Cycle slipping in nonlinear circuits under periodic nonlinearities and time delays.

Vera Smirnova¹, Anton Proskurnikov², and Natalia V. Utina³

Abstract—Phase-locked loops (PLL), Costas loops and other synchronizing circuits are featured by the presence of a nonlinear phase detector, described by a periodic nonlinearity. In general, nonlinearities can cause complex behavior of the system such as multi-stability and chaos. However, even phase locking may be guaranteed under any initial conditions, the transient behavior of the circuit can be unsatisfactory due to the cycle slipping. Growth of the phase error caused by cycle slipping is undesirable, leading e.g. to demodulation and decoding errors. This makes the problem of estimating the phase error oscillations and number of slipped cycles in nonlinear PLL-based circuits extremely important for modern telecommunications. Most mathematical results in this direction, available in the literature, examine the probability density of the phase error and expected number of slipped cycles under stochastic noise in the signal. At the same time, cycle slipping occurs also in deterministic systems with periodic nonlinearities, depending on the initial conditions, properties of the linear part and the periodic nonlinearity and other factors such as delays in the loop. In the present paper we give analytic estimates for the number of slipped cycles in PLL-based systems, governed by integro-differential equations, allowing to capture effects of high-order dynamics, discrete and distributed delays. We also consider the effects of singular small-parameter perturbations on the cycle slipping behavior.

Index Terms—Phase-locked loops, nonlinear circuits, cycle slipping, delays, singularly perturbed systems

I. INTRODUCTION

A lot of systems arising in electrical engineering, industrial electronics and telecommunications are based on the seminal idea of the phase-locked loop (PLL) and contain a digital or analog circuits, which synchronizes some internal oscillator with an exogenous periodic signal in phase (which task is sometimes referred to as phase tracking or phase recovery) [5], [9], [10]. Mathematical model of a PLL-based circuit typically may be considered as a feedback interconnection of a linear time-invariant system and a periodic nonlinearity, characterizing the phase detector. Such systems have been widely studied in mechanics and control theory under different names (“phase synchronization systems”, “pendulum-like systems” etc.), see e.g. [7], [8] and references therein.

Under natural assumptions it is often possible to prove the steady-state phase locking (that is, the phase error converges to one of the equilibria), which sometimes is called “gradient-like behavior”, see [7], [8] and references therein. Before achieving the synchronous regime, the phase error normally oscillates around some equilibrium point. However, due to large fluctuations it may leave the corresponding region of attraction and be attracted by another equilibrium point. During this transition, known as the cycle slipping, the phase shift significantly increases which is especially undesirable in data transmission systems, resulting in demodulation errors.

Appealing to a simple mechanical analogue of the PLL-based circuit, that is, the model of mathematical pendulum, the process without cycle slipping corresponds to the oscillation around the stable lower equilibrium, whereas the cycle slipping is portrayed by the passing via the upper unstable point. Starting from this basic model, J. Stoker [17] suggested a definition for the number of slipped cycles, being a crucial characteristic of the transient process in a PLL circuit. Given a Δ-periodic solution with the phase error σ(t), the number of slipped cycles is defined by k := \max_{t \geq 0} |\sigma(t) - \sigma(0)| / Δ. In other words, |σ(t) - σ(0)| < (k + 1)Δ for any t ≥ 0, however |σ(t) - σ(0)| = kΔ for some t ≥ 0.

Under persistent stochastic noise in the input signal, PLL exhibit random cycle slippings, whose statistical properties (such as the expected rate, average time of the first slip etc.) have been subject of extensive studies for more than 50 years since the pioneering paper by Viterbi [18], see [11], [12]. In the present paper, we are interested in the deterministic model of cycle slipping which is caused not by external excitations but only by the nonlinear structure of the circuit itself. We consider a very general mathematical model, described by integro-differential Volterra equations and encompassing a large class of PLL-based systems, including those with delays [3]. The offered analytic criterion, providing the upper bound for the number of cycles slipped, is based on the Popov method for the absolute stability elaborated in control theory [15], involves the initial conditions, transfer function of the linear part of the system and the periodic characteristics of the phase detector. Assumptions about the solution boil down to a frequency-domain inequality and some algebraic condition. The first “frequency-algebraic” criterion of such a type was obtained in [4] (and later was reformulated in [19]) and then extended to discrete-time PLL [16] and infinite-dimensional systems [7]. In the later papers [13], [14] the restrictions, adopted in the mentioned previous papers, were significantly relaxed under assumption of smooth periodic nonlinearity with known
a priori derivative bounds.

In this paper, we simplify and extend the results from [13], and illustrate their potential by estimating the number of slipped cycles for a delayed PLL. We also discuss robustness of our estimates against singular perturbation, introducing a higher derivative term with a small parameter (the term “singular” highlights that the order of unperturbed equation is lower than the order of the perturbed one). Singly perturbed systems describe a wide range of physical, mechanical and electrical systems [12], and their asymptotic properties and transient dynamics in general significantly differ from those of unperturbed ones, requiring a special theory [6].

II. PROBLEM SETUP

We consider a nonlinear system with a periodic nonlinearity, governed by an integro-differential Volterra equation

\[ \dot{\sigma}(t) = \alpha(t) + \rho \varphi(\sigma(t-h)) - \int_0^t \gamma(t-\tau) \varphi((\sigma(\tau))) d\tau. \]  

(1)

Here \( h \geq 0, \rho \in \mathbb{R}, \gamma, \alpha : [0, +\infty) \to \mathbb{R} \). The map \( \varphi : \mathbb{R} \to \mathbb{R} \) is assumed \( C^1 \)-smooth and \( \Delta \)-periodic with two simple isolated roots on \( [0, \Delta) \). The kernel function \( \gamma(\cdot) \) is piece-wise continuous, the function \( \alpha(\cdot) \) is continuous. The solution of (1) is defined uniquely by the initial conditions

\[ \sigma(t)|_{t \in [-h, 0]} = \sigma^0(t), \quad \sigma(0 + 0) = \sigma^0(0), \]  

(2)

where \( \sigma^0(\cdot) \) is a continuous function. The system (1) may be considered as a feedback interconnection of a linear system, whose transfer function is defined as

\[ K(p) = -\rho e^{-hp} + \int_0^t \gamma(t) e^{-pt} dt \quad (p \in \mathbb{C}), \]  

(3)

and a nonlinear block \( \sigma \mapsto \varphi(\sigma) \). Throughout the paper, we assume the linear part of (1) is exponentially stable, that is,

\[ |\alpha(t)| + |\gamma(t)| \leq M e^{-rt} \quad (M, r > 0). \]

For definiteness, we assume that

\[ \Delta \int \varphi(\sigma) d\sigma \leq 0. \]  

(4)

We assume the lower and upper slopes \( \alpha_1, \alpha_2 \) for the periodic nonlinearity \( \varphi(\cdot) \) are known:

\[ \alpha_1 := \inf_{\sigma \in [0, \Delta]} \frac{d\varphi}{d\sigma} < 0 < \alpha_2 := \sup_{\sigma \in [0, \Delta]} \frac{d\varphi}{d\sigma}. \]

Our goal is to disclose the maximal phase error deviation \( \sup_{t \geq 0} |\sigma(t) - \sigma(0)| \) along the solution, which also gives the upper bound for the number of slipped cycles.

III. PHASE ERROR ESTIMATES

The main idea of the phase error estimation, proposed in [7], [13] is a modified Popov’s method of “a priori integral indices” which was elaborated to prove absolute stability of nonlinear systems [15]. The cornerstone of this method is an integral quadratic constraint, which is guaranteed by a frequency-domain inequality.

**Lemma 1.** [13] Suppose there exist such positive \( \vartheta, \varepsilon, \delta, \tau \) that for all \( \omega \geq 0 \), the frequency-domain inequality holds:

\[ \text{Re} \left\{ \vartheta K(i\omega) - \tau (K(i\omega) + \alpha_1^{-1}i\omega)^* (K(i\omega) + \alpha_2^{-1}i\omega) \right\} - \varepsilon |K(i\omega)|^2 - \delta \geq 0 \quad (\tau^2 = -1). \]  

(5)

Then the following integral quadratic functional

\[ I_T[\sigma(\cdot)] = \int_0^T \left\{ \vartheta \dot{\sigma}(t) \varphi(\sigma(t)) + \varepsilon \dot{\sigma}^2(t) + \delta \dot{\varphi}^2(\sigma(t)) + \tau (\alpha_1^{-1} \dot{\varphi}(\sigma(t)) - \dot{\sigma}(t)) (\alpha_2^{-1} \dot{\varphi}(\sigma(t)) - \dot{\sigma}(t)) \right\} dt \]

are uniformly bounded along the solution of (1):

\[ I_T \leq Q, \]  

(6)

where \( Q \) does not depend on \( T \).

A closer analysis of the proof in [13] shows that the value of \( Q \) in fact depends on the parameters \( \vartheta, \varepsilon, \delta, \tau \), the dynamics of linear part of (1) and also the value of \( \max \varphi(\sigma(t)) \), which in practice always can be estimated from the equations (1). However, this estimates appears to be rather involved and conservative. Assuming that some estimate \( Q \) is known, the following two estimates for the number of slipped cycles main be obtained [13].

To start with, we introduce the following auxiliary functions

\[ \Phi(\sigma) := \sqrt{(1 - \alpha_1^{-1} \varphi(\sigma))(1 - \alpha_2^{-1} \varphi(\sigma))} \]

\[ P(\varepsilon, \tau, \sigma) := \varepsilon + \tau \Phi^2(\sigma) \]

\[ r_j(k, \vartheta, x) := \Delta \int_0^x \frac{\varphi(\sigma) d\sigma + (-1)^j \pi}{\Delta \int_0^x \frac{\Phi(\sigma) d\sigma}{\Phi(\sigma) \varphi(\sigma) d\sigma}}, \]  

(\( j = 1, 2 \))

\[ r_{0j}(k, \vartheta, x) := \Delta \int_0^x \frac{\varphi(\sigma) d\sigma + (-1)^j \pi}{\Delta \int_0^x \frac{\varphi(\sigma) d\sigma}{\Phi(\sigma) \varphi(\sigma) d\sigma}}, \]  

(\( j = 1, 2 \))

\[ r_{1j}(k, \vartheta, \varepsilon, \tau, x) := \frac{\Delta}{\Delta} \int_0^x |\varphi(\sigma)| P(\varepsilon, \tau, \sigma) d\sigma \]  

\( (j = 1, 2) \)

\[ Y_j(\sigma) := \varphi(\sigma) - r_{1j}(\varphi(\sigma)) P(\varepsilon, \tau, \sigma) \]  

\( (j = 1, 2) \)
and also a matrix-valued function $T_j(k, \vartheta, x, a) :=$
\[
\begin{pmatrix}
\varepsilon & a\vartheta r_j(k, \vartheta, x) \\
\frac{\varepsilon}{a} & a\vartheta r_j(k, \vartheta, x) \\
0 & a_0\vartheta r_{j0}(k, \vartheta, x) \\
0 & a_0\vartheta r_{j0}(k, \vartheta, x)
\end{pmatrix},
\]
where $a \in [0, 1]$ and $a_0 := 1 - a$.

**Theorem 1.** Suppose there exist such positive $\vartheta, \varepsilon, \delta, \tau$ and natural $k$ that the following conditions are fulfilled:
1) for all $\omega \geq 0$ the inequality (5) is valid;
2) the condition holds
\[
4\delta > (r_j(k, \vartheta, \varepsilon, \tau, Q))^2, \quad (j = 1, 2),
\]
where $Q$ is the bound from (6). Then any solution of (7) slips less than $k$ cycles, that is, the inequalities hold
\[
|\sigma(0) - \sigma(t)| < k\Delta \quad \forall t \geq 0.
\]

The conditions of Theorem 1 resemble those from [4], however, they are applicable for general infinite-dimensional system [1] and visibly improve the result from [4] even for the case of ordinary differential equations [4].

**Theorem 2.** Suppose there exist positive $\vartheta, \varepsilon, \delta, \tau, a \in [0, 1]$ and natural $k$ satisfying the conditions as follows:
1) for all $\omega \geq 0$ the inequality (5) is valid;
2) the matrices $T_j(k, \vartheta, Q, a)$ $(j = 1, 2)$ of the value of $Q$ is defined by (6), are positive definite.
Then for the solution of (4) the inequality (8) holds.

Conditions of the latter theorem may be seriously simplified in a special case where $a_1 = -a_2$ and $\varphi(\sigma(0)) = 0$. Retracing the estimates from [13], one can show that
\[
Q \leq q := \frac{1}{r} \left( \vartheta Mm + 2(\varepsilon + \tau)Mr(Mr + \rho) + (\varepsilon + \tau)Mr \right)
\]
where $m := \sup \varphi(\sigma)$. This gives rise to the following simplified version of Theorem 2.

**Theorem 3** Let $a_1 = -a_2$. Suppose there exist such positive $\vartheta, \varepsilon, \delta, \tau, a \in [0, 1]$ and natural $k$ that the following conditions are fulfilled:
1) for all $\omega \geq 0$ the frequency-domain inequality (5) holds;
2) the matrices $T_j(k, \vartheta, Q, a)$ $(j = 1, 2)$ are positive definite.
If $\sigma(0) = \sigma_0$ where $\varphi(\sigma_0) = 0$, then for any solution of (4) the estimate (8) holds. In general, the estimate (8) is valid, replacing $k$ with $k + 1$.

**Theorem 4** Let $\sigma(0) = \sigma_0$ where $\varphi(\sigma_0) = 0$. Suppose there exist such positive $\vartheta, \varepsilon, \delta, \tau, a \in [0, 1]$ and natural $k$ that the following conditions are fulfilled:
1) for all $\omega \geq 0$ the frequency-domain inequality (5) holds;
2) the matrices $T_j(k, \vartheta, Q, a)$ $(j = 1, 2)$ are positive definite.
Then for any solution of (4) the estimate (9) holds.

The first claim of Theorem 3 follows from Theorem 2, since, as was mentioned, $q$ may be used instead of $Q$ under restriction $\varphi(\sigma_0) = 0$. To prove the second claim, notice that $\varphi(\sigma)$ is $\Delta$-periodic and has zeros on the period. Therefore, if $\varphi(\sigma(t)) \neq 0$ for any $t \geq 0$, then $|\varphi(0) - \varphi(t)| < \Delta$ for any $t \geq 0$, so the solution slips no cycles. Otherwise, let $t_0 \geq 0$ be the minimal number such that $\varphi(\sigma(t_0)) \neq 0$. It is obvious that $|\sigma(t_0) - \sigma(t)| < \Delta$. Applying Theorem 2 for the solution $\sigma(t - t_0)$, one shows that $|\sigma(t_0) - \sigma(t)| < k\Delta$ for any $t \geq t_0$ and hence $|\sigma(0) - \sigma(t)| < (k + 1)\Delta$.

**IV. Example**

Let us consider a phase-locked loop (PLL) with a proportional integral low-pass filter, a sine-shaped characteristic of phase frequency detector and a time-delay in the loop. Its mathematical description is borrowed from [2]:
\[
\dot{\sigma}(t) + \frac{1}{T}\dot{\sigma}(t) + \varphi(\sigma(t - h)) + sT\dot{\varphi}(\sigma(t - h)) = 0,
\]
where $\varphi(\sigma) = \sin \sigma - \beta, \sigma(0) = 0, \beta \in (0, 1)$, $h > 0, T > 0$.

The differential equation (9) can be reduced to integro-differential equation (1) with
\[
\gamma(t) = \begin{cases} 
0, & t < h, \\
(1 - s)e^{-\frac{t}{T}}, & t \geq h \end{cases},
\]
where $\alpha(t) = e^{-\frac{t}{T}}(b - (1 - s)J)$, and
\[
J = \begin{cases} 
\int_0^{t-h} e^{\frac{t-h}{T}}\varphi(\sigma(t))dt, & t \leq h, \\
\int_{t-h}^0 e^{\frac{t-h}{T}}\varphi(\sigma(t))dt, & t > h
\end{cases}.
\]

The transfer function of the lowpass filter here has the form:
\[
K(p) = \frac{Tsp + 1}{Tp + 1}.e^{-ph}
\]

We suppose that $\varphi(\sigma(0)) = 0$ and apply Theorem 4.

Let $a_2 = -a_1 = 1, \vartheta = 1, a = 1$. The assumption 1) of Theorem 4 shapes into
\[
\Omega(\omega) \equiv \tau T^2\omega^4 + \omega^2(\tau^3 \cos\omega h - T^4 s^2(\varepsilon + \tau) + \tau - \delta^2T^2 - T^2(1 - s)\omega \sin\omega h + T\cos\omega h - (\varepsilon + \tau)T^2 - \delta \geq 0 \quad \forall \omega;
\]
whence condition 2) may be rewritten as
\[
2\sqrt{\delta} > \frac{2\pi \beta + q\delta^{-1}}{4(\delta \arcsin(\beta + \sqrt{1 - \beta^2})}
\]

Notice that for all $\omega \in \mathbb{R}$ one has
\[
\Omega(\omega) \geq \Omega(0)(\omega) \equiv (\tau T^2 - \frac{\tau T^2}{2} T^3 \sinh^2\omega)\omega^4 + (T^3 - T^4)s^2(\varepsilon + \tau) + \tau - \delta T^2 - \frac{\tau T^2}{2} T^2h^2 + (1 - s)T^2h^2 + (T - (\varepsilon + \tau)T^2 - \delta), \forall \omega
\]
and $\Omega(\omega) \approx \Omega(0)(\omega)$ when $\omega h < 1$.

We consider the case $T \leq 0.9, h_0 = \frac{T}{2} \leq 1$, since for small $T$ and small $h$ the PLL is gradient-like for all $\beta \in (0, 1)$.
Let us choose $\varepsilon = \frac{h_0}{2}, \delta = a_0 T, \tau = \tau_0 T^4$. As $\Omega(0) = \Omega(0)(0)$ it is necessary that $a_0 + \beta_0 + \gamma_0 T^4 \leq 1$. Then the optimal
values for $\alpha_0$ and $\beta_0$ are $\alpha_0 = \beta_0 = \frac{1}{2}(1 - \gamma_0 T^4)$, whence $2\sqrt{\gamma_0} = 1 - \gamma_0 T^4$. For $\gamma_0 = \max \{\frac{4}{3}b h_0^2, \frac{1}{2}(h_0 + 1 - s)^2\}$ the polynomial $\Omega_0(\omega)$ is nonnegative, $\forall \omega$. It follows from (11) that the number $k_0$ of cycles slipped satisfies the inequality

$$k_0 \leq r_0 := \lfloor 2\sqrt{\gamma_0}(\beta \arcsin \beta + \sqrt{1 - \beta^2}) - 2\pi \beta \rfloor^{-1}.$$  

Let us consider the PLL with $b = K(0)/\beta$ [4]. Then by estimating the functional $\mathcal{J}$ we conclude that the value of $q$ can be defined by the formula

$$q = T^2(A + B h_0 + C h_0^2),$$  

where

$$A = \left(\frac{7}{2}\gamma_0^2 + 3 \right),$$

$$B = 3(1 - s)(1 + \beta)(3\beta + 1),$$

$$C = \frac{1}{2}(1 - s)^2(1 + \beta)^2.$$  

It follows from (12), (13) that the number of slipped cycles increases together with $T$, with $\beta$ or with $h_0$. Let for example $h_0 = 1, s = 0.4, T = 0.1$. Then $r_0 = 1$ for $\beta = 0.9$, $r_0 = 2$ for $\beta = 0.92$, and $r_0 = 5$ for $\beta = 0.95$.

V. EXTENSION: ROBUSTNESS TO SMALL SINGULAR PERTURBATIONS

The estimates for the number of slipped cycles presented in the paper are easily shown to be robust against small variations in the parameters of the transfer function and the nonlinearity. It appears, however, that they remain robust even to a singular perturbation, introducing a high-order term into equation (1):

$$\mu \hat{\sigma}_\mu(t) + \sigma_\mu(t) = \alpha(t) + \rho \varphi(\sigma_\mu(t - h)) - \int_0^t \gamma(t - \tau) \varphi((\sigma_\mu(\tau))) d\tau \quad (t \geq 0).$$  

Here $\mu > 0$ is a small parameter. Equation (14) can be reduced to the form (1)

$$\sigma_\mu(t) = \sigma(t) - \int_0^t \gamma(t - \tau) \varphi(\sigma(t - h)) d\tau \quad (t > 0),$$  

where $\alpha(t) = \sigma(0) e^{-\frac{t}{\mu}} + \frac{1}{\mu} \int_0^t e^{-\frac{t - \tau}{\mu}} \alpha(\lambda) d\lambda + \frac{1}{\mu} J_0(t),$

$$J_0(t) = \begin{cases} 
\int_0^h e^{\frac{t - \tau}{\mu}} \varphi(\sigma(\tau)) d\lambda, & t \leq h, \\
\int_0^h e^{\frac{t - \tau}{\mu}} \varphi(\sigma(\lambda)) d\lambda, & t > h,
\end{cases}$$

$$\gamma(t) = \frac{1}{\mu} \int_0^t e^{\frac{t - \tau}{\mu}} \gamma(\lambda) d\lambda - \frac{\rho}{\mu} \int_0^t e^{\frac{t - \tau}{\mu}} \varphi(\sigma(\lambda)) d\lambda,$$

where $\gamma(t) = \frac{1}{\mu} \int_0^t e^{\frac{t - \tau}{\mu}} \gamma(\lambda) d\lambda - \frac{\rho}{\mu} \int_0^t e^{\frac{t - \tau}{\mu}} \varphi(\sigma(\lambda)) d\lambda.$

The transfer function for equation (15) is given by

$$K_\mu(p) = \frac{K(p)}{1 + \mu p}.$$  

Let $q_0 = q + (\delta M_m + 2(\varepsilon + \tau) M_m \delta M + \rho \rho M) h, \tau > 0$. Applying Theorem 3 to (15), one gets the following.

**Theorem 4.** Let $\alpha, \beta, \gamma = 0$ where $\varphi(\sigma_0) = 0$. Suppose there exist such positive $\delta, \varepsilon, \delta, \tau, a \in [0,1]$ and natural $k$ that the following conditions are fulfilled:

1) for all $\omega > 0$ the frequency-domain inequality [5] holds;
2) the matrices $T_j(k, \theta, q_0)$ ($j = 1, 2, \ldots, k$) are positive definite.

Then there exists such value $\mu_0$ that for all $\mu \in (0, \mu_0)$ and any solution of (14) the estimate (3) holds.

VI. CONCLUSION

We consider the problem of cycle slipping for PLL based circuits, governed by integro-differential Volterra equations, which model captures, in particular, potential delays in the system. Under restriction of smooth nonlinearity, we get analytic estimates for the number of slipped cycles. We also consider small singular perturbations of our equations, showing the estimates are uniform with respect to a small parameter.

REFERENCES