Chapter 3
Active power sharing in microgrids

Abstract

This chapter studies the problem of frequency regulation and active power sharing in inverter-based microgrids with time-varying voltages. Building upon the result in the previous chapter, we propose the design of internal-model-based controllers and analyze it within an (incrementally) passivity framework. A small case study indicates the effectiveness of the proposed solution.

3.1 A microgrid model

Consider a microgrid consisting of $n$ generation and $m$ load buses. Every generation bus represents an inverter, connected to a DG unit and a battery. The loads are modelled as constant impedances and corresponding load buses are eliminated, resulting in a Kron-reduced network. Consequently, the network is represented by a connected and undirected graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes (generation buses) and $\mathcal{E} = \{1, \ldots, m\}$ is the set of distribution lines connecting the nodes. The network structure can be represented by its corresponding incidence matrix $B \in \mathbb{R}^{n \times m}$. The ends of edge $k$ are arbitrary labeled with a ‘+’ and a ‘−’. Then

$$B_{ik} = \begin{cases} +1 & \text{if } i \text{ is the positive end of } k \\ -1 & \text{if } i \text{ is the negative end of } k \\ 0 & \text{otherwise.} \end{cases}$$

3.1.1 A network of inverters

The inverter at the node $i$ is assumed to be able to regulate its frequency instantaneous and its voltage in a delayed manner. The inverter is therefore represented
by

\[ \delta_i = u_i^\delta, \]
\[ \tau V_i \dot{V}_i = -V_i + u_i V_i, \]  
where \( \delta_i \) and \( V_i \) are the voltage angle and voltage respectively, and \( \tau V_i \in \mathbb{R}_{>0} \) is the time constant of the filter. The considered controllers are given by

\[ u_i^\delta = \omega_d - k_{P_i} (P^m_i - P^d_i - p_i), \]
\[ u_i V_i = V_d^i - k_{Q_i} (Q^m_i - Q^d_i - q_i), \]  
where \( \omega_d, V_d^i, P^d, Q^d \in \mathbb{R}^n \) are respectively the desired frequency, desired voltage, active power setpoint and reactive power setpoint, and are determined according to economic and technical criteria. The constants \( k_{P_i} \in \mathbb{R}_{>0} \) and \( k_{Q_i} \in \mathbb{R}_{>0} \) are the frequency and voltage droop gains respectively. \( p_i \) and \( q_i \) are additional control inputs which we will design later to achieve frequency and voltage regulation.

**Remark 3.1.1** (Inverter model and controller). The inverter model (3.1) and controller (3.2) are identical to the ones in (Schiffer et al. 2013), when we set \( p_i = 0 \) and \( q_i = 0 \). We refer the reader to (Schiffer et al. 2013) and (Schiffer et al. 2015) for a more in depth treatment of physical interpretation and underlying assumptions.

The measured active and reactive powers (\( P^m_i \) and \( Q^m_i \)) are obtained through filters with time constants \( \tau P_i \) and \( \tau Q_i \):

\[ \tau P_i \dot{P}^m_i = -P^m_i + P_i \]
\[ \tau Q_i \dot{Q}^m_i = -Q^m_i + Q_i, \]  
where the power flows are given by

\[ P_i = G_{ii} V_i^2 - \sum_{j \in N_i} (B_{ij} V_i V_j \sin(\delta_i - \delta_j) + G_{ij} V_i V_j \cos(\delta_i - \delta_j)) \]
\[ Q_i = -B_{ii} V_i^2 - \sum_{j \in N_i} (G_{ij} V_i V_j \sin(\delta_i - \delta_j) - B_{ij} V_i V_j \cos(\delta_i - \delta_j)), \]  
where \( B_{ij} \) and \( G_{ij} \) denote the susceptance and the conductance of the distribution line respectively. Additionally, we have \( B_{ii} = B_{ii} + \sum_{j \in N_i} B_{ij} \) and \( G_{ii} = G_{ii} + \sum_{j \in N_i} G_{ij} \), where \( B_{ii} \) and \( G_{ii} \) denote the shunt susceptance and shunt conductance. In the subsequent analysis we approximate the terms involving \( G \) by an unknown constant, i.e.

\[ P_i = \sum_{j \in N_i} |B_{ij}| V_i V_j \sin(\delta_i - \delta_j) + P_{0i}, \]
\[ Q_i = |B_{ii}| V_i^2 - \sum_{j \in N_i} |B_{ij}| V_i V_j \cos(\delta_i - \delta_j) + Q_{0i}. \]  

(3.5)
The approximation above (including a lossless network as a particular case by setting $P_0 = Q_0 = 0$) is common in power grid studies and is made in order to derive a suitable Lyapunov function for the stability analysis. Assuming $\tau_V \ll \tau_P \simeq \tau_Q$, we can simplify the used model by setting $\tau_V = 0$, resulting in

$$\dot{\delta}_i = \omega_i$$

$$\tau_P \dot{\omega}_i = -\omega_i + \omega_i^d - k_{P_i}(P_i - P_0^d) + u^P_i \quad (3.6)$$

$$\tau_Q \dot{V}_i = -V_i + V_i^d - k_{Q_i}(Q_i - Q_0^d) + u^Q_i,$$

with

$$u^P_i = -k_{P_i}(p_i + \tau_P \dot{p}_i)$$

$$u^Q_i = -k_{Q_i}(q_i + \tau_Q \dot{q}_i). \quad (3.7)$$

For all nodes the system writes as:

$$\dot{\delta} = \omega$$

$$\tau_P \dot{\omega} = -\omega + 1_n \omega^d - K_P(P - P_0^d) + u^P \quad (3.8)$$

$$\tau_Q \dot{V} = -V + V^d - K_Q(Q - Q_0^d) + u^Q,$$

where the used symbols follow straightforwardly from the node dynamics, and are vectors or matrices with appropriate dimensions. Since the dynamics are driven by the differences in the voltage angles we introduce $\eta = B^T \delta$ and write (3.8) as:

$$\dot{\eta} = B^T \omega,$$

$$\tau_P \dot{\omega} = -\omega + 1_n \omega^d - K_P(B\text{diag}(B_i^T V)\text{diag}(B^T V)B^c \sin(\eta) - P_0^d) + u^P,$$

$$\tau_Q \dot{V} = -V + V^d - K_Q(\text{diag}(V)B^a V

- |B|\text{diag}(B_i^T V)\text{diag}(B^T V)B^c \cos(\eta) - Q_0^d) + u^Q, \quad (3.9)$$

where $B$ is the corresponding incidence matrix. Its positive entries are given by $B_+$ and its negative entries by $B_-$, such that we have $B = B_+ + B_-$. Furthermore $|B| = B_+ - B_-$ denotes the incidence matrix with all elements positive. The susceptance of the transmission lines is denoted by $B^c = \text{diag}(B_k)$, where $B_k = |B_{ij}| = |B_{ji}|$ is the susceptance of transmission line $k$ connecting nodes $i$ and $j$. The shunt susceptance of the nodes is given by $B^a = \text{diag}(|B_{ii}|)$. To simplify the notation further we
introduce

\[ P^d_0 = P^d_0 - P_0 \]
\[ Q^d_0 = Q^d - Q_0 \]
\[ M_P = K_P^{-1} T_P \]
\[ M_Q = K_Q^{-1} T_Q \]
\[ \Theta(V) = \text{diag}(V) B^u V \]
\[ \Gamma(V) = \text{diag}(B^T V) \text{diag}(B^T V) B^e. \]

(3.10)

**Remark 3.1.2** (A new voltage dependent variable). Notice that \( \Gamma(V) \) is a diagonal matrix with \( \Gamma(V)_{kk} = |B_{ij}|V_i V_j = |B_{ji}|V_j V_i \) when edge \( k \) is incident to nodes \( i \) and \( j \). \( \Theta(V) \) is a vector with \( \Theta(V)_i = |B_{ii}| V_i^2 \).

Using the new variables (3.10), system (3.9) can be written as

\[ \dot{\eta} = B^T \omega \]
\[ M_P \dot{\omega} = -K_P^{-1} (\omega - \mathbb{1}_n \omega^d) + P^d_0 - B \Gamma(V) \sin(\eta) + u^P \]
\[ M_Q \dot{V} = -K_Q^{-1} (V - V^d) + Q^d_0 - \Theta(V) + |B| \Gamma(V) \cos(\eta) + u^Q, \]

(3.11)

which we will use throughout this chapter.

### 3.2 Stability with constant control inputs

First we investigate the stability of the equilibria of system (3.11) under the assumption \( u^P = \pi^P \) and \( u^Q = \pi^Q \) are constant. This study has been already pursued in e.g. (Schiffer et al. 2013) or in (Simpson-Porco et al. 2013) for a first order model. The approach in this work descends however from the results provided in Chapter 2 and can provide new insight into this problem. Furthermore it paves the way for allowing dynamic controlled \( u^P \) and \( u^Q \) in later sections, relying on (incremental) passivity and the internal model approach. Our analysis is based on characterizing the equilibria and providing conditions when those equilibria are locally attractive.

#### 3.2.1 Equilibria

Before characterizing the equilibria, we need the following assumption, which guarantees the existence of an equilibrium:
3.2. Stability with constant control inputs

**Assumption 3.2.1** (Feasibility). For a given \( \pi^P + P_0^d \) and \( \pi^Q + Q_0^d \), there exist \( \eta \in \text{Im}(B^T) \cap (\frac{\pi}{2}, \frac{3\pi}{2})^m, \pi \in \text{Ker}(B^T) \) and \( \overline{V} \in \mathbb{R}^n_{>0} \) such that

\[
0 = -K^{-1}_P (\overline{P} - \mathbf{1}_n \omega^d) - B \Gamma(\overline{V}) \sin(\eta) + P_0^d + \overline{\pi}^P
\]

\[
0 = -K^{-1}_Q (\overline{V} - V^d) - \Theta(\overline{V}) + |B| \Gamma(\overline{V}) \cos(\eta) + Q_0^d + \overline{\pi}^Q.
\]

(3.12)

Under this assumption the equilibria can be characterized as follows:

**Lemma 3.2.2** (Equilibria). The equilibria of (3.11) are given by

\[
\omega = \mathbf{1}_n \omega^s,
\]

(3.13)

where

\[
\omega^s = \omega^d + \frac{\mathbf{1}_n^T (P_0^d + \pi^P)}{\mathbf{1}_n^T K_1 K_P \mathbf{1}_n},
\]

(3.14)

\[
B \Gamma(\overline{V}) \sin(\eta) = \left( I_n - \frac{K_1 \mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T K_1 K_P \mathbf{1}_n} \right) (P_0^d + \pi^P),
\]

(3.15)

and \( \overline{V} \) any vector fulfilling Assumption 3.2.1.

**3.2.2 Local attractivity**

Having characterized the equilibria of system (3.11), we are now ready to investigate the stability properties of the steady state solution. Recall the underlying assumption that \( \overline{V} \in \mathbb{R}^n_{>0} \), and that therefore there exists a compact set around \( \overline{V} \) which only contains positive values of \( V \). To analyze the stability consider the incremental storage function:

\[
S(\omega, \overline{\omega}, \eta, \overline{\eta}, V, \overline{V})
\]

\[
= \frac{1}{2} (\omega - \overline{\omega})^T M_P (\omega - \overline{\omega})
\]

\[
- \mathbf{1}_m^T \Gamma(V) \cos(\eta) + \mathbf{1}_m^T \Gamma(\overline{V}) \cos(\overline{\eta}) - (\Gamma(\overline{V}) \sin(\overline{\eta}))^T (\eta - \overline{\eta})
\]

\[
+ \mathbf{1}_m^T K_Q^{-1} (V - \overline{V}) - (K_Q^{-1} V^d + Q_0^d + \overline{\pi}^Q)^T (\ln(V) - \ln(\overline{V}))
\]

\[
+ \frac{1}{2} \mathbf{1}_n^T (\Theta(V) - \Theta(\overline{V})).
\]

(3.16)

The former terms involving \( \eta \) and \( \omega \) descend from Chapter 2, whereas latter terms were used previously to analyze voltage stability (Schiffer et al. 2014). Notice that the storage function is not bounded, since unboundedness of \( \eta \) implies that the term \((\Gamma(\overline{V}) \sin(\overline{\eta}))^T \eta \) is unbounded as well. In order to invoke LaSalle’s invariance principle we therefore require condition 3.17, in the lemma below, to be satisfied.
Lemma 3.2.3 (Local minimum of (3.16)). The storage function (3.16) has, under Assumption 3.2.1, a local minimum at an equilibrium point \((\eta, \omega, V)\), if the following condition is met

\[
\left( \frac{\partial^2 S}{\partial V^2} - \frac{\partial^2 S}{\partial V \partial \eta} \frac{\partial^2 S}{\partial \eta \partial V} \right) \bigg|_{\eta=\eta, \omega=\omega, V=V} > 0,
\]

where

\[
\frac{\partial^2 S}{\partial \eta \partial V} = (|\mathcal{B}| \Gamma(V) \text{diag}(\sin(\eta)))^T \text{diag}(V)^{-1},
\]

\[
\frac{\partial^2 S}{\partial V \partial \eta} = \text{diag}(V)^{-1} (|\mathcal{B}| \Gamma(V) \text{diag}(\sin(\eta))),
\]

\[
\frac{\partial^2 S}{\partial^2 \eta} = \text{diag}(\Gamma(V) \cos(\eta)),
\]

\[
\left( \frac{\partial^2 S}{\partial^2 V} \right)_{ii} = \left( |B_{ii}| + \frac{K_Q V^d + Q_0^d + \pi_0^d}{V_i^2} \right),
\]

and \((\frac{\partial^2 S}{\partial^2 V})_{ij} = (-|B_{ij}| \cos \eta_k)\) for \(i \neq j\), where edge \(k\) is incident to nodes \(i\) and \(j\). When node \(i\) is not connected to node \(j\) we take \(|B_{ij}| = 0\).

Proof. First we consider the gradient of the storage function.

\[
\nabla S = \begin{bmatrix} \frac{\partial S}{\partial \eta}^T & \frac{\partial S}{\partial \omega}^T & \frac{\partial S}{\partial V}^T \end{bmatrix}^T = \begin{bmatrix} \Gamma(V) \sin(\eta) - \Gamma(V) \sin(\eta) \\ M_P (\omega - \overline{\omega}) \\ -V^T M_Q \text{diag}(V)^{-1} \end{bmatrix}
\]

Since \(\dot{V}|_{\eta=\eta, \omega=\omega, V=V} = 0\) it is immediate to see that we have \(\nabla S|_{\eta=\eta, \omega=\omega, V=V} = 0\). As the gradient of \(S\) is zero at an equilibrium point it is sufficient for \(S\) to have a local minimum when the Hessian is positive definite at an equilibrium. The Hessian is given by

\[
\nabla^2 S = \begin{bmatrix} \text{diag}(\Gamma(V) \cos(\eta)) & 0 & \frac{\partial^2 S}{\partial \eta \partial V} \\ 0 & M_P & 0 \\ \frac{\partial^2 S}{\partial V \partial \eta} & 0 & \frac{\partial^2 S}{\partial^2 V} \end{bmatrix},
\]

where \((\frac{\partial^2 S}{\partial V \partial \eta})^T = \frac{\partial^2 S}{\partial \eta \partial V} = (|\mathcal{B}| \Gamma(V) \text{diag}(\sin(\eta)))^T \text{diag}(V)^{-1}, \ (\frac{\partial^2 S}{\partial^2 V})_{ii} = |B_{ii}| + \frac{K_Q V^d + Q_0^d + \pi_0^d}{V_i^2}\) and \((\frac{\partial^2 S}{\partial^2 V})_{ij} = -|B_{ij}| \cos \eta_k\) for \(i \neq j\), where edge \(k\) is incident to nodes \(i\) and \(j\). Since \(M_P\) and \(\text{diag}(\Gamma(V) \cos(\eta))\) are positive definite matrices, it follows by invoking the Schur complement that \(\nabla^2 S|_{\eta=\eta, \omega=\omega, V=V} > 0\) if and only if

\[
\left( \frac{\partial^2 S}{\partial V^2} - \frac{\partial^2 S}{\partial V \partial \eta} \frac{\partial^2 S}{\partial \eta \partial V} \right) \bigg|_{\eta=\eta, \omega=\omega, V=V} > 0.
\]

\[\blacksquare\]
We are now ready to state the main result of this section, that is, any solution starting sufficiently close to an equilibrium where the storage function $S$ has a local minimum, asymptotically converges to an equilibrium fulfilling Assumption 3.2.1.

**Theorem 3.2.4** (Convergence to an equilibrium). Given system (3.11), if the equilibrium fulfills condition (3.17), then the system locally asymptotically converges to an equilibrium fulfilling Assumption 3.2.1.

**Proof.** Since the storage function has a local minimum at the equilibria we can invoke LaSalle’s invariance principle. For this we show that $\dot{S} \leq 0$, with the equality holding only at the equilibria. In fact,

$$\dot{S} = \frac{\partial S}{\partial \eta} \dot{\eta} + \frac{\partial S}{\partial \omega} \dot{\omega} + \frac{\partial S}{\partial V} \dot{V},$$

(3.22)

where

$$\frac{\partial S}{\partial \eta} \dot{\eta} = - (\omega - \overline{\omega})^T B (\Gamma(V) \sin(\eta) - \Gamma(V) \sin(\eta)),$$

$$\frac{\partial S}{\partial \omega} \dot{\omega} = (\omega - \overline{\omega})^T \left( -K_p^{-1} \omega + K_p^{-1} \overline{\omega} \right) + (\omega - \overline{\omega})^T B (\Gamma(V) \sin(\eta) - \Gamma(V) \sin(\eta)),$$

$$\frac{\partial S}{\partial V} \dot{V} = - \dot{V}^T M_Q \text{diag}(V)^{-1} \dot{V},$$

(3.23)

yielding

$$\dot{S} = -(\omega - \overline{\omega})^T K_p^{-1} (\omega - \overline{\omega}) - \dot{V}^T M_Q \text{diag}(V)^{-1} \dot{V}.$$  

(3.24)

Recalling that $V \in \mathbb{R}^n_{>0}$, we have that $\dot{S} \leq 0$ and therefore there exists a compact level set $\Upsilon$ around the equilibrium $(\overline{\eta}, \overline{\omega}, \overline{V})$, which is forward invariant. By LaSalle’s invariance principle the solution starting in $\Upsilon$ converges asymptotically to the largest invariant set contained in $\Upsilon \cap \{ (\eta, \omega, V) : \omega = \overline{\omega}, \dot{V} = 0 \}$. On such invariant set the system is

$$\dot{\eta} = B^T \overline{\omega},$$

$$\dot{0} = -K_p^{-1} (\overline{\omega} - 1_n \omega^d) - B \Gamma(V) \sin(\eta) + P_0^d + \overline{\pi}^p,$$

$$\dot{0} = -K_Q^{-1} (V - V^d) - \Theta(V) + |B| \Gamma(V) \cos(\eta) + Q_0^d + \overline{\pi}^Q.$$  

(3.25)

Recall that $\overline{\omega} = 1_n \overline{\omega}^*$, such that $\dot{\eta} = B^T 1_n \overline{\omega}^* = 0$. Since on the invariant set $\dot{\eta} = \dot{\omega} = \dot{V} = 0$, system (3.11) approaches the set of equilibria contained in $\Upsilon$. Consider a forward invariant set $\Omega \subseteq \Upsilon$ around $(\overline{\eta}, \overline{\omega}, \overline{V})$, where it holds that $\frac{\partial^2 S}{\partial \eta \partial \omega \partial V} > 0$. Since every equilibrium in $\Omega$ is Lyapunov stable, it then follows from Lemma 1.4.8 that the solution starting in $\Omega$ converges to a point. I.e., we can conclude that the system approaches the set where where $V = \tilde{V}$ and $\eta = \tilde{\eta}$ are constants. ■
3. Active power sharing in microgrids

3.3 Frequency regulation by dynamic control inputs

In the previous section we investigated the stability of system (3.11) with constant $u^P = \pi^P$. As shown in Lemma 3.2.2 this generally results in a steady state frequency $\omega$ deviating from the desired frequency $\omega^d$. In this section we adopt the approach pursued in Chapter 2, and based on incremental passivity and the internal model approach, we design a dynamic and distributed controller providing a control input $u^P(t)$ such that $\omega = \frac{1}{n} \omega^d$ is locally asymptotically stable, i.e. the frequency is stabilized at its desired value. Before proposing a stabilizing controller we first discuss possible values of the steady state control input.

3.3.1 Power sharing

From Lemma 3.2.2 it is immediate to see that any input $\pi^P$, satisfying $1_T^T (P_0^d + \pi^P) = 0$, implies that a stable equilibrium satisfies $\omega = \frac{1}{n} \omega^d$. Among the possible choices of $\pi^P$, we focus on the steady state input solving the following optimization problem:

The control input $u^P$ is a solution to the following optimization problem:

$$\min_{u^P, v} \frac{1}{2} (u^P)^T R u^P = \min_{u^P, v} \sum_{i \in V} \frac{1}{2} r_i(u^P_i)^2$$

s.t. $0 = -B \Gamma(V) v + P_0^d + u^P$, 

(3.26)

where we have set $\sin(\eta) = v$. The equality constraint in the optimization problem coincides with the frequency dynamics in (3.11) at steady state where $\omega = \frac{1}{n} \omega^d$. Following standard literature on convex optimization (see e.g. Boyd and Vandenberghe 2004)) we introduce the Lagrangian function

$$L(u^P, v, \lambda) = (u^P)^T R u^P + \lambda^T (-B \Gamma(V) v + P_0^d + u^P).$$

(3.27)

Assume that $R$ is a positive diagonal matrix and therefore $R$ is strictly convex. It follows that $L(u^P, v, \lambda)$ is convex in $(u^P, v)$ and concave in $\lambda$. Therefore there exists a saddle point solution to

$$\max_{\lambda} \min_{u^P, v} L(u^P, v, \lambda).$$

Applying first order optimality conditions, the saddle point $(\pi^P, \lambda, \bar{\lambda})$ must satisfy

$$R \pi^P + \bar{\lambda} = 0,$$

$$\Gamma(V) B^T \bar{\lambda} = 0,$$

$$-B \Gamma(V) \pi + P_0^d + \pi^P = 0.$$ 

(3.28)
Solving this set of equations for \( \pi^P \) it is straightforward to show that

\[
\pi^P = -R_{-1}^{-1} \frac{1}{n} P_{-1} \begin{pmatrix} P \end{pmatrix}_0 \begin{pmatrix} T \end{pmatrix}_n \begin{pmatrix} n \end{pmatrix}^{-1} K_{-1}^{-1} 1_n. \quad (3.29)
\]

**Remark 3.3.1** (Active power sharing). Active power sharing is an important aspect in microgrids. Some results on obtaining a desirable power sharing are provided e.g. in (Simpson-Porco et al. 2013) and (Schiffer et al. 2013). Note that in the present setting, the matrix \( R \) can be chosen to obtain an appropriate active power sharing. For instance the choice \( R = I_n \) results in \( \pi^P_i = \pi^P_j = \pi^P \ast \) for all \( i, j \), whereas \( R = K_{-1}^{-1} P \) results in \( \pi^P_i \) proportional to its droop gain.

### 3.3.2 Stability

Relying on results of the previous section and our previous work (Bürger et al. 2014), we can design a controller which makes the system (3.11) locally asymptotically converging to an equilibrium where \( \varpi = 1_1 \omega^d \), and thus provides frequency control, which is comparable to secondary control in the classic power grid. The stability result of the previous section assumed \( u^P = \pi^P \) and \( u^Q = \pi^Q \) are constants. If we let \( u^P \) be any control input, it is immediate to see from Lemma 3.2.4 that system (3.11) is incrementally passive from the input \( u = u^P \) to the output \( y = \omega \).

**Corollary 3.3.2** (Incremental passivity). Let condition (3.17) hold. Then for system (3.11) there exists a regular storage function \( S'(\eta, \bar{\eta}, \omega, \bar{\omega}) \) which satisfies the following incremental dissipation inequality

\[
\dot{S}' \leq - (y - \bar{y}) T K_{-1}^{-1} (y - \bar{y}) + (y - \bar{y}) T (u - \bar{u}), \quad (3.30)
\]

where \( u = u^P \) and \( y = \omega \).

As the main result of this section we propose a dynamic controller that converges asymptotically to the feedforward input (3.29) and guarantees an asymptotical convergence of the frequency \( \omega \) to its desired value \( 1_n \omega^d \).

**Theorem 3.3.3** (Frequency regulation and active power sharing). Given system (3.11), and assuming that condition (3.17) holds, the controllers at the nodes

\[
\begin{align*}
\dot{\theta}_i &= \sum_{j \in N^\text{comm}_i} (\theta_j - \theta_i) - r_i^{-1} (\omega_i - \omega^d), \\
\end{align*}
\]

\[
\begin{align*}
\frac{P}{T} &= r_i^{-1} \frac{P}{T_i}, \quad i \in V
\end{align*}
\]

where \( N^\text{comm}_i \) denotes the set of neighbors of node \( i \) in a graph describing the exchange of information among the controllers, guarantee locally the solutions to the closed-loop system
to converge asymptotically to the largest invariant set where \( \omega_i = \omega^d \) for all \( i \in V \), and \( \theta = \vec{\theta}, \vec{\theta} \) being the vector \[
\vec{\theta} = \begin{bmatrix} 1_n \end{bmatrix}_T P^d_0 \begin{bmatrix} 1_n \end{bmatrix}_R^{-1} \begin{bmatrix} 1_n \end{bmatrix}.
\] (3.32)
such that \( \pi^P = -R^{-1} \vec{\theta} \), is as in (3.29).

**Proof.** Bearing in mind Corollary 3.3.2 we have that system (3.11), with dynamic \( u^P \) and constant \( \pi^Q \) is incrementally passive from the input \( u^P \) to the output \( \omega \). The internal model principle design pursued in (De Persis 2013), (Bürger and De Persis 2013) and (Bürger and De Persis 2015) prescribes the design of a controller able to generate the feedforward input \( \pi^P \). To this purpose, we introduce the overall controller

\[
\dot{\theta} = -L^\text{com} \theta + \bar{H}^T v,
\]
\[
u^P = \bar{H} \theta,
\] (3.33)

where \( \theta \in \mathbb{R}^n \), \( L^\text{com} \) the Laplacian associated with a graph that describes the exchange of information among the controllers, and with the term \( \bar{H}^T v \) needed to guarantee the incremental passivity property of the controller (see (Bürger and De Persis 2013), (Bürger and De Persis 2015) for details). Here \( v \in \mathbb{R}^n \) is an extra control input to be designed later, while \( \bar{H} = \bar{H}^T = -R^{-1} \).

If \( v = 0 \) and \( \vec{\theta}(0) = \begin{bmatrix} 1_n \end{bmatrix}_T P^d_0 \begin{bmatrix} 1_n \end{bmatrix}_R^{-1} \begin{bmatrix} 1_n \end{bmatrix} \), then \( \vec{\theta}(t) := \vec{\theta}(0) \) satisfies the differential equation in (3.33) and moreover the corresponding output \( \bar{H} \vec{\theta}(t) \) is identically equal to the feedforward input \( \pi^P(t) \) defined in (3.29), provided that \( \bar{H} = -R^{-1} \). More explicitly, we have

\[
\dot{\vec{\theta}} = -L^\text{com} \vec{\theta},
\]
\[
\pi^P = -R^{-1} \vec{\theta}.
\] (3.34)

Consider now the incremental storage function

\[
\Phi(\theta, \vec{\theta}) = \frac{1}{2} (\theta - \vec{\theta})^T (\theta - \vec{\theta})
\]

It satisfies along the solutions to (3.33)

\[
\dot{\Phi}(\theta, \vec{\theta}) = (\theta - \vec{\theta})^T (-L^\text{com} \theta - R^{-1} v + L^\text{com} \vec{\theta})
\]
\[
\leq - (\theta - \vec{\theta})^T R^{-1} v = (u^P - \pi^P)^T v.
\]
(3.35)
We now interconnect system (3.11) and the controller (3.33), obtaining

\[
\begin{align*}
\dot{\eta} &= B^T \omega, \\
M_P \dot{\omega} &= -K_P^{-1}(\omega - \mathbb{I}_n \omega^d) - B(\Gamma(V) \sin(\eta)) + P_0^d - R^{-1} \theta, \\
M_Q \dot{V} &= -K_Q^{-1}(V - V^d) - \Theta(V) + |B| \Gamma(V) \cos(\eta) + Q_0^d + \pi_Q, \\
\dot{\theta} &= -\mathbf{L}^{com} \theta - R^{-1} v.
\end{align*}
\]  

(3.36)

Consider the incremental storage function

\[ Z(\omega, \mathbb{I}_n \omega^d, \eta, \eta, V, V, \theta, \theta) = S(\omega, \mathbb{I}_n \omega^d, \eta, \eta, V, V) + \Phi(\theta, \theta), \]

(3.37)

where \((\eta^T, \mathbb{I}_n \omega^d, V^T)^T\) fulfills Assumption 3.2.1. Following the argumentation of Lemma 3.2.3, it is immediate to see that under condition (3.17) we have that \(\nabla Z|_{\eta=\eta^d, \omega=\omega^d, V=V, \theta=\theta} = 0\) and \(\nabla^2 Z|_{\eta=\eta^d, \omega=\omega^d, V=V, \theta=\theta} > 0\), such that \(Z\) has a local minimum at its equilibrium. It turns out that

\[
\begin{align*}
\dot{Z} &= -(\omega - \mathbb{I}_n \omega^d)^T K_P^{-1}(\omega - \mathbb{I}_n \omega^d) \\
&\quad - \dot{V}^T M_Q \text{diag}(V)^{-1} \dot{V} \\
&\quad + (\omega - \mathbb{I}_n \omega^d)^T (u - \bar{u}) \\
&\quad - (\theta - \bar{\theta})^T \mathbf{L}^{com}(\theta - \bar{\theta}) + (u - \bar{u})^T v.
\end{align*}
\]

(3.38)

As we are still free in designing \(v\), the choice \(v = -(\omega - \mathbb{I}_n \omega^d)\) returns

\[
\begin{align*}
\dot{Z} &= -(\omega - \mathbb{I}_n \omega^d)^T K_P^{-1}(\omega - \mathbb{I}_n \omega^d) \\
&\quad - \dot{V}^T M_Q \text{diag}(V)^{-1} \dot{V} \\
&\quad - (\theta - \bar{\theta})^T \mathbf{L}^{com}(\theta - \bar{\theta}) \leq 0
\end{align*}
\]

(3.39)

As \(\dot{Z} \leq 0\), there exists a compact level set \(\Upsilon\) around the equilibrium \((\bar{\eta}, \mathbb{I}_n \omega^d, \nabla, \bar{\theta})\), which is forward invariant. By LaSalle’s invariance principle the solution starting in \(\Upsilon\) asymptotically converges to the largest invariant set contained in \(\Upsilon \cap \{(\eta, \omega, V, \theta) : \omega = \mathbb{I}_n \omega^d, \dot{V} = 0, \theta = \bar{\theta} + \mathbb{I}_n \alpha(t)\}\). On such invariant set the system is

\[
\begin{align*}
\dot{\eta} &= B^T \mathbb{I}_n \omega^d, \\
0 &= -B(\Gamma(V) \sin(\eta) - \Gamma(\nabla) \sin(\eta)) - R^{-1} \mathbb{I}_n \alpha(t), \\
0 &= -K_Q^{-1}(V - V^d) + Q_0^d - \Theta(V) + |B| \Gamma(V) \cos(\eta) + \pi_Q, \\
\dot{\theta} &= -\mathbf{L}^{com}(\theta - \bar{\theta} + \mathbb{I}_n \alpha(t)).
\end{align*}
\]

(3.40)

Premultiplying the second line in (3.40) by \(\mathbb{I}_n^T\) yields \(\mathbb{I}_n^T R^{-1} \mathbb{I}_n \alpha(t) = 0\). As \(R^{-1}\) is a positive definite diagonal matrix it follows that necessary \(\alpha(t) = 0\) and therefore the control input \(u^P\) converges to the optimal control input \(\pi^P\) given by (3.29).
Recall that $\omega = \mathbf{1}_n \omega^d$, such that $\dot{\omega} = B^T \mathbf{1}_n \omega^d = 0$. Since on the invariant set $\dot{\omega} = \dot{V} = \dot{\theta} = 0$, the solutions to system (3.11) controlled by (3.31) approach the set of equilibria contained in $\Upsilon$. Consider a forward invariant set $\Omega \subseteq \Upsilon$ around $(\eta, \omega, V, \theta)$, where it holds that $\frac{\partial^2 Z}{\partial \eta \partial \omega \partial V \partial \theta} > 0$. Since every equilibrium in $\Omega$ is Lyapunov stable, it then follows from Lemma 1.4.8 that the solution starting in $\Omega$ converges to a point.

Bearing in mind that $p_i$ is applied to the physical system, which is related to $u_i^P$ via (3.7), the following corollary provides the overall controller:

**Corollary 3.3.4 (Applied control input).** The control input $p$, applied to the controllers (3.2) is generated by the following system:

$$
\begin{align*}
\dot{\theta} &= -L^{\text{com}} \theta - R^{-1}(\omega - \mathbf{1}_n \omega^d), \\
u^P &= -R^{-1} \theta, \\
T_P \dot{x}^P &= -x^P - K^{-1}_P u^P, \\
p &= x^P.
\end{align*}
$$

(3.41)

### 3.4 Case study

We will illustrate the performance of the controller proposed in Theorem 3.3.3 on an academic example of a microgrid. Consider a network of four interconnected inverters, as shown in Figure 3.1. Noticing that $\omega$ is independent of the voltages, we propose additionally the following decentralized controller aiming at voltage regulation:...
regulation:

\[
\begin{align*}
\hat{\mu}_i &= -(V_i - V_i^d) \\
u_i^Q &= \mu_i.
\end{align*}
\] (3.42)

The control objective chosen in this simulation is to let the system converge to the desired values \(\omega^d = 50\) and \(V^d = (1, 1, 1)^T\). The system is initially at steady state with \(P_0^d = (0.02, 0.15, 0.1, 1)^T\) for \(t \in [0, 25]\) and \(Q^d = (0.202, 0.014, 0.0131, 0.089)^T\) for \(t \in [0, 45]\). At timestep \(t > 25\), \(P_0^d\) is changed to \(P_0^d = (0.04, 0.4, 0.2, 0.2)^T\) and at timestep \(t > 45\), \(Q_0^d\) is changed to \(Q_0^d = (3.78, 1.8, 0, 4)^T\). The frequency and voltage response to the control inputs is provided in Figure 3.2. From Figure 3.2 we can see how the controller proposed in Theorem 3.3.3 regulates the frequency after a disturbance back to its desired value. Furthermore we notice that controller (3.42) is able regulate the voltage to its desired value. Although the proven results hold only locally, the simulations provide evidence that the proposed controllers perform well in a wide range around the desired values.
Figure 3.2: Frequency and voltage response to changing values of $P_0^d$ and $Q_0^d$ and its corresponding control inputs $u^P$ and $u^Q$. The value of $P_0^d$ is changed at timestep 25 and $Q_0^d$ at timestep 45.