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If $d$ is Super-Metric, Then $d/(1 + d)$ is Super-Metric

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Abstract

If a function $d$ is metric, a well-known result is that $d/(1 + d)$ is also metric. We consider $m$-ary analogs of the binary notion of semi-metric, called hemi-metrics and super-metrics. The metrics are totally symmetric maps from $X^{m+1}$ into $\mathbb{R}_{\geq 0}$. It is shown that, if $d$ is super-metric, then $d/(1 + d)$ is also super-metric.

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Keywords: Hemi-metric, simplex inequality, tetrahedron inequality

1 Hemi-metrics and super-metrics

A metric is a function that defines a distance between two elements of a set. We consider generalizations of the notion of metric in the direction of distances between three or more elements.

Deza and Rosenberg [4] introduced the following notion. Let $m$ be a positive integer and $X$ a set with at least $m+2$ elements. A function $d : X^{m+1} \rightarrow \mathbb{R}$ is called $m$-hemi-metric if (see, also [1,2,5]):

1. $d$ is non-negative, i.e., $d(x_1, \ldots, x_{m+1}) \geq 0$ for all $x_1, \ldots, x_{m+1} \in X$.

2. $d$ is totally symmetric, i.e., satisfies $d(x_1, \ldots, x_{m+1}) = d(x_{\pi(1)}, \ldots, x_{\pi(m+1)})$ for all $x_1, \ldots, x_{m+1} \in X$ and for any permutation $\pi$ of $\{1, \ldots, m+1\}$. 
3. \(d\) is zero conditioned, i.e. \(d(x_1, \ldots, x_{m+1}) = 0\) if and only if \(x_1, \ldots, x_{m+1}\) are not pairwise distinct.

4. For all \(x_1, \ldots, x_{m+2} \in X\), \(d\) satisfies the \(m\)-simplex inequality:

\[
d(x_1, \ldots, x_{m+1}) \leq \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}).
\] (1)

The notion of \(m\)-hemi-metric is an \(m\)-ary analog of the binary notion of semi-metric. An important special case of the \(m\)-hemi-metric is the following notion obtained for \(m = 2\). A function \(d : X^3 \to \mathbb{R}\) is called a 2-metric if \(d\) is non-negative, totally symmetric, zero conditioned, and satisfies the tetrahedron inequality:

\[
d(x_1, x_2, x_3) \leq d(x_1, x_2, x_4) + d(x_1, x_3, x_4) + d(x_2, x_3, x_4).
\] (2)

Interpreting \(d(x_1, x_2, x_3)\) as the area of the triangle with vertices \(x_1, x_2\) and \(x_3\), the tetrahedron inequality specifies that the area of each triangle face of the tetrahedron formed by \(x_1, x_2, x_3\) and \(x_4\) does not exceed the sum of the areas of the remaining faces. Alternative axiom systems are considered in [6-11].

Deza and Dutour [3] introduced the following notion. Let \(s\) be a positive real number. A function \(d : X^{m+1} \to \mathbb{R}\) is called \((m, s)\)-super-metric if \(d\) is non-negative, totally symmetric, zero conditioned, and satisfies the \((m, s)\)-simplex inequality:

\[
sd(x_1, \ldots, x_{m+1}) \leq \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}).
\] (3)

An \((m, s)\)-super-metric is an \(m\)-hemi-metric if \(s \geq 1\). Furthermore, a \(m\)-hemi-metric is a \((m, 1)\)-super-metric and a semi-metric is a \((1, 1)\)-super-metric.

For the ordinary metric, a well-known result is that, if \(d\) is metric, then \(d/(1 + d)\) and \(\min\{1, d\}\) are also metric. In Section 2 we present an analogous result for the function \(d/(1 + d)\) for hemi-metrics and super-metrics. In Section 3 we present an analogous result for the function \(\min\{1, d\}\) for hemi-metrics and the \((2, 2)\)-super-metric.

## 2 Function \(d/(1 + d)\)

Lemma 2.1 considers the notion of \(m\)-hemi-metric. Lemma 2.3 considers the notion of \((m, s)\)-super-metric for \(s \geq 1\). Lemma 2.2 is used in the proof of Lemmas 2.3 and 3.2.

**Lemma 2.1.** Let \(d\) be \(m\)-hemi-metric. Then \(d/(1 + d)\) is \(m\)-hemi-metric.
If \( d \) is super-metric, then \( d/(1 + d) \) is super-metric.

**Proof.** Non-negativity of \( d/(1 + d) \) follows from the non-negativity of \( d \). Furthermore, total symmetry and axiom 3 follow from the identity

\[
\frac{d(x_1, \ldots, x_{m+1})}{1 + d(x_1, \ldots, x_{m+1})} = 1 - \frac{1}{1 + d(x_1, \ldots, x_{m+1})},
\]

and the fact that \( d \) is totally symmetric and zero conditioned. Thus, we must show that \( d/(1 + d) \) satisfies (1).

Because \( d/(1 + d) \) is strictly increasing in \( d \), and since \( d \) satisfies (1), we have

\[
d(x_1, \ldots, x_{m+1}) \leq \frac{1}{1 + d(x_1, \ldots, x_{m+1})} \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}).
\]

Furthermore, for all \( i \in \{1, \ldots, m+1\} \) we have the inequality

\[
d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}) \leq \frac{1}{1 + d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})} \sum_{j=1}^{m+1} d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2}).
\]

Summing (6) over all \( i \in \{1, \ldots, m+1\} \), and combining the resulting inequality with inequality (5), completes the proof.

**Lemma 2.2.** Suppose \( s > 1 \) and let \( d \) be \((m, s)\)-super-metric. Then \( d \) satisfies the inequality

\[
(s - 1)d(x_1, \ldots, x_{m+1}) \leq \sum_{i=2}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}).
\]

**Proof.** Interchanging the roles of \( x_1 \) and \( x_{m+2} \) in (3), and dividing the result by \( s \), we obtain

\[
d(x_2, \ldots, x_{m+2}) \leq \frac{1}{s} \sum_{i=2}^{m+2} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}).
\]

Adding inequalities (3) and (8) yields

\[
(s - 1) \leq \left( 1 + \frac{1}{s} \right) \sum_{i=2}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}),
\]

which is equivalent to (7). □
Lemma 2.3. Suppose $s \geq 1$ and let $d$ be $(m, s)$-super-metric. Then $d/(1+d)$ is $(m, s)$-super-metric.

Proof. The case $s = 1$ is proved in Lemma 2.1. Therefore, suppose $s > 1$. The proof of non-negativity, total symmetry and axiom 3 is analogous to the proof of Lemma 2.1. We must show that $d$ satisfies (3).

Because $d/(1 + d)$ is strictly increasing in $d$, and since $d$ satisfies (3), we have

$$\frac{d(x_1, \ldots, x_{m+1})}{1 + d(x_1, \ldots, x_{m+1})} \leq \frac{\frac{1}{s} \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})}{1 + \frac{1}{s} \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})}. \quad (10)$$

After multiplying both sides of (10) by $s$, we may write the result as

$$\frac{sd(x_1, \ldots, x_{m+1})}{1 + d(x_1, \ldots, x_{m+1})} \leq \sum_{i=1}^{m+1} \frac{d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})}{1 + \frac{1}{s} \sum_{j=1}^{m+1} d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2})}. \quad (11)$$

Due to Lemma 2.2, combined with the total symmetry of $d$, we have for all $i \in \{1, \ldots, m+1\}$,

$$(s - 1)d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}) \leq \sum_{j=1}^{m+1} d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2}) - d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}). \quad (12)$$

Adding $d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})$ to both sides of (12), and dividing the result by $s$, we have for all $i \in \{1, \ldots, m+1\}$,

$$d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}) \leq \frac{1}{s} \sum_{j=1}^{m+1} d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2}). \quad (13)$$

Furthermore, using (13), we have, for all $i \in \{1, \ldots, m+1\}$, the inequality

$$\frac{d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})}{1 + \frac{1}{s} \sum_{j=1}^{m+1} d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2})} \leq \frac{d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})}{1 + d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})}. \quad (14)$$

Summing (14) over all $i \in \{1, \ldots, m+1\}$, and combining the result with (11), completes the proof. \qed
If $d$ is super-metric, then $d/(1 + d)$ is super-metric.

**3 Function $\min \{1, d\}$**

Lemma 3.1 considers the notion of $m$-hemi-metric. Lemma 3.2 considers the notion of $(2, 2)$-super-metric.

**Lemma 3.1.** Let $d$ be $m$-hemi-metric. Then $\min \{1, d\}$ is $m$-hemi-metric.

*Proof.* Non-negativity, symmetry and axiom 3 of $\min \{1, d\}$ follow from the analogous properties of $d$. Thus, we must show that $\min \{1, d\}$ satisfies (1).

We go through the various cases.

Suppose there is an $j \in \{1, \ldots, m + 1\}$ such that

$$d(x_1, \ldots, x_{m+1}) \leq d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2}).$$

(15)

In this case we have

$$\min_{m+1} \{1, d(x_1, \ldots, x_{m+1})\} \leq \min_{m+1} \{1, d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2})\} \leq \sum_{i=1}^{m+1} \min \{1, d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})\}.$$  

(16)

Thus, we may assume that, for all $i \in \{1, \ldots, m + 1\}$, we have

$$d(x_1, \ldots, x_{m+1}) \geq d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}).$$

(17)

Suppose $d(x_1, \ldots, x_{m+1}) \leq 1$. In this case we have, for all $i \in \{1, \ldots, m + 2\}$,

$$\min_{m+1} \{1, d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})\} = d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}),$$

and it follows that $\min \{1, d\}$ satisfies (1) because $d$ satisfies (1).

Next, suppose $d(x_1, \ldots, x_{m+1}) > 1$ because $d$ satisfies (1).

Furthermore, suppose there is an $j \in \{1, \ldots, m + 1\}$ such that $d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2}) \geq 1$. In this case we have

$$\min_{m+1} \{1, d(x_1, \ldots, x_{m+1})\} = 1 = \min_{m+1} \{1, d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2})\} \leq \sum_{i=1}^{m+1} \min \{1, d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})\}.$$ 

(18)

Therefore, suppose that $d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}) \leq 1$ for all $i \in \{1, \ldots, m + 1\}$. In this final case we have, since $d$ satisfies (1),

$$\min_{m+1} \{1, d(x_1, \ldots, x_{m+1})\} = 1 < d(x_1, \ldots, x_{m+1}) \leq \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}) = \sum_{i=1}^{m+1} \min \{1, d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})\}.$$  

(19)

This completes the proof. □
Lemma 3.2. Let $d$ be $(2,2)$-super-metric. Then $\min \{1,d\}$ is $(2,2)$-super-metric.

Proof. Non-negativity, symmetry and axiom 3 of $\min \{1,d\}$ follow from the analogous properties of $d$. Thus, we must show that $\min \{1,d\}$ satisfies

$$2d(x_1,x_2,x_3) \leq d(x_1,x_2,x_4) + d(x_1,x_3,x_4) + d(x_2,x_3,x_4),$$

which is a strong version of tetrahedron inequality (2) [6,8,9,11]. We go through the various cases.

First, suppose $d(x_1,x_2,x_3) \leq 1$. In addition, suppose at least two of the three quantities on the right-hand side of (20) $\geq 1$. In this case we have

$$2 \min \{1,d(x_1,x_2,x_3)\} = 2d(x_1,x_2,x_3) \leq 2 = 1 + 1$$

$$\leq \min \{1,d(x_1,x_2,x_4)\} + \min \{1,d(x_1,x_3,x_4)\} + \min \{1,d(x_2,x_3,x_4)\}.$$ 

Furthermore, without loss of generality, suppose that $d(x_1,x_2,x_4) > 1$ and $d(x_1,x_3,x_4),d(x_2,x_3,x_4) \leq 1$. In this case we have

$$\min \{1,d(x_1,x_2,x_3)\} = d(x_1,x_2,x_3) \leq 1 = \min \{1,d(x_1,x_2,x_4)\}. \quad (21)$$

We also have, using Lemma 2.2,

$$\min \{1,d(x_1,x_2,x_3)\} = d(x_1,x_2,x_3) \leq d(x_1,x_3,x_4) + d(x_2,x_3,x_4)$$

$$= \min \{1,d(x_1,x_3,x_4)\} + \min \{1,d(x_2,x_3,x_4)\}. \quad (22)$$

Combining (21) and (22) gives the desired inequality.

Moreover, suppose all three quantities on the right-hand side of (20) $\leq 1$. In this case we have, since $d$ satisfies (20),

$$2 \min \{1,d(x_1,x_2,x_3)\} = 2d(x_1,x_2,x_3)$$

$$\leq d(x_1,x_2,x_4) + d(x_1,x_3,x_4) + d(x_2,x_3,x_4)$$

$$= \min \{1,d(x_1,x_2,x_4)\} + \min \{1,d(x_1,x_3,x_4)\} + \min \{1,d(x_2,x_3,x_4)\}.$$ 

Second, suppose $d(x_1,x_2,x_3) > 1$. In addition, suppose at least two of the three quantities on the right-hand side of (20) $\geq 1$. In this case we have

$$2 \min \{1,d(x_1,x_2,x_3)\} = 2 = 1 + 1$$

$$\leq \min \{1,d(x_1,x_2,x_4)\} + \min \{1,d(x_1,x_3,x_4)\} + \min \{1,d(x_2,x_3,x_4)\}.$$ 

Furthermore, without loss of generality, suppose that $d(x_1,x_2,x_4) \geq 1$ and $d(x_1,x_3,x_4),d(x_2,x_3,x_4) \leq 1$. In this case we have

$$2 \min \{1,d(x_1,x_2,x_3)\} = 2 < d(x_1,x_2,x_3) + \min \{1,d(x_1,x_2,x_4)\}. \quad (23)$$
If \( d \) is super-metric, then \( d/(1 + d) \) is super-metric.

We also have, using Lemma 2.2,

\[
d(x_1, x_2, x_3) \leq d(x_1, x_3, x_4) + d(x_2, x_3, x_4) \\
= \min \{1, d(x_1, x_3, x_4)\} + \min \{1, d(x_2, x_3, x_4)\}.
\]

Combining (23) and (24) gives the desired inequality.

Finally, suppose all three quantities on the right-hand side of (20) \( \leq 1 \). In this case we have, since \( d \) satisfies (20),

\[
2 \min \{1, d(x_1, x_2, x_3)\} = 2 < 2d(x_1, x_2, x_3) \\
\leq d(x_1, x_2, x_4) + d(x_1, x_3, x_4) + d(x_2, x_3, x_4) \\
= \min \{1, d(x_1, x_2, x_4)\} + \min \{1, d(x_1, x_3, x_4)\} + \min \{1, d(x_2, x_3, x_4)\}.
\]

This completes the proof.

\[\square\]

References


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