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Publication date:
2017

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):
Ruiz Duarte, E. (2017, Mar 1). An invitation to algebraic number theory and class field theory.

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An invitation to algebraic number theory and class field theory

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March 1, 2017

Abstract

This informal document was motivated by a question here at my university by a bachelor student. I will try to expose something that personally I think is impressive. The aim is to do it in such a way that is understandable with a basic knowledge of algebra.

We will examine without any rigor the "generalization" in some sense of the concept of "factorization", not just in the integers \( \mathbb{Z} \) but in some "generalized" integers from number fields of the form \( \mathbb{Q}(\sqrt{-n}) \) for \( n > 0 \) which are called imaginary quadratic number fields. The problems in these new rings that will arise will give us interesting mathematics.

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First of all we will assume that the rings that we will use here are commutative and they will have a multiplicative neutral element 1.

We begin with basics, most of the people here are familiar with the ring of integers:

\[ Z = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \]

We know that in \( Z \) are prime numbers and composite numbers.

Then we have the field \( \mathbb{Q} \) with all the "fractions" in it gotten from numbers of \( Z \). In \( \mathbb{Q} \) numbers like \( \pi, e \) are missing. There are just numbers that can be represented as \( \frac{a}{b} \) such that \( a, b \in \mathbb{Z} \).

The most usual ring of integers is of course \( \mathbb{Z} \), but we can add a new algebraic element in it and experiment with this new space. For example if we add the root of a polynomial which is not rational (an algebraic number); and then we calculate all the possible combinations of it with the field \( \mathbb{Q} \), we will get a new field.

For example if we adjoin one of the roots of \( x^2 + 5 \) which is \( \sqrt{-5} = i\sqrt{5} \) where \( i \) is the imaginary unit we get the imaginary quadratic field:

\[ Q(\sqrt{-5}) = \{ a + b\sqrt{-5} : a, b \in \mathbb{Q} \} \] (1)

This space is like the usual \( \mathbb{Q} \), but we added a new element not in \( \mathbb{Q} \) (the algebraic number \( \sqrt{-5} \)). Is easy to see that every nonzero element there has inverse and that the product of two elements also belongs to \( Q(\sqrt{-5}) \).

Take \( a + b\sqrt{-5}, c + d\sqrt{-5} \in Q(\sqrt{-5}) \) then the product of both numbers is given by \( ac - 5bd + (ad + bc)\sqrt{-5} \in Q(\sqrt{-5}) \) and the inverse of \( a + b\sqrt{-5} \) is given by:

\[ \frac{1}{a + b\sqrt{-5}} = \frac{a}{a^2 + 5b^2} + \frac{-b\sqrt{-5}}{a^2 + 5b^2} \in Q(\sqrt{-5}) \]

So, \( Q(\sqrt{-5}) \) is a well defined field, just as \( \mathbb{Q} \).

The ring of integers which will be the equivalent to \( \mathbb{Z} \) in \( Q \), but now for \( Q(\sqrt{-5}) \) is:

\[ \mathbb{Z}[\sqrt{-5}] = \{ a + b\sqrt{-5} : a, b \in \mathbb{Z} \} \] (2)

Is well known that the usual ring of integers \( \mathbb{Z} \) is a Unique Factorization Domain, this means that any number can be factored in prime numbers of \( \mathbb{Z} \) in a unique way.
For a general number field (extensions of \( \mathbb{Q} \) by other algebraic elements like our example) this is not always true, in our example \( \mathbb{Z}[\sqrt{-5}] \) is not a unique factorization domain.

Example:

\[
6 = 3 \cdot 2 = (1 + \sqrt{-5})(1 - \sqrt{-5})
\]

(3)

So we will see what is happening here, but first we will need some theory, I begin with the essential, and finish with the deep theory.

0.1 Rings

We know that a general ring has two operations \( \langle R, +, \cdot \rangle \) where with the addition \( \langle R, + \rangle \) all elements have an additive inverse, a neutral element and here particularly is commutative. Also it has a product \( \langle R, \cdot \rangle \) which interacts via the law of distributivity with the addition. Here in \( R \) not necessary every element is invertible.

The common example is again \( \langle \mathbb{Z}, +, \cdot \rangle \) because the multiplicative inverse of the nonzero \( a \in \mathbb{Z} \) is an integer \( a^{-1} \in \mathbb{Z} \) such that \( a \cdot a^{-1} = 1 \), such number does not exist in \( \mathbb{Z} \) unless \( a = \pm 1 \) because there is no \( \frac{1}{a} \in \mathbb{Z} \) for \( a \neq \pm 1 \). For the addition for each integer \( a \) there is always a ”negative” \( -a \) such that under + it gives us 0, this means \( a + (-a) = 0 \) (neutral element).

0.2 Ideals of a Ring

A ring \( \langle R, +, \cdot \rangle \) is an algebraic structure that has some subalgebraic structures called ideals. If \( I \subset R \) is an ideal then their elements form a group under +, namely \( \langle I, + \rangle \leq \langle R, + \rangle \) and for \( a \in I \) multiplication by every element \( r \in R \), namely \( a \cdot r \in I \), that is \( aI = I \).

For example in \( \mathbb{Z} \) consider the ”multiples of \( n \)” there are the ideals and we denote them as \( n\mathbb{Z} := \{ n \cdot a : a \in \mathbb{Z} \} \).

The reason that \( n\mathbb{Z} \) is closed under addition is rather obvious as \( a, b \in n\mathbb{Z} \) implies that \( a + b = n\alpha + n\beta = n(\alpha + \beta) \in n\mathbb{Z} \) for some \( \alpha, \beta \in \mathbb{Z} \). The other property of the ideal is also obvious as for \( r \in \mathbb{Z} \) and \( a \in n\mathbb{Z} \) we have that \( ra = nar \in n\mathbb{Z} \) for some \( a \in \mathbb{Z} \).

Said this, these ideals \( I \subset R \) behave more-less like a vector space over \( R \) (an \( R \)-module), as \( rI = I \) for all \( r \in R \), just like in \( \mathbb{R}^2 \) which can be seen as a \( \mathbb{R} \)-vector space, as any element \( v \in \mathbb{R}^2 \), can be multiplied by an \( \alpha \in \mathbb{R} \).
such that $\alpha v \in \mathbb{R}^2$.

Note that if $1 \in I$ then $I = R$, so much more interesting examples arise when the ideal is not the ring, which is our case.

So now, we said that the ideals $I \subset R$ behave "like a vector space", well then we can think in a "basis" of them, which means, a set of elements that generate the ideal. This is denoted by $I = \langle a, b \rangle$ If you just need 1 element we call the ideal principal, that is $I = \langle a \rangle = aR$ for some $a \in R$.

In $\mathbb{Z}$ all ideals as principal.

So, now $\mathbb{Z}$ is starting to be boring, because an ideal of $\mathbb{Z}$ given by $\langle m, n \rangle$ consists in all the linear combinations $xm + yn$ and is generated by one element.

More precisely $\langle m, n \rangle = \langle \gcd(m, n) \rangle = \gcd(m, n)\mathbb{Z}$.

The proof that every ideal in $\mathbb{Z}$ is principal can be seen here for the curious https://proofwiki.org/wiki/Ring_of_Integers_is_Principal_Ideal_Domain.

Returning to our example with the ring of integers $\mathbb{Z}[\sqrt{-5}]$ of the field $\mathbb{Q}(\sqrt{-5})$, there are ideals, like $I = \langle 3, 1 + \sqrt{-5} \rangle$ that are not principal.

Principal ideals play a crucial role in unique factorization of $R$ as we will see in the next theorem.

**Theorem 1:**

Let $K = \mathbb{Q}(\alpha)$ be an algebraic number field and $R$ its ring of integers, then:

All ideals of $R$ are principal if and only if $R$ has unique factorization for its elements.

We will prove in the next section that $I = \langle 3, 1 + \sqrt{-5} \rangle$ is not principal and hence, using the latter theorem $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain. We already knew that it was not a factorization domain as 6 as an element of $\mathbb{Z}[\sqrt{-5}]$ has two different non trivial factorizations.

The next section can be skipped as it has more technicalities, but is just to fill a big hole that a curious person may find in this informal text.

**0.3 Proving that an ideal of a ring of integers is not principal**

Recall that this is informal, so I will give the essential ideas.

The basic tool to factorize in the ring of integers $\mathbb{Z}[\sqrt{D}]$ (where $D$ is non
square) is the following:

**Definition:** The norm of a quadratic field of the form $\mathbb{Q}(\sqrt{D})$ for a non-square $D$ is given by the function $N : \mathbb{Q}(\sqrt{D}) \to \mathbb{Q}$ such that:

$$N(a + b\sqrt{D}) = (a + b\sqrt{D})(a - b\sqrt{D}) = a^2 - Db^2$$

The following property is the key of why the norm is important to factorize when applied to elements $\mathbb{Z}[\sqrt{D}]$.

**Proposition 2:** Let $\alpha, \beta \in \mathbb{Q}(\sqrt{D})$ then if $N$ is its associated norm then $N(\alpha \beta) = N(\alpha)N(\beta)$.

Now using this proposition the next theorem tells us information about the "dimension" of the ideal of a number field in general, i.e. the number of generators of the ideal.

**Theorem 3:** Let $R$ be a ring of integers of a number field $K$, consider $z \in R$ and the principal ideal generated by $z$, namely $(z) = zR$ then $N(z) = \#R/(z)$

With this theorem 3 now we can prove something about the factorization of the integers in $K = \mathbb{Q}(\sqrt{-5})$ when we combine it with theorem 2. So here $R = \mathbb{Z}[\sqrt{-5}]$ and $N(a + b\sqrt{-5}) = a^2 + 5b^2$.

We want to prove that $I = \langle 3, 1 + \sqrt{-5} \rangle$ is not principal. So using the previous theorem this reduces to prove that $\#R/I \neq N(z)$ for all $z \in R$. This is easy as $R/(3) \cong \mathbb{Z}/3\mathbb{Z}[\sqrt{-5}]$ then $\#R/(3) = 9$ and this implies that for the quotient with the bigger ideal $I$ we have $\#R/I = 3$, then we would like an element $a + b\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ with norm 3, which means we want an integer solution to $a^2 + 5b^2 = 3$ which is impossible. and then $I$ cannot be principal.

### 0.4 Operations with ideals of a ring

Consider now set of all ideals $I \subseteq R$ (counting the trivial ideal $R$), call them $\mathfrak{I}$, now if you take $I, J \in \mathfrak{I}$ there is a way of multiply them $K = I \otimes J$ in such a way that $K \in \mathfrak{I}$, that is, $K$ is also an ideal of $\langle R, +, \cdot, \rangle$.

We define

$$I \otimes J := \left\{ \sum_{i=1}^{n} a_i \cdot b_i : a_i \in I, b_i \in J \forall n \in \mathbb{N} \right\}$$  \hspace{1cm} (4)

This new ideal $I \otimes J$ is intuitively all the possible products $ab$ that can be constructed for $a \in I$ and $b \in J$. 

5
Example: Consider the ring of polynomials in 4 variables with complex coefficients $R = \mathbb{C}[x, y, z, w]$ and consider the following ideals generated by different combinations of variables $I = (z, w), J = (x + z, y + w), K = (x + z, w)$. Then:

$IJ = (z(x + z), z(y + w), w(x + z), w(y + w)) = (z^2 + xz, zy + wy, wx + wz, wy + w^2)$

$IK = (xz + z^2, zw, xw + zw, w^2)$

Now we need to drop redundant ideals in $\mathcal{I}$ so we define an equivalence relation $\sim$ in the ideals of $\mathcal{I}$, namely $I \sim J$ if there are $a, b \in R$ such that $aI = bJ$. So here, ideals which are related by $\sim$ are denoted as $[I]$. If you choose a representative of $[I]$ and $[J]$ then $[I] \otimes [J] = [IJ]$ is well defined and commutative.

So here we have a new structure $\mathcal{I}/\sim$ which has multiplicative structure with the ideals.

0.5 Extending the ideals to fractional ideals

Here comes an interesting part, we are limiting ourselves to rings of the form $\mathbb{Z}[\sqrt{D}]$ for a non square $D$ which are the integers of the quadratic field $\mathbb{Q}(\sqrt{D})$.

Now we would like to define for an ideal $I \subset \mathbb{Z}[\sqrt{D}]$ another ideal $I^{-1}$ which we will call fractional Ideal, that will give us an "identity", the identity will be the boring ideal we mentioned before, which is the whole $R$. So we are trying to find a good structure for $I^{-1}$ such that $I \oplus I^{-1} = R$.

This $I^{-1}$ will depend now in the elements of $\mathbb{Q}(\sqrt{D})$, as we need some "fractions", to "clear" denominators and get the $1 \in I \otimes I^{-1}$ which makes an ideal trivial, i.e. it makes it $R$.

$I^{-1} := \{ z \in \mathbb{Q}(\sqrt{D}) : z \cdot I \subseteq \mathbb{Z}[\sqrt{D}] \}$

This is exactly as we need, as the $z$ are the "fractions" in the fields where multiplied by $I$ stays in the integer ring $\mathbb{Z}[\sqrt{D}]$. Then we have that $I \otimes I^{-1} = R$.

This fractional ideals are not exactly ideals of $\mathbb{Z}[\sqrt{D}]$ as they are not elements of $\mathbb{Z}[\sqrt{D}]$, they are $\mathbb{Z}[\sqrt{D}]-$submodules of the field $\mathbb{Q}(\sqrt{D})$ but this
is a matter of terminology.

A big remark is that for a general integer ring $R$ is not always possible to invert the ideals, but in our world of number fields $\mathbb{Q}(\sqrt{\alpha})$ where $\alpha$ is an algebraic integer, the ideals of the corresponding integer ring are always invertible.

Here we are studying Dedekind Rings which means that every ideal $I \subset \mathbb{Z}[\sqrt{D}]$ can be factored by prime ideals using the multiplication $\otimes$. This means that it is always possible to find a unique finite set of prime ideals $\{P_i\}$ of $\mathbb{Z}[\sqrt{D}]$ such that $I = \otimes P_i$.

### 0.6 Class group of a ring of integers

The final object is built from the multiplicative structure given before, namely by all the ideals $I \subseteq R$ with the equivalence relation $\sim$, and with multiplication of ideals $\otimes$ namely $(I/\sim, \otimes)$.

Now extend $I/\sim$ to all the fractional its ideals (we can do this as we are supposing that $R$ is a Dedekind domain), call this extension to fractional ideals $\mathfrak{F}$.

Consider now the same equivalence classes of $\mathfrak{F}$ using $\sim$ defined in the previous section.

We define the Class group of $R$ by

$$\text{Cl}(R) := (\mathfrak{F}/\sim, \otimes)$$

**Theorem 4**: Let $R$ be a ring of integers of a number field $K$ then $\#\text{Cl}(R) = 1$ if and only if $R$ has unique factorization

### 0.7 Weird and impressive facts about $\text{Cl}(R)$ and conclusion

The class group $\text{Cl}(R)$ as the theorem 4 says, has only 1 element then the ring $R$ has unique factorization in its elements (not only on ideals), so it measures in some way how complicated the factorization behaves in a ring $R$.

A Weird and impressive thing is that for rings of the form $\mathbb{Z}[\sqrt{-n}]$ there is only a finite number of values for $n$ where $\mathbb{Z}[\sqrt{-n}]$ has Unique factorization of its elements (not only in on the ideals). This is the same as saying
that \( \#\text{Cl}(\mathbb{Z}[\sqrt{-n}]) = 1 \) by theorem 4. The numbers for \( n \) such that we get Unique factorization are: \( \{1, 2, 3, 7, 11, 19, 43, 67, 163\} \).

These weird numbers that generate Unique factorization in their associated imaginary quadratic fields are called Heegner numbers, and our example \( \mathbb{Z}[\sqrt{-5}] \) as we saw does not have Unique factorization as 5 is not a Heegner number, and in fact \( \#\text{Cl}(\mathbb{Z}[\sqrt{-5}]) = 2 \).

These Heegner numbers also are related to something that could be a mathematical coincidence but, it isn’t, and is the fact that the numbers \( e^{\pi \sqrt{43}}, e^{\pi \sqrt{67}}, e^{\pi \sqrt{163}} \) are ”almost an integer”:

\[
\begin{align*}
  e^{\pi \sqrt{43}} &\approx 884736743.999777466 \\
  e^{\pi \sqrt{67}} &\approx 147197952743.999998662454 \\
  e^{\pi \sqrt{163}} &\approx 262537412640768743.999999999925007
\end{align*}
\]

This is not a coincidence but proving it, requires a little more analytic theory and the theory of Eisenstein series, but I hope this move your mind to be more interested in algebraic number theory.