Summary

In this thesis we study the integral manifolds of the charged three-body problem. Our aim is to give a mathematical analysis of the mechanical system that consists of three charged bodies in the space $\mathbb{R}^3$ interacting via a Coulomb potential. This physical system is mathematically described as a Hamiltonian system on an 18-dimensional phase space. Like any $N$-body system the charged three-body system has a large symmetry group and as a result a large number of conserved quantities or integrals. The invariance under the translation symmetry gives rise to three integrals given by the three components of the linear momentum. When the total linear momentum is assumed to be zero then the center of mass is also conserved and hence we get three more integrals. Rotational symmetry give rise to further three integrals in the form of the three components of the angular momentum. Finally, the absence of an explicit time dependence implies the conservation of energy. In total we thus have 10 integrals which can be viewed to define a map from the phase space to $\mathbb{R}^{10}$. The level sets of this map of integrals are referred to as integral manifolds. The bifurcations of the integral manifolds depend on the single scalar parameter $\nu = -C^2 h$, where $C$ and $h$ are the values of the angular momentum and the energy, respectively. For non-critical values of $\nu$ the integral manifold is a 8-dimensional smooth manifold. At critical values the integral manifolds can bifurcate and change their topology.

For the gravitational three-body problem it has been shown that there are nine critical values of the parameter $\nu$. One critical value corresponds to the energy and hence $\nu$ being zero. The remaining critical values all have negative energy and hence positive $\nu$. Four of these values correspond to so called central configurations which are given by the three Euler collinear central configurations and the Lagrange equilateral triangle configuration. Three further critical values correspond to critical points at infinity consisting of a corotating two-body system with infinite distance to a third body. A ninth critical value results from a configuration for which the biggest and middle principal moments of inertia become equal. In the seminal work by Christopher K. McCord, Kenneth R. Meyer and Quidong Wang it has been shown that the integral manifolds bifurcate at
the first eight of these critical values whereas there is no bifurcation at the last
mentioned critical value. The projection of the integral manifolds from the phase
space to the configuration space defines the so called Hill region. It turns out that
all eight bifurcations of the integral manifolds also entail bifurcations of the Hill
region.

Our aim in this thesis is to find the critical values of the map of integrals of
the charged three-body problem and study whether and how the Hill regions
change at the critical values. The contributions of this thesis can be summarised
as follows. Following a general introduction to the problem in Chapter 1 we have:

Chapter 2: This chapter contains a brief introduction to $N$-body systems and
provides the basic mathematical background required for the study of such
systems. The $N$-body systems can be described as a Hamiltonian system. We
therefore briefly describe some basic facts from the theory of Hamiltonian sys-
tems. Moreover, we introduce a class of potentials which comprise both the
Newton and the Coulomb potentials, the relationship between symmetries and
integrals which leads to the definition of the map of integrals, the integral mani-
folds and Hill regions. We also review results on the critical values of the map of
integrals of the gravitational three-body problem.

Chapter 3: In this chapter we introduce some special solutions of $N$-body
systems. For these so called homographic solutions we give three examples for the
case of the gravitational three-body problem. The homographic solutions lead to
the notion of central configurations. We give several equivalent definitions and
properties of central configurations of which some depend on the homogeneity
of the potential and others do not.

Chapter 4: One type of critical points of the map of integrals is given by central
configurations. In this chapter we determine the number of collinear and non-
collinear central configurations in a system of three charged bodies. For the
collinear case, we first determine the space of collinear configurations reduced by
the symmetries of translation, rotation and dilation where the latter is a symmetry
following from the homogeneity of the potential. For the special case of two
equal masses, we show that the parameter space is divided into 13 regions with
different numbers of collinear central configurations. We show that there is
exactly one non-collinear central configuration if all charges have the same sign
and no non-collinear configuration otherwise. We also see that as opposed to
the gravitational three-body problem not all relative equilibria project to central
configurations in the charged three-body problem.
Chapter 7. Summary

Chapter 5: For the gravitational three-body problem, the integral manifolds are known to bifurcate also due to critical points at infinity. In this chapter we develop the mathematical theory to grasp what a critical point at infinity is. We here follow an approach by Alain Albouy. We provide several details for his approach and make various aspects of this approach more clear. The existence of critical points at infinity is then studied for the charged three-body problem.

Chapter 6: The purpose of this chapter is to investigate bifurcations of the Hill regions in the charged three-body system due to the critical points found in the previous chapters. To this end we review the abstract reduction of Hamiltonian systems describing $N$ bodies by the symmetries of translation and rotation. We show how this reduction can be made explicit for three-body systems. The resulting space is then further reduced by the dilation symmetry of charged three-body systems. The Hill region can then be thought of as a shape space endowed with information on the admissible orientations of the three-body system for given shape and constants of motion. We illustrate the procedure for the gravitational three-body problem and then study the charged three-body system given by a compound of two electrons and one proton and the Helium atom, respectively.

In Chapter 7 we give conclusions and an outlook.