Distributed control of power networks
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Chapter 8

Distributed control with discrete communication

Abstract

This chapter investigates the digital nature of the consensus algorithm, that has been employed throughout this thesis, to achieve an optimal allocation of inputs to a network. We focus on a particular digital implementation of the communication to which we will refer as a ‘discrete broadcasting protocol’. That is, a node in a network broadcasts a certain value to its neighbors at discrete time instances. The contribution of this chapter includes the design of a storage function, taking into account the digital nature of the communication, that allows us to determine a lower bound on the broadcasting frequency guaranteeing stability of an underlying system, that is proven to be stable in the presence of continuous communication. First, for illustrative purposes we consider a simple example of a consensus algorithm, whereafter we discuss the consensus algorithm in closed loop with a nonlinear system.

8.1 From continuous consensus to discrete broadcasting

In this chapter we study a ‘discrete broadcasting’ implementation of the thoroughly studied consensus algorithm

\[
\dot{\theta} = -B^T \Gamma B^T \theta = -L \theta ,
\]

(8.1)

where \( L \) is the (weighted) Laplacian matrix associated to a connected and undirected graph \( G = (V, E) \) consisting of \( n \) nodes and where \( \theta \in \mathbb{R}^n \) is the state. With broadcasting we, informally, mean that a node \( i \) sends (broadcasts) its current value of \( \theta_i \) to its neighbouring nodes at discrete time instances. Despite the fact that many result on this topic are available, there are still some open questions (Nowzari et al. 2016) and we believe that the provided results in this chapter contribute to
a further understanding of interconnected physical and digital systems. To facilitate a discrete broadcasting implementation, we impose a few assumptions on the communication network.

**Assumption 8.1.1 (Communication network).** Throughout this chapter we make the following assumptions:

- The communication network is connected and undirected, i.e. if node $i$ can communicate with node $j$, then also the converse is possible.
- Each node knows (an upper bound on) its (weighted) degree.
- Each node broadcasts its current state, with a (possibly varying) frequency, to all its neighbouring nodes.
- The communication is without delays and errors.

It is worth mentioning that in comparison with (De Persis and Postoyan 2017), where clocks or events are defined on the edges of the network, we focus on the design of ‘rules’, determining the broadcasting instances, that are implemented at the nodes, which is more in line with implementation practises. Within a setting of wireless transmission, the set of neighbouring nodes of node $i$ is generally the set of nodes that is within the communication range of node $i$. The main objective is to determine a scheduling of information transmission on each node, using only information that is locally available at a node, such that convergence to a consensus state is guaranteed. In this chapter we first study the autonomous system (8.1) that provide preparatory results for the remainder, whereafter we will study system (8.1) interconnected with output strictly incrementally passive distribution systems that include e.g. the various power networks that have appeared in this thesis.

**Remark 8.1.2 (Relaxing Assumption 8.1.1).** Although Assumption 8.1.1 covers a large class of networks, important extensions include incorporating delays and time-varying communication topologies. These extensions have been studied extensively within continuous consensus protocols, and an interesting endeavor is to study if the developed methods for the continuous protocol, can be incorporated within the setting studied here.

### 8.2 Average preserving consensus

In this section we propose a modification of the continuous consensus algorithm (8.1). Specifically, we consider the case where every node broadcasts at certain time
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instances, determined by a local clock at each node, its actual value of $\theta_i$ to its adjacent nodes, resulting in the system

$$\dot{\theta} = -L\hat{\theta}, \quad (8.2)$$

where, $\hat{\theta}_i$, is the latest broadcasted value of $\theta_i$. First, note that (8.2) preserves the average of initial values, since

$$\mathbf{1}_n^T \dot{\theta} = -\mathbf{1}_n^T L\hat{\theta} = 0. \quad (8.3)$$

Therefore, we aim at characterizing a scheduling of information transmission under which system (8.2) convergence to consensus, i.e.

$$\lim_{t \to \infty} \theta(t) = \overline{\theta}, \quad (8.4)$$

where

$$\overline{\theta} = \frac{1}{n} \sum_{i=1}^{n} \theta_i(0) \in \text{Im}(\mathbf{1}_n). \quad (8.5)$$

8.2.1 Hybrid system

To formalize the idea outlined above we introduce at every node $i \in \mathcal{V}$ a local 'clock variable' $\phi_i$ whose dynamics are given by

$$\dot{\phi}_i = -2\text{deg}(i)(\frac{1}{4} + \phi_i + \phi_i^2) - \epsilon_i, \quad (8.6)$$

with $\epsilon_i \in \mathbb{R}_{>0}$ a constant. With $\text{deg}(i)$ we denote the (weighted) degree of node $i \in \mathcal{V}$, i.e.

$$\text{deg}(i) = \sum_{k \in \mathcal{E}_i} \Gamma_k, \quad (8.7)$$

with $\mathcal{E}_i$ being the set of edges incident to node $i$ and $\Gamma_k$ the weight of edge $k$. Broadcasting of $\theta_i$ occurs when $\phi_i = a_i$, whereafter the clock is reset to $\phi_i^+ = b_i$, where $0 < a_i < b_i$ for all $i \in \mathcal{V}$. Furthermore, as we will show in Subsection 8.2.4, there exists a minimum time between two broadcasting instances of node $i \in \mathcal{V}$.

Remark 8.2.1 (Broadcasting with time-varying frequencies). For a given $b_i$ and $a_i$, equation (8.6) determines the broadcasting frequency. The analysis in this chapter permits to include a time-varying broadcasting frequency by considering instead of (8.6), the differential inclusion

$$\dot{\phi}_i \in \left[ -M_i, -2\text{deg}(i)(\frac{1}{4} + \phi_i + \phi_i^2) - \epsilon_i \right], \quad (8.8)$$
where $\epsilon_i \in \mathbb{R}_{> 0}$, and $M_i \in \mathbb{R}_{> 0}$ are constants. The corresponding analysis is pursued in a future publication. However, periodic broadcasting has its advantages due to its simplicity and can be implemented straightforwardly and permits e.g. a 'round robin' scheduling of transmission.

**Remark 8.2.2 (Clock dynamics).** The choice of (8.6) is a result of the stability analysis performed later in this section. We provide the clock dynamics here for the sake of exposition, but its rational will become clear in the proof of Lemma 8.2.5. It is important to note that every node possesses its own clock, allowing for asynchronous communication and that (8.6) does not depend on any global information of the network. Furthermore, (8.6) allows us to determine the minimum broadcasting frequency and we do so in Subsection 8.2.4.

To analyze the convergence properties we formulate the system within the hybrid system framework given in (Goebel et al. 2012), for which a few basic concepts are recalled in Subsection 1.4.3. For the present study, the corresponding flow set is

$$ C := \{(\theta, \hat{\theta}, \phi) \in \mathbb{R}^n \times \mathbb{R}^n \times [a, b] \} $$

with $[a, b] = [a_1, b_1] \times \ldots \times [a_n, b_n]$. The flow map $F(\theta, \hat{\theta}, \phi)$ is

$$
\begin{align*}
\dot{\theta} &= -L\hat{\theta} \\
\dot{\hat{\theta}} &= 0 \\
\dot{\phi} &= \alpha(\phi)
\end{align*}
$$

where $\alpha_i(\phi_i) = -2\deg(i)(1 + \phi_i + \phi_i^2) - \epsilon_i$, given by (8.6) above. The jump set is

$$ D := \{(\theta, \hat{\theta}, \phi) \in \mathbb{R}^n \times \mathbb{R}^n \times [a, b] : \exists i \in \{1, \ldots, n\} \text{ s.t. } \phi_i = a_i \} $$

The jump map $G(\theta, \hat{\theta}, \phi)$ is defined by

$$ G(\theta, \hat{\theta}, \phi) := \{G_i(\theta, \hat{\theta}, \phi) : i \in \{1, \ldots, n\} \text{ and } \phi_i = a_i \} $$

with

$$
\begin{align*}
\theta^+ &= \theta \\
\hat{\theta}_i^+ &= \theta_i \\
\hat{\theta}_j^+ &= \hat{\theta}_j \quad j \neq i \\
\phi_i^+ &= \phi_i \\
\phi_j^+ &= \phi_j \quad j \neq i
\end{align*}
$$

The definition of the jump map (8.10) ensures that at each jump, only one clock variable is reset. In case multiple clocks have reached their lower bound, multiple, but a finite number, successive jumps occur without flows in between. We provide further details on this in the proof of Theorem 8.2.8. The hybrid system with the data
above will be represented by the notation $H_1 = (C, F, D, G)$ or, briefly, by $H_1$. The stability analysis in the next subsection 8.2.3 is based on an invariance principle for hybrid systems that requires the system to be nominally well posed. This property is established for system at hand in the following lemma:

**Lemma 8.2.3** (Nominally well posed). The system $H_1$ is nominally well posed.

**Proof.** Following (Goebel et al. 2012, Theorem 6.8) it is sufficient that the system $H_1$ satisfies the following hybrid basic conditions:

1. $C$ and $D$ are closed subsets of $X := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$,
2. $F : X \rightarrow X$ is outer semicontinuous and locally bounded relative to $C$, $C \subset \text{dom } F$, and $F(z)$ is convex for every $z \in C$,
3. $G : X \rightarrow X$ is outer semicontinuous and locally bounded relative to $D$, $D \subset \text{dom } G$.

Condition 1) can be readily verified. Condition 2) is satisfied by $F : X \rightarrow X$ being a continuous mapping. Condition 3) is satisfied since $G$ is locally bounded and is given by the union of graphs of the continuous mappings $G_i$, $i \in \{1, \ldots, n\}$, which is closed. ■

### 8.2.2 Precompactness of solutions

In this subsection we show that the maximal solutions to $H_1$ are precompact, i.e. they are complete and the closure of their range is compact. Before proving the precompactness of the solutions we establish an important lemma that is essential to the remainder of this section. Consider the storage function

$$V_1(\theta, \hat{\theta}, \phi) = \frac{1}{2}(\theta - \bar{\theta})^T(\theta - \bar{\theta}) + (\theta - \hat{\theta})^T[\phi](\theta - \hat{\theta}),$$  

(8.12)

with $\bar{\theta}$ given by (8.5).

**Remark 8.2.4** (An alternative expression of $V_1$). Noting that $\frac{1}{2}(\theta - \bar{\theta})^T(\theta - \bar{\theta})$ can be written as $||\Pi \theta||^2$, with $\Pi = I - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T$, we can write 8.12 as

$$V_1 = \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} \Pi + [\phi] & -[\phi] \\ -[\phi] & [\phi] \end{bmatrix} \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix}.$$  

(8.13)

\begin{footnote}{We denote diag$(\phi_1, \ldots, \phi_n)$ by $[\phi]$.} \end{footnote}
Lemma 8.2.5 (Evolution of $V_1$). The storage function $V_1$ given in (8.12) satisfies

$$
\dot{V}_1(\theta, \hat{\theta}, \phi) = \begin{bmatrix} \theta^T \\ \hat{\theta} \end{bmatrix} Z_1 \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix} \leq 0 \quad (\theta, \hat{\theta}, \phi) \in C \quad (8.14)
$$

along the solutions to $H_1$, where $Z_1$ is given by

$$
Z_1 = \begin{bmatrix}
-\alpha & -\frac{1}{2}((I + 2[\phi])\mathcal{L})^T \\
\alpha & [\phi] + \mathcal{L}[\phi]
\end{bmatrix}.
$$

Proof. First, we consider the evolution of $V_1(\theta, \hat{\theta}, \phi)$ during the flows. We have that

$$
\dot{V}_1(\theta, \hat{\theta}, \phi) = -\theta^T \mathcal{L} \hat{\theta} + \hat{\phi}^T [\theta - \hat{\theta}](\theta - \hat{\theta}) + 2\phi^T [\theta - \hat{\theta}](-\mathcal{L} \hat{\theta})
$$

$$
= -\theta^T \mathcal{L} \hat{\theta} + \alpha^T [\theta - \hat{\theta}](\theta - \hat{\theta}) + 2\phi^T [\theta - \hat{\theta}](-\mathcal{L} \hat{\theta})
$$

$$
= \begin{bmatrix} \theta^T \\ \hat{\theta} \end{bmatrix} Z_1 \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix},
$$

where the matrix $Z_1$ is given by (8.15). Note that $[\alpha]$ is negative definite, and in order to have that $Z_1 \leq 0$ it is sufficient that the Schur complement $S_1$ of block $[\alpha]$ of matrix $Z_1$ satisfies $S_1 \leq 0$. This Schur complement is given by

$$
S_1 = [\alpha] + [\phi] \mathcal{L} + \mathcal{L}[\phi]
$$

$$
- \left( -\alpha - \frac{1}{2}((I + 2[\phi])\mathcal{L})^T \right) [\alpha]^{-1} \left( -\alpha - \frac{1}{2}((I + 2[\phi])\mathcal{L}) \right)
$$

$$
= [\alpha] + [\phi] \mathcal{L} + \mathcal{L}[\phi] - [\alpha][\alpha]^{-1}[\alpha]
$$

$$
- \frac{1}{4}((\mathcal{L} + 2[\phi])\mathcal{L})^T [\alpha]^{-1}(\mathcal{L} + 2[\phi] \mathcal{L}) - [\alpha][\alpha]^{-1}\frac{1}{2}(I + 2[\phi])\mathcal{L})
$$

$$
- \frac{1}{2}((I + 2[\phi])\mathcal{L})^T [\alpha]^{-1}[\alpha]
$$

$$
= -\mathcal{L} - \frac{1}{4}((\mathcal{L} + 2[\phi])\mathcal{L})^T [\alpha]^{-1}(\mathcal{L} + 2[\phi] \mathcal{L})
$$

$$
= -\mathcal{L} - \mathcal{L}([\alpha]^{-1}(\frac{1}{4}I + [\phi] + [\phi][\phi]))\mathcal{L}
$$

$$
= -B\Gamma^\frac{1}{2}((I + \Gamma^\frac{1}{2}B^T ([\alpha]^{-1}(\frac{1}{4}I + [\phi] + [\phi][\phi]))B\Gamma^\frac{1}{2}))\Gamma^\frac{1}{2}B^T.
$$

---

We write $\Gamma = \Gamma^\frac{1}{2}\Gamma^\frac{1}{2}$, where $\Gamma = \text{diag}(\Gamma_1, \ldots, \Gamma_k)$; the diagonal matrix consisting of the weights of the edges.
A sufficient condition for $S_1 \leq 0$ is

$$I + \Gamma_{\hat{z}}^T([\alpha]^{-1}(\frac{1}{4}I + [\phi] + [\phi][\phi]))B\Gamma_{\hat{z}}^T > 0. \quad (8.18)$$

This requires that all eigenvalues of $\Gamma_{\hat{z}}^T([\alpha]^{-1}(\frac{1}{4}I + [\phi] + [\phi][\phi]))B\Gamma_{\hat{z}}^T$ are greater than $-1$. Studying this problem is equivalent to finding the smallest eigenvalue of the weighted edge-Laplacian $\Gamma_{\hat{z}}^TBX\Gamma_{\hat{z}}^T$, with $X = [\alpha]^{-1}(\frac{1}{4}I + [\phi] + [\phi][\phi])$. This is equivalent to find $\lambda_{\text{min}}(XB\Gamma_{\hat{z}}^T) = \lambda_{\text{min}}(XL)$. By Gershgorin circle theorem we have that every eigenvalue of $XL$ lies within at least one of the Gershgorin discs with center $(XL)_{ii}$ and radius, $R_i = \sum_{j \in N_i} (XL)_{ij}$, where in this case we have that $(XL)_{ii} = \deg(i)X_{ii} = R_i$. To ensure that $\lambda_{\text{min}}(XL) > -1$ we need to enforce $2\deg(i)X_{ii} > -1$ for all $i$. Bearing in mind the definition of $X$ it follows that if $\alpha_i < -2\deg(i)(\frac{1}{4} + \phi_i + \phi_i^2)$, then $S_1 \leq 0$ and consequently $Z_1 \leq 0$. Since we picked in $(8.6)$ $\alpha_i = -2\deg(i)(\frac{1}{4} + \phi_i + \phi_i^2) - \epsilon_i$, with $\epsilon_i \in \mathbb{R}_{>0}$, we have indeed $V_i(\theta, \hat{\theta}, \phi) \leq 0$.

Second, $V_i(\theta^+, \hat{\theta}^+, \phi^+) = V_i(\theta, \hat{\theta}, \phi)$, since $\theta_i = \hat{\theta}_i$ whenever the value of $\phi_i$ is increased during a jump. \hfill \blacksquare

Exploiting the previous lemma we can now establish the following result.

**Lemma 8.2.6 (Precompactness of solutions).** Every maximal solution to system $\mathcal{H}_1$ is precompact.

**Proof.** For any $(\theta, \hat{\theta}, \phi) \in C \cup D = C$, we have that $(\theta, \hat{\theta}, \phi) \in D$ or that $(\theta, \hat{\theta}, \phi)$ is in the interior of $C$. Furthermore, $G(D) \subset C \cup D$. Since the domain of a maximal solution to $\mathcal{H}_1$ is unbounded it follows from (Goebel et al. 2012, Proposition 6.10) that there exists from every initial condition a nontrivial solution and every maximal solution is complete. Furthermore, we note that by construction $\phi \in \times [a, b]$ is bounded and positive. Exploiting Lemma 8.2.5, where we have proven that

$$V_i(\theta, \hat{\theta}, \phi) \leq 0 \quad (\theta, \hat{\theta}, \phi) \in C$$

$$V_i(\theta^+, \hat{\theta}^+, \phi^+) \leq V_i(\theta, \hat{\theta}, \phi) \quad (\theta, \hat{\theta}, \phi) \in D$$

it follows from (8.12) that both $\theta - \overline{\theta}$ and $\theta - \hat{\theta}$ are bounded over time. Since $\overline{\theta}$ is a constant, $\theta$ is bounded, and consequently $\hat{\theta}$ is bounded as well. We can conclude that the solutions to system $\mathcal{H}_1$ remain bounded and that the closure of their range is compact. We can therefore conclude that every maximal solution is precompact. \hfill \blacksquare
8.2.3 Stability analysis

According to the previous subsections the system $\mathcal{H}_1$ is nominally well posed and its maximal solutions are precompact. To infer that the solutions approach the desired (consensus) state, we rely on an invariance principle for hybrid systems (Lemma 1.4.17). Before we do so, we establish the following lemma that we will use to characterize the set that the solutions to $\mathcal{H}_1$ approach.

**Lemma 8.2.7** (Nullspace of $Z_1$). *The nullspace of $Z_1$ given in (8.15) satisfies*

$$\text{Ker}(Z_1) = \text{Im}(I_{2n}).$$  \tag{8.20}

**Proof.** (i) $\text{Im}(I_n) \subseteq \text{Ker}(Z_1)$.

Take $v = \beta I_n \in \text{Im}(I_n)$, with some scalar $\beta \in \mathbb{R}$. The claim follows from

$$Z_1 v = \beta \begin{bmatrix}
[\alpha]I_n & [\alpha]I_n - \frac{1}{2}(I + 2[\phi])L I_n \\
-\frac{1}{2}L I_n - L[\phi]I_n + L[\phi]I_n
\end{bmatrix} = 0,$$

or

$$A z_1 + B z_2 = 0 \quad S z_2 = 0.$$

(ii) $\text{Im}(I_n) \supseteq \text{Ker}(Z_1)$.

Note first the following factorization of the matrix $Z_1$, where indeed $\det(A) \neq 0$:

$$Z_1 = [A \quad B] C = \begin{bmatrix} I & A \quad 0 \\ CA^{-1} & 0 \quad S \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

Let $v \in \text{Ker}(Z_1)$. Then $Z_1 v = 0$. The latter is equivalent to

$$\begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} A^{-1} B \\ I \end{bmatrix} z = 0,$$

or

$$A z_1 + B z_2 = 0 \quad S z_2 = 0.$$

Recall from (8.17), that $S = -BYB^T$, with

$$Y = I + \Gamma \frac{1}{2} B^T([\alpha]^{-1} \left( \frac{1}{4} I + [\phi] \right) B) \Gamma \frac{1}{2},$$  \tag{8.22}
If \( Y > 0 \), then \( \text{Ker}(S) = \text{Im}(I_n) \). The latter and \( S z_2 = 0 \) imply \( z_2 = I_n \beta \) for some \( \beta \). Replacing it in \( A z_1 + B z_2 = 0 \) and bearing in mind the expressions for \( A, B \), we have
\[
0 = ([\alpha] z_1 - ([\alpha] + \frac{1}{2}(I + 2[\phi])L)z_2
= ([\alpha] z_1 - ([\alpha] + \frac{1}{2}(I + 2[\phi])L)I \beta
= ([\alpha] z_1 - [\alpha] I_n \beta,
\]
from which it follows that \( z_1 = I_n \beta \), that is
\[
z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = I_{2n} \beta,
\]
and therefore \( z \in \text{Im}(I_{2n}) \). This proves claim (ii) and the thesis is complete.

Using the lemma above, we are now ready to state the main result of this section.

**Theorem 8.2.8** (Approaching consensus). *The maximal solutions of system \( \mathcal{H}_1 \) approach the set where \( \theta = \hat{\theta} = \bar{\theta} \), with
\[
\bar{\theta} = I_n \sum_{i} \frac{\theta_i(0)}{n} \in \text{Im}(I_n).
\] (8.23)

**Proof.** We first note that the average value of \( x(0) \) is preserved since during flows
\[
I_n^T \dot{\theta} = - I_n^T L \hat{\theta} = 0,
\] (8.24)
and at a the jumps \( \theta^+ = \theta \). According to Lemma 8.2.5,
\[
V_1(\theta, \hat{\theta}, \phi) \leq u_c(\theta, \hat{\theta}, \phi) \quad \forall (\theta, \hat{\theta}, \phi) \in C
\] (8.25)
\[
V_1(\theta^+, \hat{\theta}^+, \phi^+) - V_1(\theta, \hat{\theta}, \phi) \leq u_d(\theta, \hat{\theta}, \phi) \quad \forall (\theta, \hat{\theta}, \phi) \in D,
\] (8.26)
where
\[
u_c(\theta, \hat{\theta}, \phi) = \begin{cases}
\begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix}^T Z_1 \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix} & (\theta, \hat{\theta}, \phi) \in C \\
-\infty & \text{otherwise}
\end{cases}
\] (8.27)
\[
u_d(\theta, \hat{\theta}, \phi) = \begin{cases}
0 & (\theta, \hat{\theta}, \phi) \in D \\
-\infty & \text{otherwise}
\end{cases}
\] (8.28)

\footnote{Trivially \( \text{Im}(I_n) \) \subseteq \text{Ker}(S). Let now \( v \in \text{Ker}(S) \). Then \( B^T v = 0 \), which implies \( v^T B^T v = 0 \). Since \( Y > 0 \), then necessarily \( B^T v = 0 \) and hence \( v \in \text{Im}(I_n) \), which proves \( \text{Ker}(S) \subseteq \text{Im}(I_n) \). Hence \( \text{Ker}(S) = \text{Im}(I_n) \).}
Here, we have introduced \( u_c \) and \( u_d \) to define the evolution of \( V_1 \) outside the flow and jump set, respectively, as is required to invoke the invariance principle (Goebel et al. 2012, Theorem 8.2). Now, in view of Lemmas 8.2.3, 8.2.5, 8.2.6, and (Goebel et al. 2012, Theorem 8.2), any maximal solutions to system \( H_1 \) approach the largest weakly invariant subset of

\[
\mathcal{Y} = V_1^{-1}(r) \cap \mathcal{X} \cap [u_c^{-1}(0) \cup (u_d^{-1}(0) \cap G(u_d^{-1}(0)))],
\]

where \( r \in V_1(\mathcal{X}) \) and \( \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^n \times [a,b] \). Note that \( u_d^{-1}(0) = D \), but that not necessarily \( D \cap G(D) = \emptyset \), as potentially multiple clocks reach at the same (hybrid) time their lower bound. We continue by showing that any maximal solution to \( H_1 \) in a weakly forward invariant subset of \( \mathcal{Y} \) is in the subset \( V_1^{-1}(r) \cap \mathcal{X} \cap u_c^{-1}(0) \). Assume, ad absurdum, that there exists a maximal solution \( q \) in \( \mathcal{Y} \) with initialization \( q(t_0,k_0) /in u_c^{-1}(0) \), for which \( V_1^{-1}(r) \cap \mathcal{X} \cap [u_d^{-1}(0) \cap G(u_d^{-1}(0))] \) is a weakly forward invariant subset. In this case, \( q(t_0,k_0) \in D \cap G(D) \). The solution \( q \) experiences a finite number of \( m \) jumps until all clocks which are equal to their lower bound are reset. After the jumps \( q(t_0,k_0 + m) \in C \setminus D \). This implies that \( q(t_0,k_0 + m) \in u_c^{-1}(0) \), since it would be otherwise no longer in the set \( \mathcal{Y} \). This contradicts the original claim, and we can conclude that any maximal solution in approaches the largest weakly invariant subset of \( u_c^{-1}(0) \). Then, in view of (8.27), any maximal solution to \( H_1 \) approaches the set

\[
\{ \theta, \dot{\theta}, \phi : \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}^T Z_1 \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = 0 \}. \tag{8.29}
\]

Since, according to Lemma 8.2.7, \( \ker(Z_1) = \text{Im}(I_{2n}) \) and since

\[
\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \in \text{Im}(I_{2n}), \tag{8.30}
\]

we conclude that \( \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \) approaches the set where

\[
\{ \theta, \dot{\theta}, \phi : \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + c I_{2n} \}, \tag{8.31}
\]

where \( c \in \mathbb{R} \) is some scalar. Since we also have that

\[
I_{2n}^T \dot{\theta}(t), \tag{8.32}
\]

is a conserved quantity, we necessarily have that \( c = 0 \).


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8.2.4 Minimum broadcasting frequency

The choice of \(a_i, b_i\) and the clock dynamics

\[
\dot{\phi}_i = -2 \deg(i) \left( \frac{1}{4} + \phi_i + \phi_i^2 \right) - \epsilon_i,
\]

(8.33)
determine the broadcasting frequency of node \(i \in V\). To determine the required broadcasting frequency, we study the clock dynamics given by

\[
\dot{\phi}_i = -2 \deg(i) \left( \frac{1}{4} + \phi_i + \phi_i^2 \right),
\]

(8.34)
that provides the upper bound on the maximum inter sampling time for a given \(a_i\) and \(b_i\). Now we turn our attention to the question how much time \(T_i\) elapses between a clock reset \(\phi_i(t) = b_i\) until the following reset at \(\phi_i(t + T_i) = a_i\).

**Theorem 8.2.9 (Inter broadcasting time).** The inter broadcasting time \(T_i\), induced by dynamics (8.34), is given by

\[
T_i = \frac{b_i - a_i}{\deg(i)(2a_i b_i + a_i + b_i + \frac{1}{2})}.
\]

(8.35)

**Proof.** We solve

\[
\dot{\phi}_i = -2 \deg(i) \left( \frac{1}{4} + \phi_i + \phi_i^2 \right),
\]

satisfying the boundary condition \(\phi_i(0) = b_i\). It can be readily confirmed that the solution is

\[
\phi_i(t) = \frac{b_i(1 - \deg(i)t - \frac{1}{2}\deg(i)t^2)}{\deg(i)(2b + 1)t + 1}.
\]

(8.36)
Solving (8.74) with boundary condition \(\phi_i(T_i) = a_i\) then yields

\[
T_i = \frac{b_i - a_i}{\deg(i)(2a_i b_i + a_i + b_i + \frac{1}{2})}.
\]

(8.37)

We define the upper bound on the allowed time between to broadcasting instances \(T_{i}^{\text{max}}\), as

\[
T_{i}^{\text{max}} = \lim_{b_i \to \infty, a_i \to 0} T_i(a_i, b_i).
\]

From Theorem 8.2.9 the following result is immediate:

**Corollary 8.2.10 (Minimum broadcasting frequency).** The maximum allowed time between two sampling instances satisfies

\[
T_{i}^{\text{max}} < \frac{1}{\deg(i)},
\]

such that the broadcasting frequency of node \(i \in V\) needs to be faster than the (weighted) degree of node \(i\).

The implication of this section is that despite the nonlinear clock dynamics \(\dot{\phi}_i = -2 \deg(i) \left( \frac{1}{4} + \phi_i + \phi_i^2 \right) - \epsilon_i\), we can rely on a simple counter to determine the broadcasting instances.


8. Distributed control with discrete communication

8.3 Coordination of distributed dynamical systems

In this section we investigate the broadcasting implementation of the consensus algorithm for (optimal) coordination of distributed dynamical systems. Consider a network of dynamical systems defined on a connected, undirected graph \( G = (V, \mathcal{E}) \). Each node represents a nonlinear system

\[
\begin{align*}
\dot{x}_i &= f_i(x_i, u_i, \theta_i, d_i) \\
y_i &= h_i(x_i),
\end{align*}
\]

where \( x_i \in \mathbb{R}^r \) is the state, \( u_i \in \mathbb{R} \) is the input from other nodes in the system and \( y_i \in \mathbb{R} \) is the output. The unknown constant disturbance at node \( i \) is \( d_i \in \mathbb{R} \), whereas \( \theta_i \) is a controllable external input to node \( i \). Compactly, the dynamics at the nodes are represented by

\[
\begin{align*}
\dot{x} &= f(x, u, \theta, d) \\
y &= h(x).
\end{align*}
\]

The nodes in the network are interconnected by edges, where edge \( k \) is described by

\[
\begin{align*}
\dot{\xi}_k &= F_k(\xi_k, v_k) \\
\lambda_k &= H_k(\xi_k, v_k),
\end{align*}
\]

leading to a compact notation for all edges as

\[
\begin{align*}
\dot{\xi} &= F(\xi, v) \\
\lambda &= H(\xi, v).
\end{align*}
\]

The interconnection structure is described by the incidence matrix \( B \), and

\[
\begin{align*}
u &= B \lambda \\
v &= -B^T y,
\end{align*}
\]

such that the overall system is given by

\[
\begin{align*}
\dot{x} &= f(x, BH(\xi, B^T h(x)), \theta, d) \\
\dot{\xi} &= F(\xi, B^T h(x)).
\end{align*}
\]

In the same spirit as the previous chapters, we aim at output regulation, by an optimal allocation of the inputs \( \theta_i, i \in V \) to the network. To do so, we make the following assumption, ensuring the existence of a steady state of (8.43):
Assumption 8.3.1 (Existence of steady state with input sharing). For a given $d$, there exists a constant $(\bar{x}, \bar{\xi})$ and a constant $\bar{\theta} \in \text{Im}(\mathbb{I}_n)$ satisfying,

$$0 = f(\bar{x}, BH(\bar{\xi}, B^T h(\bar{x})), \bar{\theta}, d)$$

$$0 = F(\bar{\xi}, B^T h(\bar{x})).$$

(8.44)

Note that we have taken $\bar{\theta} \in \text{Im}(\mathbb{I}_n)$ in the assumption above. This is instead of explicitly posing an optimization problem that prescribes the optimal input $\bar{\theta}$, that generally depends on the particular network dynamics. We consider the following control objective:

Objective 8.3.2 (Output regulation and input sharing). The output of system (8.43) satisfies

$$\lim_{t \to \infty} y(t) = \bar{y} = h(\bar{x}),$$

(8.45)

while the input to the system satisfies

$$\lim_{t \to \infty} \theta(t) \in \text{Im}(\mathbb{I}_n).$$

(8.46)

Furthermore, we assume that the physical network enjoys some useful passivity property that we have established e.g. for several power network models in the previous chapters.

Assumption 8.3.3 (Output strictly incremental passivity). There exists a radially unbounded incremental storage function $V_2(x, x, \xi, \bar{\xi})$ that satisfies

$$V_2 = -(h(x) - h(\bar{x}))^T Y(h(x) - h(\bar{x})) + (h(x) - h(\bar{x}))^T K(\bar{\theta} - \bar{\theta}),$$

(8.47)

along the solutions to (8.43), with constant $\bar{\theta} \in \text{Im}(\mathbb{I}_n)$, $Y \in \mathbb{R}^{n \times n}$ positive definite and diagonal matrix and $K \in \mathbb{R}^{n \times n}$ is a diagonal matrix. I.e, system (8.43) is output strictly incrementally passive with respect to the steady state (8.44).

Consider now the controller

$$\dot{\bar{\theta}} = -L\bar{\theta} - K^T (h(x) - h(\bar{x})).$$

(8.48)

The following lemma shows that controller (8.48) achieves Objective 8.3.2.

Lemma 8.3.4 (Achieving Objective 8.3.2 by a continuous consensus protocol). Let Assumption 8.3.1 and Assumption 8.3.3 hold. Controller (8.48) achieves Objective 8.3.2 for system (8.43).
Proof. The incremental storage function
\[
\frac{1}{2}(\theta - \bar{\theta})^T (\theta - \bar{\theta}) + V_2(x, \xi, \bar{\xi}),
\]
(8.49)
is radially unbounded and satisfies
\[
\dot{\Theta} + \dot{V}_2 = -(h(x) - h(\bar{x}))^T Y(h(x) - h(\bar{x})) - \theta^T L \theta,
\]
(8.50)
along the solutions to (8.43), (8.48). By LaSalle’s invariance principle, the solutions to the closed loop system (8.43), (8.48) approach the largest invariant set where \( h(x) = h(\bar{x}) \) and \( \theta \in \text{Im}(1) \). ■

Similar as in Sections 8.1 and 8.2, we consider the following broadcasting implementation of (8.48)
\[
\dot{\theta} = -L \hat{\theta} - K^T (h(x) - h(\bar{x})),
\]
(8.51)
and define a clock variable \( \phi \) having the following dynamics
\[
\dot{\phi} = \alpha + \beta,
\]
(8.52)
where
\[
\alpha_i = -2\text{deg}(i)(\frac{1}{4} + \phi_i + \phi_i^2) - \epsilon_i \quad \forall i \in \mathcal{V},
\]
(8.53)
and
\[
\beta_i = -\frac{4(K_i \phi_i)^2}{Y_i} - \epsilon_{2i} \quad \forall i \in \mathcal{V}.
\]
(8.54)
Here, \( \epsilon_i \) and \( \epsilon_{2i} \) are again two arbitrarily small scalars. Accordingly, we study, similarly to the previous section, the following hybrid system with flow set \( C := \{(x, \xi, \theta, \hat{\theta}, \phi) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \times [a, b]\} \). The flow map \( F(x, \xi, \theta, \hat{\theta}, \phi) \) is
\[
\begin{align*}
\dot{x} &= f(x, BH(\xi, B^T h(x)), \theta, d) \\
\dot{\xi} &= F(\xi, B^T h(x)) \\
\dot{\theta} &= -L \hat{\theta} - K^T (h(x) - h(\bar{x})) \\
\dot{\hat{\theta}} &= 0 \\
\dot{\phi} &= \alpha + \beta
\end{align*}
\]
(8.55)
where $\alpha_i$ and $\beta_i$ are given by (8.34) and (8.54), respectively. The jump set is $D := \{(x, \xi, \theta, \hat{\theta}, \phi) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \times [a, b] : \exists i \in \{1, \ldots, n\} \text{ s.t. } \phi_i = a_i\}$. The jump map $G(x, \xi, \theta, \hat{\theta}, \phi)$ is defined by

$$G(x, \xi, \theta, \hat{\theta}, \phi) := \{G_i(x, \xi, \theta, \hat{\theta}, \phi) : i \in \{1, \ldots, n\} \text{ and } \phi_i = a_i\}, \quad (8.56)$$

with

$$\begin{align*}
x^+ &= x \\
\xi^+ &= \xi \\
\theta^+ &= \theta \\
\hat{\theta}_i^+ &= \hat{\theta}_i \\
\hat{\theta}_j^+ &= \hat{\theta}_j \quad j \neq i \\
\phi_i^+ &= \phi_i \\
\phi_j^+ &= \phi_j \quad j \neq i
\end{align*} := G_i(x, \xi, \theta, \hat{\theta}, \phi). \quad (8.57)$$

The hybrid system with the data above will be represented by the notation $H = (C, F, D, G)$ or, briefly, by $H$. Before analyzing the asymptotical behaviour of $H$, we establish the following useful lemma:

**Lemma 8.3.5 (Evolution of $V = V_1 + V_2$).** The storage function $V = V_1 + V_2$, with $V_1$ as (8.12) satisfies

$$\dot{V} = \begin{bmatrix} \theta^T \\ \hat{\theta}^T \end{bmatrix} Z_1 \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix} + \begin{bmatrix} \theta - \hat{\theta} \\ \frac{h(x) - h(\Gamma)}{h(x) - h(\Gamma)} \end{bmatrix} Z_2 \begin{bmatrix} \theta - \hat{\theta} \\ \frac{h(x) - h(\Gamma)}{h(x) - h(\Gamma)} \end{bmatrix} \leq 0 \quad (x, \xi, \theta, \hat{\theta}, \phi) \in C$$

$$V(x^+, \xi^+, \theta^+, \hat{\theta}^+, \phi^+) \leq V(x, \xi, \theta, \hat{\theta}, \phi) \quad (x, \xi, \theta, \hat{\theta}, \phi) \in D, \quad (8.58)$$

along the solutions to $H$, where $Z_1$ and $Z_2$ are given by

$$Z_1 = \begin{bmatrix} [\alpha] & -[\alpha] - \frac{1}{2}(I + 2[\phi]\mathcal{L})^T \\
-\frac{1}{2}(I + 2[\phi]\mathcal{L}) & [\alpha] + [\phi]\mathcal{L} + \mathcal{L}[\phi] \end{bmatrix}, \quad (8.59)$$

$$Z_2 = \begin{bmatrix} [\beta] & [\phi]K + K^T[\phi] \\
[\phi]K + K^T[\phi] & -Y \end{bmatrix}. \quad (8.60)$$

**Proof.** First, evaluating 8.12 along the flows of system $H$ yields
\[ V_1 = -\theta^T \mathcal{L} \hat{\theta} - (\theta - \bar{\theta})^T K(h(x) - h(\bar{x})) \\
+ \phi^T[\theta - \bar{\theta}] - L \hat{\theta} - K(h(x) - h(\bar{x})) \\
= -\theta^T \mathcal{L} \hat{\theta} + (\alpha + \beta)^T[\theta - \bar{\theta}] - L \hat{\theta} + K(h(x) - h(\bar{x})) \\
= \left[ \begin{array}{c} \theta \\ \hat{\theta} \end{array} \right]^T Z_1 \left[ \begin{array}{c} \theta \\ \hat{\theta} \end{array} \right] - (\theta - \bar{\theta})^T K(h(x) - h(\bar{x})) \\
- 2\phi^T[\theta - \bar{\theta}(-K(h(x) - h(\bar{x}))) + \beta^T[\theta - \bar{\theta}](\theta - \hat{\theta}), \tag{8.61} \right. \\
\]

where the matrix \( Z_1 \) is identical to (8.15). Bearing in mind that
\[ \dot{V}_2 = -(h(x) - h(\bar{x}))^T Y(h(x) - h(\bar{x})) + (h(x) - h(\bar{x}))^T K(\theta - \bar{\theta}), \tag{8.62} \]
the incremental storage function \( V \) satisfies along the flows to system \( H \)
\[ \dot{V} = \left[ \begin{array}{c} \theta \\ \hat{\theta} \end{array} \right]^T Z_1 \left[ \begin{array}{c} \theta \\ \hat{\theta} \end{array} \right] + \left[ \begin{array}{c} \theta - \hat{\theta} \\ h(x) - h(\bar{x}) \end{array} \right]^T Z_2 \left[ \begin{array}{c} \theta - \hat{\theta} \\ h(x) - h(\bar{x}) \end{array} \right]. \tag{8.63} \]

Furthermore, \( Z_2 < 0 \) if for all \( i \in V \)
\[ \beta_i < -\frac{4(K_i \phi_i)^2}{Y_i}. \tag{8.64} \]

Second, \( V(x^+, \xi^+, \theta^+, \hat{\theta}^+, \phi^+) \leq V(x, \xi, \theta, \hat{\theta}, \phi) \), since \( \theta_i = \hat{\theta}_i \) whenever the value of \( \phi_i \) is increased during a jump.

The following two lemmas can be established in the same way as Lemma 8.2.3 and Lemma 8.2.6 in the previous section and we omit the details.

**Lemma 8.3.6 (Nominally well posed).** The system \( H \) is nominally well posed.

**Lemma 8.3.7 (Precompactness of solutions).** Every maximal solution to system \( H \) is precompact.

We are now ready to prove the main result of this section.

**Theorem 8.3.8 (Achieving output regulation and input sharing).** The maximal solutions of system \( H \) approach the set where \( h(x) = y = \bar{y} \) and \( \theta = \bar{\theta} \in \text{Im}(1_n) \).

**Proof.** According to Lemma 8.3.5
\[ \dot{V}(x, \xi, \theta, \hat{\theta}, \phi) \leq u_c(x, \xi, \theta, \hat{\theta}, \phi) \quad \forall (x, \xi, \theta, \hat{\theta}, \phi) \in C \tag{8.65} \]
\[ V(x^+, \xi^+, \theta^+, \hat{\theta}^+, \phi^+) - V(x, \xi, \theta, \hat{\theta}, \phi) \leq u_d(x, \xi, \theta, \hat{\theta}, \phi) \quad \forall (x, \xi, \theta, \hat{\theta}, \phi) \in D. \tag{8.66} \]
where

\[ u_c(x, \xi, \theta, \hat{\theta}, \phi) = \begin{cases} 
\begin{bmatrix} \theta \\ \hat{\theta} \\ -\infty \end{bmatrix}^T Z_1 \begin{bmatrix} \theta \\ \hat{\theta} \\ h(x) - h(\pi) \end{bmatrix}^T + \begin{bmatrix} \theta - \hat{\theta} \\ h(x) - h(\pi) \end{bmatrix}^T Z_2 \begin{bmatrix} \theta - \hat{\theta} \\ h(x) - h(\pi) \end{bmatrix} & (x, \xi, \theta, \hat{\theta}, \phi) \in C \\
0 & \text{otherwise}
\end{cases} \] (8.67)

\[ u_d(x, \xi, \theta, \hat{\theta}, \phi) = \begin{cases} 
0 & (x, \xi, \theta, \hat{\theta}, \phi) \in D \\
-\infty & \text{otherwise.}
\end{cases} \] (8.68)

In view of Lemmas 8.3.5, 8.3.6, 8.3.7, and (Goebel et al. 2012, Theorem 8.2), the maximal solutions to system \( \mathcal{H} \) approach the largest weakly invariant subset of

\[ \Upsilon = V^{-1}(r) \cap \mathcal{X} \cap \left[ u_c^{-1}(0) \cup (u_d^{-1}(0) \cap G(u_d^{-1}(0))) \right], \] (8.69)

where \( r \in V(\mathcal{X}) \) and \( \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [a, b] \). Similarly as in the proof of Theorem 8.2.8, we can argue that any maximal solution approaches the set \( u_c^{-1}(0) \).

In view of (8.67), this subset is given by

\[ \{ x, \xi, \theta, \hat{\theta}, \phi : \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix}^T Z_1 \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix} + \begin{bmatrix} \theta - \hat{\theta} \\ h(x) - h(\pi) \end{bmatrix}^T Z_2 \begin{bmatrix} \theta - \hat{\theta} \\ h(x) - h(\pi) \end{bmatrix} = 0 \}. \] (8.70)

Since \( Z_1 \leq 0 \) with \( \text{Ker}(Z_1) = \text{Im}(I_n) \) and since \( Z_2 < 0 \), we conclude that the maximal solutions to system \( \mathcal{H} \) approach the set where \( h(x) = h(\pi) = \bar{y} \) and \( \theta = \hat{\theta} = \bar{\theta} \in \text{Im}(I_n) \).

### 8.3.1 Inter broadcasting time

This section follows Section 8.2.4, where we consider here the clock dynamics

\[ \dot{\phi}_i = -2\deg(i)\left( \frac{1}{4} + \phi_i + \phi_i^2 \right) - \frac{4K_i^2 \phi_i^2}{Y_i} - \epsilon_i - \epsilon_{2i}. \] (8.71)

Again, \( a_i, b_i \) and (8.34) determine the broadcasting frequency of node \( i \). To determine the required broadcasting frequency, we study the clock dynamics given by

\[ \dot{\phi}_i = -2\deg(i)\left( \frac{1}{4} + \phi_i + \phi_i^2 \right) - \frac{4K_i^2 \phi_i^2}{Y_i}, \] (8.72)

that provides the upper bound on maximum inter sampling time for a given \( a_i \) and \( b_i \). Now we turn our attention to the question how much time \( T_i \) elapses between a clock reset \( \phi_i(t) = b_i \) until the following reset at \( \phi_i(t + T_i) = a_i \).
Theorem 8.3.9 (Inter broadcasting time). The inter broadcasting time $T_i$, induced by dynamics (8.34), is given by

$$T_i = \frac{2}{\sqrt{c_{1i}c_{2i}}} \left( \arctan \left( \frac{2b_i(c_{1i} + c_{2i}) + c_{1i}}{\sqrt{c_{1i}c_{2i}}} \right) - \arctan \left( \frac{2a_i(c_{1i} + c_{2i}) + c_{1i}}{\sqrt{c_{1i}c_{2i}}} \right) \right)$$

(8.73)

Proof. We solve $\dot{\phi}_i = -2\text{deg}(i)(\frac{1}{4} + \phi_i + \phi_i^2) - \frac{4K_i^2\phi_i^2}{Y_i}$, satisfying the boundary condition $\phi_i(0) = b_i$. It can be readily confirmed that the solution is

$$\phi_i(t) = -\frac{0.5\sqrt{c_{1i}c_{2i}} \tan \left( \frac{1}{4} \sqrt{c_{1i}c_{2i}} t - \arctan \left( \frac{2b_i(c_{1i} + c_{2i}) + c_{1i}}{\sqrt{c_{1i}c_{2i}}} \right) \right)}{c_{1i} + c_{2i}}$$

(8.74)

with

$$c_{1i} = 2\text{deg}(i)$$
$$c_{2i} = \frac{4K_i^2}{Y_i}$$

(8.75)

Solving (8.74) with boundary condition $\phi_i(T_i) = a_i$ then yields

$$T_i = \frac{2}{\sqrt{c_{1i}c_{2i}}} \left( \arctan \left( \frac{2b_i(c_{1i} + c_{2i}) + c_{1i}}{\sqrt{c_{1i}c_{2i}}} \right) - \arctan \left( \frac{2a_i(c_{1i} + c_{2i}) + c_{1i}}{\sqrt{c_{1i}c_{2i}}} \right) \right).$$

(8.76)

We define the upper bound on the allowed time between to broadcasting instances $T_i^{\text{max}}$, as $T_i^{\text{max}} = \lim_{b_i \to \infty, a_i \to 0} T_i(a_i, b_i)$. From Theorem 8.2.9 the following result is immediate:

Corollary 8.3.10 (Inter broadcasting time). The maximum allowed time between two sampling instances satisfies

$$T_i^{\text{max}} < \frac{2}{\sqrt{c_{1i}c_{2i}}} \left( \frac{\pi}{2} - \arctan \left( \frac{\sqrt{c_{1i}}}{\sqrt{c_{2i}}} \right) \right)$$

(8.77)

A further analysis shows also that $T_i^{\text{max}}$ given above reduces to the expression in Corollary 8.2.10, if one takes $c_{2i} \to 0$, as one would expect.
We perform two case studies to illustrate the results obtained in this chapter. First we study the consensus algorithm in Section 8.2, whereafter we apply the results of Section 8.3 to the power network.

**8.4.1 Average preserving consensus**

In this case study we perform a simulation of the hybrid system (8.9)–(8.11). Consider a cycle graph of 4 nodes, where the flow dynamics are given by \( \dot{\theta} = -L \hat{\theta} \), with \( L \) the (unweighed) Laplacian matrix. The broadcasting instances are determined by the clock dynamics

\[
\dot{\phi}_i = -2 \deg(i) \left( \frac{1}{4} + \phi_i + \phi_i^2 \right),
\]

where for all nodes \( i \in \{1, 2, 3, 4\} \) the lower and upper bound are \( a_i = 0 \) and \( b_i = 50 \), respectively. Note that in comparison with (8.6), we have omitted \( \epsilon_i \), as this is implicitly incorporated by taking \( b_i < \infty \).

Initially, the state variables satisfy

\[
\theta(0) = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix}, \quad \hat{\theta}(0) = \begin{bmatrix} 5 \\ 1 \\ -10 \\ 4 \end{bmatrix}, \quad \phi(0) = \begin{bmatrix} 1 \\ 10 \\ 20 \\ 40 \end{bmatrix}.
\]

The evolution of the system is given in Figure 8.1, from where we notice that all states \( \theta_i, i \in \{1, 2, 3, 4\} \) approach the average of the elements of \( \theta(0) \).

**8.4.2 Optimal Load Frequency Control**

This subsection we perform a numerical study on the same power network as in Chapter 2, where we assume for the sake of exposition that the voltages are constant. We incorporate the results from this chapter to obtain discrete time communication among the controllers and we formulate the resulting hybrid system as follows:

Consider the flow set \( C := \{ (\eta, \omega, \theta, \hat{\theta}, \phi) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [1^{-10}, 2^n] \} \). Here, we have chosen \( a_i = 1^{-10}, b_i = 2 \) for all \( i \in \{1, \ldots, 4\} \). The flow map \( F(\eta, \omega, \theta, \hat{\theta}, \phi) \) is

\[
\begin{align*}
\dot{\eta} &= B^T \omega \\
M \dot{\omega} &= Q^{-1} \theta - B \Gamma \sin(\eta) - D \omega - P_d \\
\dot{\theta} &= -L \hat{\theta} - Q^{-1} \omega \\
\dot{\hat{\theta}} &= 0 \\
\dot{\phi} &= -2 \deg \cdot (\frac{1}{4} \mathbb{I}_n + \phi + [\phi] \phi) - 4 D^{-1} Q^{-2} [\phi] \phi
\end{align*}
\]

where \( \mathbb{I}_n \) is the identity matrix.
where $\text{Deg} = \text{diag}(\text{deg}(1), \ldots, \text{deg}(4))$, $\mathcal{B}$ is the incidence matrix reflecting the topology of the power network and $\Gamma = \text{diag}(\Gamma_1, \ldots, \Gamma_k)$ describes the line characteristics, with $\Gamma_k = B_{ij} V_i V_j$ for line $k$ connecting areas $i$ and $j$. The jump set is $D := \{(\eta, \omega, \theta, \hat{\theta}, \phi) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \times [1^{-10}, 2]^n : \exists i \in \{1, \ldots, n\} \text{ s.t. } \phi_i = 1^{-10}\}$. The jump map $G(\eta, \omega, \theta, \hat{\theta}, \phi)$ is defined by

$$G(\eta, \omega, \theta, \hat{\theta}, \phi) := \{G_i(\eta, \omega, \theta, \hat{\theta}, \phi) : i \in \{1, \ldots, n\} \text{ and } \phi_i = 1^{-10}\},$$

(8.81)
with

\[
\begin{align*}
\eta^+ &= \eta \\
\omega^+ &= \omega \\
\theta^+ &= \theta \\
\dot{\theta}_i^+ &= \dot{\theta}_i \\
\dot{\theta}_j^+ &= \dot{\theta}_j \quad j \neq i \\
\phi_i^+ &= 0 \\
\phi_j^+ &= \phi_j \quad j \neq i \\
\end{align*}
\]

\[G_i(\eta, \omega, \theta, \dot{\theta}, \phi). \tag{8.82}\]

The stability analysis is omitted here, but follows essentially the proof of Theorem 8.3.8, under the assumption that steady state differences in voltage angles satisfy \(\eta_k \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\), for all \(k \in \mathcal{E}\), and taking

\[V_2 = \frac{1}{2} \omega^T M \omega - \mathbb{1}_{\mathcal{M}}^T \Gamma \cos(\eta) + \mathbb{1}_{\mathcal{M}}^T \Gamma \cos(\bar{\eta}) - (\Gamma \sin(\eta))^T (\eta - \bar{\eta}). \tag{8.83}\]

We illustrate the performance of the controllers on a connected four area network introduced in Section 2.6. The network topology is shown in Figure 8.2. For simplicity, we assume here that the voltages are constant. An overview of the numerical values of the relevant parameters is provided in Table 8.1 below. The communication among the controllers is depicted in Figure 8.2 as well and differs from the topology of the power grid. The system is initially at steady state with a constant load \(P_d(t) = (2.00, 1.00, 1.50, 1.00)^T, t \in [0, 30]\) and according to their cost functions generators take a different share in the power generation such that the total costs are

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig8.2}
\caption{A four area equivalent network of the power grid, where \(B_{ij}\) denotes the susceptance of the transmission line connecting two areas. The dashed lines represent the communication links.}
\end{figure}
minimized. $P_d(t) = (2.20, 1.05, 1.55, 1.10)^T$, $t \geq 30$. The frequency response to the control input is given in Figure 8.3. From Figure 8.3 we can see how the frequency drops due to the increased load. Furthermore we note that the controller regulates the power generation such that a new steady state condition is obtained where the frequency deviation is again zero and costs are minimized. Since we have chosen $Q_i = 1$ for all $i \in \{1, \ldots 4\}$, we have indeed an identical generation at all nodes at steady state.

---

**Table 8.1:** An overview of the numerical values used in the simulations.

<table>
<thead>
<tr>
<th></th>
<th>Area 1</th>
<th>Area 2</th>
<th>Area 3</th>
<th>Area 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_i$</td>
<td>5.22</td>
<td>3.98</td>
<td>4.49</td>
<td>4.22</td>
</tr>
<tr>
<td>$D_i$</td>
<td>1.60</td>
<td>1.22</td>
<td>1.38</td>
<td>1.42</td>
</tr>
<tr>
<td>$V_i$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$Q_i$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Figure 8.3: Time evolution of the frequency deviation $\omega$, clock $\phi$, control input $\theta$, and information exchange $\hat{\theta}$. 