 , \qquad \rho(\lambda) \geq 0 .$

Suppose now that the experiment is repeated with new settings for the polarizers, **a'** and **b'** instead of **a** and **b**, generating new probabilities. Moreover, the combinations { **a , b'**} and { **a', b**} can also be realized, resulting finally in measurements of relative frequencies that estimate the joint probabilities P(**a,b**), P(**a',b**), P(**a,b'**) and P(**a',b'**).

We shall define the Bell coefficient, **B,** which involves the analogous probabilities for the four possible combinations of settings, as follows:

$$\mathbf{B} = P(\mathbf{a,b}) + P(\mathbf{a',b}) + P(\mathbf{a,b'}) + P(\mathbf{a',b'})$$
$$= \int d\lambda\, \rho(\lambda)\, \{P(\mathbf{a} \mid \lambda)\, P(\mathbf{b} \mid \lambda)] + P(\mathbf{a'} \mid \lambda)\, P(\mathbf{b} \mid \lambda) + [\, P(\mathbf{a} \mid \lambda) - P(\mathbf{a'} \mid \lambda)]\, P(\mathbf{b'} \mid \lambda) \,\}$$

We propose to obtain an upper bound on **B**. If $P(\mathbf{a} \mid \lambda) - P(\mathbf{a'} \mid \lambda) \leq 0$, we majorize the integrand above by omitting the term involving this difference, which is negative or zero, and further we majorize $P(\mathbf{a} \mid \lambda)\, P(\mathbf{b} \mid \lambda)$ by $P(\mathbf{a} \mid \lambda)$ and $P(\mathbf{a'} \mid \lambda)\, P(\mathbf{b} \mid \lambda)$ by $P(\mathbf{b} \mid \lambda)$. Thus

$\mathbf{B} \leq \int d\lambda\, \rho(\lambda)\, [P(\mathbf{a} \mid \lambda) + P(\mathbf{b} \mid \lambda)]$

On the other hand, if $P(\mathbf{a} \mid \lambda) - P(\mathbf{a'} \mid \lambda) > 0$, we majorize by replacing $P(\mathbf{b'} \mid \lambda)$ by 1, which is allowed, since its coefficient is in this case positive. After transposition of some terms, we find

$\mathbf{B} = \int d\lambda\, \rho(\lambda)\, \{P(\mathbf{a} \mid \lambda) + P(\mathbf{a} \mid \lambda)\, P(\mathbf{b} \mid \lambda) - P(\mathbf{a'} \mid \lambda)[1 - P(\mathbf{b'} \mid \lambda)\,]\} .$

Here the term involving $P(\mathbf{a'} \mid \lambda)$ is negative or zero, and so may be omitted, and moreover we now choose to replace $P(\mathbf{a} \mid \lambda)\, P(\mathbf{b} \mid \lambda)$ by $P(\mathbf{b} \mid \lambda)$. In this way we have shown that

$$\mathbf{B} \leq \int d\lambda\, \rho(\lambda)\, [P(\mathbf{a} \mid \lambda) + P(\mathbf{b} \mid \lambda)]$$

is valid also in this case. Rewriting the result in terms of the unconditional probabilities, we obtain the Bell inequality in the form that we shall use it later:

$$\mathbf{B} = P(\mathbf{a,b}) + P(\mathbf{a',b}) + P(\mathbf{a,b'}) + P(\mathbf{a',b'}) \leq P(\mathbf{a}) + P(\mathbf{b})$$

Suppose that the settings at A and B are chosen such that the angle between **a** and **b**, between **a** and **b'** and between **a'** and **b** are all the same, say θ, while that between **a'** and **b'** is $3\,\theta$. In this case we obtain, as the Bell inequality,

$3 \cos^2 \theta - \cos^2 3\,\theta < 2 .$

With the choice $\theta = \pi/6$, we evaluate the left-hand side as 9/4, showing indeed that quantum mechanics predicts a violation of the Bell inequality. This prediction has been confirmed in the experiments of Aspect and of others.

4. Noncontextuality and Bell's Inequality

A different derivation of the inequality starts from the supposition that separate joint probabilities exist for all of the four combinations of polarizer settings. Let us now write P(**a**+, **b**+) in place of P(**a**,**b**) , to emphasize that this is the probability of *transmission* at A and B, with the polarizer settings **a** and **b** respectively. The corresponding probability for *absorption* at A and B is written P(**a**−,**b**−), while P(**a**+, **b** −) and P(**a**−,**b**+) are the probabilities for transmission at one polarizer and absorption at the other. We set

P(**a**+, **b**+) = P(**a**+, **a'**+, **b'**+, **b**+) + P(**a**+, **a'**+, **b'**−, **b**+) + P(**a**+, **a'**−, **b'**+, **b**+) + P(**a**+, **a'**−, **b'**−, **b**+)

which might be given the following Kolmogorovian, counterfactual interpretation. Consider the set of pairs of photons that are transmitted, one at A and one at B, when the settings are respectively **a** and **b.** Imagine that this set is divided into four disjoint sets, according to *what supposedly would have happened* if the settings had been **a'** and **b'** at A and B. Axiom V. of Kolmogorov is invoked to justify the addition of probabilities for these exclusive situations. One supposes that each photon pair has, *at the same time,* the proclivity to be transmitted if **a** and **b** are the settings, *and* one or other of the four exclusive proclivities with respect to the counterfactual settings **a'** and **b'**.

Noncontextuality means here that, if the settings really are **a'** and **b'**, instead of **a** and **b**, then the corresponding joint probability can now be divided into the following counterfactual subsets:

P(**a'**+, **b'**+) = P(**a**+, **a'**+, **b'**+, **b**+) + P(**a**+, **a'**+, **b'**+, **b**−) + P(**a**−, **a'**+, **b'**+, **b**+) + P(**a**−, **a'**+, **b'**+, **b**−)

Here the first term on the right is supposed to be the same as the first term on the right of the expression for P(**a**+, **b**+). That is, the counterfactual probability that a photon pair would have been transmitted if the settings had been **a** and **b**, given that they are **a'** and **b'**, is the same as the corresponding probability if the settings had been **a'** and **b'**, given that they are **a** and **b** (and similarly for all the other possible combinations). This assumption is natural from Einstein's realist viewpoint: the idea is that a given pair of photons either does, or does not have the necessary properties to ensure transmission when the settings are *either* **a** and **b** *or* **a'** and **b'**. On the other hand, the assumption would have been anathema to Bohr, for whom the proclivities are joint properties of the photons and of the macroscopic measuring system. The choice of **a** and **b** for the settings specifies one macroscopic measuring system, and the choice of **a'** and **b'** specifies another. For him the counterfactual probabilities have no meaning, since if the photons are detected with one setting, they cannot be detected with another. The following derivation of the Bell inequality from the assumption of noncontextuality, together with the violation of the inequality in the Aspect experiment, is often taken to support Bohr's view at the expense of Einstein's Weltanschauung.

Let us streamline the notation before proceeding further. We write P_{++} in place of the probability P(**a**+, **b**+), and ρ_{+jk+} for the four counterfactual probabilities, where j and k can take on the values + or − . We write $P_{il} = \Sigma_{jk}\, \rho_{ijkl}$. Here i and l go over the values + and −. We have here four probabilities, P, and moreover sixteen counterfactual conditional probabilities ρ. In accordance with Kolmogorov's axiom III, all these probabilities are non-negative, the ρ as well as the P. Next, we shall consider the alternative probabilities P(**a**±, **b'**±), which we shall rewrite $Q_{\pm\pm}$. Evidently $Q_{ik} = \Sigma_{jl}\, \rho_{ijkl}$. Writing $R_{\pm\pm}$ in place of P(**a'**±, **b**±), we have $R_{jl} = \Sigma_{ik}\, \rho_{ijkl}$. The fourth option, namely with $S_{\pm\pm}$ in place of P(**a'**±, **b'**±), reads $S_{il} = \Sigma_{jk}\, \rho_{ijkl}$ .It is easy to check the following expression for the Bell coefficient:

$$\mathbf{B} = P(++) + Q(++)+R(++) - S(++) = 2\, \rho_{++++} + 2\, \rho_{++-+} + 2\, \rho_{+-++} + \rho_{+--+} + \rho_{-+-+} - \rho_{-++-}$$

On the other hand, the probability that the photon has the proclivity to be transmitted at A when the setting there is **a,** irrespective of what happens at B, is

$$P(\mathbf{a}+)=P(\mathbf{a}+,\mathbf{b}+)+P(\mathbf{a}+,\mathbf{b}-) = \Sigma_{jkl}\, \rho_{+jkl} .$$

The probability that the photon has the proclivity to be transmitted with the setting at B when the setting there is **b,** irrespective of what happens at B, is

$$P(\mathbf{b}+)=P(\mathbf{a}+,\mathbf{b}+)+P(\mathbf{a}-,\mathbf{b}+) = \Sigma_{ijk+}\, \rho_{ijk+} .$$

A short calculation shows that

$$P(\mathbf{a}+) + P(\mathbf{b}+) - B = \rho_{+--+} + \rho_{+++-} + \rho_{++--} + \rho_{+-+-} + \rho_{+---} + \rho_{-+++} + \rho_{--++} + \rho_{---+} ..+ \rho_{-++-}$$

This is non-negative, since none of the ρ_{ijkl} are negative. In terms of the original notation, we have shown that

$$\mathbf{B} = P(\mathbf{a}+,\mathbf{b}+) + P(\mathbf{a}+,\mathbf{b'}+) + P(\mathbf{a'}+,\mathbf{b}+) - P(\mathbf{a'}+,\mathbf{b'}+) \leq P(\mathbf{a}+) + P(\mathbf{b}+)$$

which is precisely the Bell inequality, in the notation of this section. It has been shown to be a consequence of the assumed existence of (noncontextual) joint probabilities that satisfy the Kolmogorov axioms.

5. Representation Theorem

In this section, we start with the sixteen probabilities, P_{ij} , Q_{ij} , R_{ij} , S_{ij} , subject to the normalization conditions

$$P_{++} + P_{+-} + P_{-+} + P_{--}= Q_{++} + Q_{+-} + Q_{-+} + Q_{--}= R_{++} + R_{+-} + R_{-+} + R_{--}= S_{++} + S_{+-} + S_{-+} + S_{--}= 1$$

The question is whether these quantities admit a representation in terms of the sixteen weights ρ_{ijkl} (noncontextual joint probabilities). In the first place, the answer is certainly <u>no</u>, unless, in addition to the normalization conditions, the following constraints are satisfied:

$$P_{++} + P_{+-} = Q_{++} + Q_{+-} \qquad P_{++} + P_{-+} = R_{++} + R_{-+}$$
$$S_{++} + S_{+-} = R_{++} + R_{+-} \qquad S_{++} + S_{-+} = Q_{++} + Q_{-+}$$

So let us restate the question: given that the positive P, Q, R and S satisfy the above constraints, is there always a representation in terms of the ρ? We shall show that, if we drop the requirement that the ρ are positive, then there is indeed a solution, but it is not unique. Moreover, for some P, Q, R and S, we shall show that there are no solutions for which all the ρ are non-negative. That this must be so follows from the fact that the Bell inequality is violated for some P, Q, R and S, whereas if the ρ were positive in such cases, one could derive that inequality.

Let us first ask the restricted question: is it possible always to find ρ_{ijkl} if we only specify the P, Q and R as given, positive quantities, obeying those of the constraints that do not involve the S? That this *is* possible we now show by construction. Set

$$\rho_{ijkl} = (P_{il}\; Q_{ik}\; R_{jl}) / (Q_i\, R_l)$$

With

$$Q_i = \Sigma_k\, Q_{ik} = \Sigma_l\, P_{il} \qquad\qquad R_l = \Sigma_j\, R_{jl} = \Sigma_i\, P_{il} .$$

These expressions are consistent with the constraints given above, and they can readily be shown to be a valid representation of the P, Q and R. Moreover, the constructed ρ are non-negative, since all the conditional probabilities are positive. The above construction shows that, for any acceptable P, Q and R, there is a representation of the required form. However, although the four probabilities S_{jk} could be calculated using the ρ_{ijkl} that have been constructed, there is no guarantee that they would agree with the S that are given (or measured).

6. Negative Probabilities

We have seen that it is always possible to construct a set ρ_{ijkl} that fits any specified, acceptable set of probabilities, P_{il} , Q_{ik} and R_{jl} but that the corresponding values of S_{jk} are not guaranteed to be as specified. What must we add to ρ to rectify the S values? Evidently we must add something with care, for the P, Q and R are already correct and so must not be disturbed. In order to change S while leaving P, Q and R unchanged, the changes in S must satisfy

$$S_{++} + S_{+-} = 0 \qquad S_{++} + S_{-+} = 0 \qquad S_{++} + S_{+-} + S_{-+} + S_{--} = 0$$

which implies that the allowed changes are related as follows:

$$\Delta S_{++} = -\Delta S_{+-} = -\Delta S_{-+} = \Delta S_{--} \ .$$

Consider the example given at the end of section 3, corresponding to $\theta = \pi/6$. This gives

$$P_{++} = P_{--} = Q_{++} = Q_{--} = R_{++} = R_{--} = 3/8$$
$$P_{+-} = P_{-+} = Q_{+-} = Q_{-+} = R_{+-} = R_{-+} = 1/8$$
$$S_{++} = S_{--} = 0$$
$$S_{+-} = S_{-+} = ½ \ .$$

From the representation theorem given in section 5, we find the following, non-negative values:

$$\rho_{++++} = \rho_{----} = 27/128$$
$$\rho_{++-+} = \rho_{+-++} = \rho_{+-+-} = \rho_{-+-+} = \rho_{--+-} = \rho_{-+--} = 9/128$$
$$\rho_{+--+} = \rho_{---+} = \rho_{+---} = \rho_{-++-} = 3/128$$
$$\rho_{++--} = \rho_{--++} = 1/128 \ .$$

These assignments reproduce the P, Q and R correctly, but we find

$$S_{++} = 3/32$$
$$S_{+-} = S_{-+} = 9/32$$
$$S_{--} = 27/32 \ ,$$

which are indeed quite wrong. However, by adding

$$\Delta S_{++} = -\Delta S_{+-} = -\Delta S_{-+} = \Delta S_{--} = 9/32$$

we can rectify the mismatch, without spoiling the consistency relations with the P, Q and R . However, this changes four of the ρ to the following values:

$$\rho_{-++-} = -33/128 \qquad \rho_{-+--} = 45/128 \qquad \rho_{--+-} = 45/128 \qquad \rho_{----} = -9/128 \ .$$

As can be seen, ρ_{-++-} and ρ_{----} are negative, which is inconsistent with their interpretation as probabilities.

Is there any way to remove the negativity of these ρ, without spoiling the fit to P, Q, R and S? It may be shown directly by manipulating sixteen-dimensional matrices that there is not (see [Atkinson 1999]). Although there is a seven-dimensional manifold of solutions of the problem of representing the sixteen P, Q, R and S, at least one of the sixteen ρ must be negative. This is confirmation of what we already knew indirectly, for on the one hand we have shown that the Bell inequality can be proved if none of the ρ are negative, and on the other hand we know that the inequality is in fact violated for the choice of angles in question.

7. Conclusions

In the above treatment of the Aspect experiment, we have shown that the the physically testable, non-negative probabilities, $P_{\pm\pm}$, $Q_{\pm\pm}$, $R_{\pm\pm}$, and $S_{\pm\pm}$, may be written in separable form, in terms of noncontextual, counterfactual probabilities, ρ_{ijkl} , but only on condition that some of these counterfactual quantities are allowed to be negative. Why should they not be negative? Because they are called probabilities? Call them then 'indefinite weights' that have to be added, four or eight at a time, to yield genuine probabilities that can stand the full rigour of empirical confrontation! It is in the very nature of the separate weights that they do not refer to measurable quantities.

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