Chapter 7

Conclusions and Future Research

7.1 Conclusions

This thesis deals with the integral manifolds of the charged three-body system. As such it uses the analysis of McCord et al \[34\] of the three-body system with gravitational interaction as a starting point. The main similarity is that these systems have the same symmetry group and the main difference is that in the charged three-body system the bodies may repel each other. Both belong to the class of Hamiltonian mechanical systems where the potential function is the key ingredient. Indeed the potential function determines the symmetry group and we define a class of potentials with translation and rotation symmetry and moreover the asymptotic properties of the potential of gravitation also shared by the Coulomb potential. The integral manifolds are the fibers of the integral map and the latter is in principle determined by the symmetry group. Thus the study of the integral manifolds of the charged three-body system is part of a larger program studying the integral manifolds of three-body systems with a potential from the aforementioned class.

As mentioned above the integral manifolds are the fibers of the integral map, thus the topology may change at critical values of the latter. Finding these values and the accompanying topological changes has been carried out in some detail by McCord, Meyer and Wang \[34\] for the potential of gravitation. There are two noteworthy facts. The first is that there is a simple bifurcation parameter which is a function of the values of the integral map, indicating the critical values. It is given by \(-C^2 h\), where \(C\) is the value of angular momentum and \(h\) is the value of
the Hamiltonian. The second is that critical values occur at two types of critical points, namely ‘ordinary critical points’ and ‘critical points at infinity’. However, the latter need a precise definition.

Projecting the integral manifolds on the configuration space one obtains the Hill regions. Changes in topology of the former can result in changes in topology of the latter. In fact there is a one to one correspondence between these changes as shown in [34]. In a symmetry reduced system the Hill regions become lower dimensional and their bifurcations can be visualized.

In this thesis we have carried out a part of this programme for the charged three-body system, a system with a Coulomb potential. Our aim is to find the critical values of the integral map. We are primarily interested in relative equilibria because other critical points occur for zero angular momentum. Here a difference with the case of gravitation turns up. In that case every relative equilibrium projects onto a central configuration and vice versa for every central configuration there is a relative equilibrium. But for the Coulomb potential there may exist relative equilibria not projecting onto a central configuration. We concentrate on the central configurations, collinear and non-collinear. Their existence is not only controlled by the masses but also by the charges. Thus we find a number of relative equilibria corresponding to these central configurations and thus critical values of the integral map. In a number of examples we recognize these critical values as bifurcation points. But in these examples there is also a critical value related to a relative equilibrium that does not project onto a central configuration. We have not yet been able to identify this relative equilibrium in general. Furthermore we determine the critical values related to critical points at infinity. But first we put some effort in properly defining such points following [2]. Also these values can be recognized in our examples.

In the examples we look at the Hill regions of a reduced system. However the reduction does not commute with the projection on configuration space. The reduction we use borrows ideas from gauge theory and aims to separate the motion of a frame in which the bodies move and the evolution of the shape the bodies form inside that frame. After the reduction we redefine the notion of configuration space. Then we are able to give a new definition of the Hill regions. Their bifurcations nicely illustrate the general results from the earlier chapters. But also a critical value corresponding to a relative equilibrium not related to a central configuration shows up.

7.2 Future research

We addressed several questions related to the charged three-body system, but there are still many questions left. Moreover the present analysis suggests several
1. Although we now know the critical values of the integral map for the charged three-body system, we still need to characterize the topological type of the integral manifolds. In conjunction with that we also want to understand the topological changes at the critical values. The methods of McCord, Meyer and Wang [34] may well be applicable here.

2. Once the task of the previous item has been accomplished it would be nice to generalize these results for a class of potentials – for example, to the class we introduced in section 2.4. The details of possibly many relative equilibria in a generalized potential might complicate this. But if the potential and its derivative tend to zero sufficiently fast near infinity, we conjecture that the topological changes of the integral manifolds are universal for this class.

3. To have critical points at infinity of the integral map we need at least that the integral manifolds can be unbounded. Still the question remains what the origin of critical points at infinity is. We conjecture that asymptotic homogeneity is a key ingredient.

4. In a more general class of potentials, like we introduced in section 2.4, relative equilibria may exist that do not project onto a central configuration. We wish to characterize these and moreover assess whether the bifurcation of the integral manifold at the accompanying critical value differs from a bifurcation at a relative equilibrium that does project onto a central configuration.

5. In a three-body system every central configuration is planar. In general, for every planar central configuration there exists a relative equilibrium that projects onto this central configuration, at least for the class of potentials we introduced. Then a question is: what is the role of non-planar central configurations in a $N$-body system with $N > 3$? Each central configuration corresponds to a special solution of the dynamical system, namely the homographic solutions discussed in section 3.3.1. But do these solutions in some sense differ from nearby solutions not related to a central configuration? In other words, in the high dimensional phase portrait stationary points and relative equilibria play a key role. Do these special solutions play a similar role?

6. Discussing dynamics, is it possible to make general statements about the stability of relative equilibria in the charged three-body system? Or even in systems with a more general class of potentials?
7.2. Future research

7. So far we considered potentials that allow for rotations and translations as symmetry groups. What can be said if the potential not only depends on distance but also on orientation? A physical example could be electrostatic dipole-dipole interaction. This will change the symmetry group and thus the integral map.

8. In a somewhat different setting we could also consider the earlier mentioned class of potentials but also include an external force with only $SO(1)$ and translation symmetry. Again this changes the symmetry group and the integral map.

9. There is the general question of what the special solutions and relative equilibria studied in this thesis imply for the global dynamics of the $N$-body system. This also applies to the previous item which is relevant for many applications. We mention the study of double versus sequential ionization of the helium atom in external fields.