Chapter 6

Bifurcation of the Hill Regions

6.1 Introduction

In chapter 2 we have seen that $N$-body systems have ten constants of motion or integrals see [7, 35]. The level sets of the integrals give the integral manifolds. Due to homogeneity of the potential function it is useful to parametrize the integral manifolds by the parameter $\nu = -C^2 h$, where $C$ is the angular momentum and $h$ is the energy. The bifurcation values at which the integral manifolds change their topology can be obtained from the critical points of $H$ restricted to the level sets of the angular momentum $C$. For the gravitational three-body problem, there are nine critical values that lead to nine critical values of the parameter $\nu$ (see chapter 1). The critical values result from (ordinary) critical points which in turn are related to central configurations (see chapter 4) and critical points at infinity (see chapter 5). The integral manifolds can change topology at the critical values. For the gravitational three-body problem, the topological changes of the integral manifolds also entail changes of the Hill regions, see [33]. The purpose of this chapter is to understand bifurcations of the Hill regions in a systems of three charged bodies.

This chapter is organised as follows. Section 6.2 start with the general reduction of symmetries of systems of $N$-bodies. This is then done in more detail for the case of three bodies in section 6.3. In section 6.4 we introduce charged body systems and atomic units. In section 6.5 the reduction is carried out one step further by taking into account the dilation symmetry of charged particle systems. In section 6.6 we define what we mean by Hill region. This requires one to study critical points of certain functions which are related relative equilibria as we discuss in section 6.7. In section 6.8 we discuss three examples consisting of the gravitational three-body problem, the compound of two electrons and one
6.2 Reduction of $N$-body systems

Before we can say what we mean by a Hill region we recall the reduction of the translational and rotational symmetries of a general closed $N$-body system. We will follow to a large extent the presentation in [30] which contains many references to the literature on reduction. A system is called closed if all forces are internal, i.e., they only depend on the mutual distances between the bodies and are directed parallel to the mutual difference vectors of positions. Such systems are conservative [7]. If $x_i, i = 1, \ldots, N$, denote the position vectors of the $N$ particles of masses $m_i, i = 1, \ldots, N$, in $\mathbb{R}^3$ and $p_i$ are the conjugate momenta, then the Hamiltonian is of the form

$$H(x, p) = \sum_{i=1}^{N} \frac{1}{2} m_i p_i \cdot p_i + V(x_1, \ldots, x_N) = \frac{1}{2} p M^{-1} p + V(x), \quad (6.1)$$

where $x = (x_1, \ldots, x_N)$ and $p = (p_1, \ldots, p_N)$ are considered as vectors in $\mathbb{R}^{3N}$ and $M$ is the $3N \times 3N$ diagonal matrix with diagonal $(m_1, m_1, m_1, m_2, m_2, m_2, \ldots, m_N, m_N, m_N)$. The phase space of an $N$-body system is the cotangent bundle $T^* X$ over the configuration space $X = \mathbb{R}^{3N}$.

For charged $N$-body systems, the potential $V$ is singular at configurations where the position vectors of two or more particles coincide. Taking out these collisions the configuration space becomes $X \setminus \Delta_c$, where

$$\Delta_c = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^{3N} : x_i = x_j \text{ for some } 1 \leq i < j \leq N \} \quad (6.2)$$

is the collision set. The phase space is then the cotangent bundle $T^* (X \setminus \Delta_c)$ which we identify with $(\mathbb{R}^{3N} \setminus \Delta_c) \times \mathbb{R}^{3N}$.

Such systems are symmetric under the Euclidian group $SE(3)$ which is the semi-direct product of the additive group of translations $(\mathbb{R}^3, +)$ which for each $a \in \mathbb{R}^3$ acts as $(x, p) \mapsto ((x_1 + a, \ldots, x_N + a), p)$ and the group of rotations $SO(3)$ which for $R \in SO(3)$ acts as $(x, p) \mapsto ((Rx_1, \ldots, Rx_N), (Rp_1, \ldots, Rp_N))$. Here we think of the elements $R$ of $SO(3)$ as $3 \times 3$ orthogonal matrices with unit determinant. The symmetries of translations and rotations can be reduced successively. The symmetry of translations is easily reduced by choosing Jacobi vectors $s_i, i = 1, \ldots, N - 1$ (see, e.g., [19] and Sec. 6.3 for a concrete definition in the case of $N = 3$). Together with the position vector of the center of mass of the $N$-body system the Jacobi vectors uniquely determine the positions of the $N$ bodies in space. As the center of mass of a closed $N$ body system is moving
Chapter 6. Bifurcation of the Hill Regions

with a constant velocity due to the absence of external forces we can choose an inertial frame of reference which has the center of mass at its origin. We view the \( s_i, i = 1, \ldots, N - 1 \), to be the coordinate vectors of the Jacobi vectors with respect to this center of mass frame. If we now ignore the trivial position of the center of mass then we can view the space of the Jacobi vectors \( J = \mathbb{R}^3(N-1) \) as the translation reduced configuration space. Taking out the collision set this space becomes \( J \setminus \Delta_c \) where \( \Delta_c \) now is the collision set in terms of the Jacobi vectors (see Sec. 6.3 for an example). If we moreover ignore the trivial constant momentum of the center of mass we have reduced the phase space to \( T^*J \cong \mathbb{R}^3(N-1) \times \mathbb{R}^3(N-1) \) or \( (\mathbb{R}^3(N-1) \setminus \Delta_c) \times \mathbb{R}^3(N-1) \), respectively.

We note that different choices of Jacobi vectors can be parametrized by the kinematic group which is thoroughly studied in [36]. The metric associated with expressing the kinetic energy \( K \) in terms of the velocities corresponding to the Jacobi vectors is diagonal. We will choose the Jacobi vectors to be mass-weighted. Then the metric associated with the kinetic energy becomes Euclidean, i.e. the kinetic energy becomes

\[
K = \frac{1}{2} \sum_{i=1}^{N-1} \dot{s}_i^2.
\]  

(6.3)

To address the rotational symmetry we consider a body fixed coordinate frame which is a frame that is related to the center of mass frame by a rotation. Let us denote the coordinate vectors of the Jacobi vectors in the body fixed frame by \( r_i, i = 1, \ldots, N - 1 \). We then have \( s_i = R r_i \) for some rotation matrix \( R \in SO(3) \). If we identify all configurations that are related by a rotation about the center of mass we get the so called internal space which we denote by \( Q \) and which is formally given by the quotient \( \mathbb{R}^3(N-1)/SO(3) \). As the rotation group \( SO(3) \) has dimension 3 (for example Euler angles or Cayley-Klein parameters are coordinates on \( SO(3) \)) the internal space \( Q \) has dimension \( 3(N-2) \). Collinear configurations of the \( N \) particles in \( \mathbb{R}^3 \) lead to singularities in the reduction. Away from collinear configurations the internal space has a smooth structure (see Sec. 6.3 for more details). The coordinates on the internal space are called internal coordinates. We will denote the internal space coordinates vectors by \( q \) and their components by \( q_\mu \) with the convention that Greek indices run from 1 to \( 3(N-2) \). We note that the internal space and the internal coordinates are sometimes also referred to as shape space and shape space coordinates, respectively. We however reserve the term ‘shape’ for definitions that we make below.

We remark that specifying a body-fixed frame can be phrased in the language of gauge theory [30]. One specific choice gives rise to one specific gauge. The reduced Hamiltonian function on the phase space reduced by translations and rotations is again the sum of the kinetic energy and the potential energy. The
6.2. Reduction of $N$-body systems

reduced potential is then a function of the internal coordinates only. The reduced kinetic energy is a function of the internal space coordinates and their conjugate momenta and the rotational degrees of freedom. The reduction can be phrased in such a way that the dependence on the gauge becomes apparent (see also \[14, 15\]). To see this we make the following definitions.

Let $R \in SO(3)$ denote the rotation from the center of mass frame to the body frame and $L$ denote the angular momentum with respect to the center of mass which in terms of the mass weighted Jacobi vectors is given by

$$L = \sum_{i=1}^{N-1} s_i \times \dot{s}_i.$$  \hfill (6.4)

Then the body velocities and body angular momentum are defined, respectively, by

$$\dot{r}_i = R^T \dot{s}_i,$$  \hfill (6.5)

and

$$J = R^T L.$$  \hfill (6.6)

The moment of inertia tensor $M(q)$ of the $N$-body system is the tensor with components

$$M_{ij}(q) = \sum_{k=1}^{N-1} (r_k^2 \delta_{ij} - r_{ki} r_{kj}),$$  \hfill (6.7)

where $r_k = (r_{k1}, r_{k2}, r_{k3})$ in body coordinates. With the so called gauge potential

$$A_{\mu}(q) = M^{-1}(q) \cdot \sum_{i=1}^{N-1} \left( r_i \times \frac{\partial r_i}{\partial q_{\mu}} \right)$$  \hfill (6.8)

and the metric

$$g_{\mu\nu}(q) = \frac{\partial r_{\alpha}(q)}{\partial q_{\mu}} \frac{\partial r_{\alpha}(q)}{\partial q_{\nu}} - A_{\mu}(q) \cdot M(q) \cdot A_{\nu}(q)$$  \hfill (6.9)

(where here and in the following we use the Einstein convention of summation over repeated indices) the kinetic energy becomes

$$K = \frac{1}{2} (\Omega + A_{\mu} \dot{q}_{\mu}) \cdot M \cdot (\Omega + A_{\nu} \dot{q}_{\nu}) + \frac{1}{2} g_{\mu\nu} \dot{q}_{\mu} \dot{q}_{\nu}.$$  

Here $\Omega$ is the angular velocity which is the vector corresponding to the skew-symmetric matrix $R^T \dot{R}$ by the isomorphism
Chapter 6. Bifurcation of the Hill Regions

\[
\begin{pmatrix}
0 & -\Omega_3 & \Omega_2 \\
\Omega_3 & 0 & -\Omega_1 \\
-\Omega_2 & \Omega_1 & 0
\end{pmatrix} \mapsto \begin{pmatrix}
\Omega_1 \\
\Omega_2 \\
\Omega_3
\end{pmatrix}.
\] (6.10)

By using the equation
\[
J = \frac{\partial K}{\partial \Omega} = M(\Omega + A_\mu \dot{q}_\mu),
\] (6.11)
the conjugate momenta of the internal space coordinates are obtained to be
\[
p_\mu = \frac{\partial K}{\partial \dot{q}_\mu} = g_{\mu\nu} \dot{q}_\nu + J \cdot A_\mu.
\] (6.12)

Definition 6.1. The reduced ro-vibrational Hamiltonian is defined as
\[
H(q, p, J) = \frac{1}{2} J \cdot M^{-1} \cdot J + \frac{1}{2} g^{\mu\nu} (p_\mu - J \cdot A_\mu)(p_\nu - J \cdot A_\nu) + V(q_1, ..., q_{3N-6}),
\] (6.13)
where in order to keep the notation reasonably short we omitted the argument \(q\) for \(M, A_\mu,\) and \(g^{\mu\nu}\) even though these are functions of the internal coordinates (see Equations (6.7) (6.8) (6.9)). The first term on the right hand side of (6.13) is called the rotational or centrifugal kinetic energy and the second term is called the vibrational kinetic energy.

The equations of motion are given by
\[
\dot{q}_\mu = \frac{\partial H}{\partial p_\mu}, \quad \dot{p}_\mu = -\frac{\partial H}{\partial q_\mu}, \quad \dot{J} = J \times \nabla_J H,
\] (6.14)
where \(\nabla_J H = (\partial_J J_1 H, \partial_J J_2 H, \partial_J J_3 H), J = (J_1, J_2, J_3)\) and \(\mu = 1, ..., 3N - 6\).

The magnitude \(r = \|J\|\) of the body angular momentum \(J\) is a constant of motion. The phase space of the reduced system with a magnitude of the total angular momentum equal to \(r\) then has the structure of a product space given by the product of the angular momentum sphere
\[
S_r^2 = \{ J \in \mathbb{R}^3 | J_1^2 + J_2^2 + J_3^2 = r^2 \}
\] (6.15)
and the cotangent bundle over the internal space, \(T^*Q\), with coordinates \((q_\mu, p_\mu), \mu = 1, ..., 3N - 6\). As the angular momentum sphere is two-dimensional, the reduced system can be viewed to have 1 rotational degree of freedom and \(3N - 6\) vibrational degrees of freedom. The rotational and vibrational degrees of freedom are coupled via the gauge potentials (see Eq. 6.13) which give rise to Coriolis terms in the equations of motion.
6.3 Reduction of 3-body systems

In the following we make the reduction described in the previous section more concrete for the case of 3-body systems. Consider a system of three bodies with masses \( m_1, m_2 \) and \( m_3 \) and position vectors \( x_1, x_1, x_1 \in \mathbb{R}^3 \). We define mass-weighted Jacobi vectors according to

\[
\begin{align*}
    s_1 &= \sqrt{\mu_1} (x_1 - x_3), \\
    s_2 &= \sqrt{\mu_2} \left( x_2 - \frac{m_1 x_1 + m_3 x_3}{m_1 + m_3} \right),
\end{align*}
\]

where

\[
\begin{align*}
    \mu_1 &= \frac{m_1 m_3}{m_1 + m_3}, \\
    \mu_2 &= \frac{m_2 (m_1 + m_3)}{m_1 + m_2 + m_3}
\end{align*}
\]

are the reduced masses of the two-body systems with masses \( m_1 \) and \( m_3 \) and \( m_2 \) and \( m_1 + m_3 \), respectively (see Fig. 6.1).

As mentioned in Sec. 6.2 we can view the space \( \mathcal{J} = \mathbb{R}^3 \times \mathbb{R}^3 \) of Jacobi vectors \( s_1 \) and \( s_2 \) as the translation reduced configuration space. Viewing the Jacobi vectors as column vectors of \( 3 \times 2 \) matrices we can identify the configuration space with the space of \( 3 \times 2 \) matrices \( \mathbb{R}^{3 \times 2} \). The translation reduced configuration space \( \mathcal{J} \) can then be viewed as the disjoint union \[26\]

\[
\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \mathcal{J}_2,
\]

where for \( k = 0, 1, 2 \),

\[
\mathcal{J}_k := \{ A \in \mathbb{R}^{3 \times 2} : \text{rank} A = k \}.
\]

Here \( \mathcal{J}_2 \) contains the non-collinear configurations, \( \mathcal{J}_1 \) contains the collinear configurations, and \( \mathcal{J}_0 \) the triple collision (which is the centre of mass located at the origin). We note that for systems with more than three particles, \( \mathcal{J} \) in \[6.17\] also contains the union with \( \mathcal{J}_3 \) (defined in an analogous way). For three particles, \( \mathcal{J}_3 \) is empty. Moreover, \( \mathcal{J}_2 \) is a smooth manifold whose boundary is formed by \( \mathcal{J}_0 \cup \mathcal{J}_1 \), i.e. \( \partial \mathcal{J}_2 = \mathcal{J}_0 \cup \mathcal{J}_1 \).

The collision set is given by

\[
\Delta_c = \{ (s_1, s_2) \in \mathbb{R}^3 \times \mathbb{R}^3 : s_1 = 0 \text{ or } s_2 = \frac{\sqrt{\mu_2}}{\sqrt{\mu_1}} \frac{m_1}{m_1 + m_3} s_1 \text{ or } s_2 = \frac{\sqrt{\mu_2}}{\sqrt{\mu_1}} \frac{m_3}{m_1 + m_3} s_1 \},
\]

(6.19)
where the conditions defining the set correspond to collisions of particles 1 and 3, 2 and 3, and 1 and 2, in this order. The collision set is contained in the boundary of \( J_2 \), i.e. \( \Delta_c \subset \partial J_2 \). Let

\[
\Delta_{c,k} := \Delta \cap J_k
\]

for \( k = 0, 1, 2 \). Then \( \Delta_{c,2} \) is empty, and \( \Delta \) equals the disjoint union of \( \Delta_{c,1} \) and \( \Delta_{c,0} \), where \( \Delta_{c,1} \) contains the double collisions which are no triple collisions and \( \Delta_{c,0} \) contains the triple collision.

The rotation group \( SO(3) \) acts on \( J \) according to

\[
(s_1, s_2) \mapsto (Rs_1, Rs_2), \quad R \in SO(3).
\]

The quotient space \( J/\text{SO}(3) \) which consists of the equivalence classes of configurations that can be mapped to one another via a rotation \( R \in SO(3) \) is called the *internal space*. Let \( \pi : J \to J/\text{SO}(3) \) denote the quotient map. For \( (s_1, s_2) \in J \), let \( \mathcal{O}_{(s_1, s_2)} := \{(Rs_1, Rs_2) : R \in SO(3)\} \) be the \( SO(3) \) orbit through \( (s_1, s_2) \) and \( G_{(s_1, s_2)} := \{R \in SO(3) : (Rs_1, Rs_2) = (s_1, s_2)\} \) the isotropy group at \( (s_1, s_2) \). Then

\[
G_{(s_1, s_2)} = \begin{cases} 
\{e\} & \text{for } (s_1, s_2) \in J_2, \\
SO(2) & \text{for } (s_1, s_2) \in J_1, \\
SO(3) & \text{for } (s_1, s_2) \in J_0,
\end{cases}
\]

and

\[
\mathcal{O}_{(s_1, s_2)} = \begin{cases} 
SO(3) & \text{for } (s_1, s_2) \in J_2, \\
S^2 & \text{for } (s_1, s_2) \in J_1, \\
\{0\} & \text{for } (s_1, s_2) \in J_0.
\end{cases}
\]

The configuration space can hence be viewed to be stratified into three strata defined via the orbit type, i.e., \( J = J_2 \cup J_1 \cup J_0 \), and the projection is similarly stratified according to

\[
J_2 \to J_2/\text{SO}(3), \quad J_1 \to J_1/\text{SO}(3), \quad \text{and } J_0 \to J_0/\text{SO}(3).
\]

As coordinates on the internal space we can take the *Jacobi coordinates* \( (\rho_1, \rho_2, \phi) \) defined as

\[
\rho_1 = \|s_1\|, \quad \rho_2 = \|s_2\|, \quad s_1 \cdot s_2 = \rho_1 \rho_2 \cos \phi,
\]

where \( 0 \leq \phi \leq \pi \) (see Fig. 6.1).

Let \( \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \) be the standard basis in \( \mathbb{R}^3 \). Then we can define a section \( \sigma : J/\text{SO}(3) \to J \) as

\[
\sigma(\rho_1, \rho_2, \phi) = (\bar{r}_1, \bar{r}_2) := (\rho_1 \hat{e}_1, \rho_2 \cos \phi \hat{e}_1 + \rho_2 \sin \phi \hat{e}_2).
\]

This section is called the *xyx* gauge in \( [30] \) as it corresponds to the choice of a body frame where two bodies (bodies 1 and 3 in our case) are located on the \( x \)
6.3. Reduction of 3-body systems

![Diagram of mass weighted Jacobi vectors](Image)

**Figure 6.1:** Directions of the mass weighted Jacobi vectors $s_1$ and $s_2$. The vector $s_2$ has its tail at the centre of mass of the particles 1 and 3. The Jacobi coordinates $\rho_1$ and $\rho_2$ are the lengths of the vectors $s_1$ and $s_2$, respectively, and $\phi$ is the angle between the two vectors.

axis and the third body (body 2) is contained in the $xy$ plane. The mass weighted Jacobi vectors $(s_1, s_2)$ of the three-body system are then given by

$$ (s_1, s_2) = (R \bar{r}_1, R \bar{r}_2) = (\rho_1 R \bar{\xi}_1, \rho_2 \cos \phi R \bar{\xi}_1 + \rho_2 \sin \phi R \bar{\xi}_2) \quad (6.26) $$

for some $R \in SO(3)$.

Collinear configurations are given in terms of the Jacobi coordinates by either of the equalities $\phi = 0$, $\phi = \pi$ or $\rho_2 = 0$. For $\rho_1 = 0$ (in which case $\phi$ is not defined), particles 1 and 3 collide. For $\phi = 0$ combined with $\rho_2 = \frac{m_3}{m_1 + m_3} \sqrt{\mu_2} \rho_1$, particles 1 and 2 collide. For $\phi = \pi$ combined with $\rho_2 = \frac{m_1}{m_1 + m_3} \sqrt{\mu_2} \rho_1$, particles 1 and 3 collide. At the triple collision $\rho_1 = \rho_2 = 0$.

Besides the Jacobi coordinates another natural choice of coordinates is given by the inter particle distances $d_{12}$, $d_{13}$ and $d_{23}$ which besides being nonnegative need to satisfy the triangle inequality $d_{12} + d_{13} \geq d_{23}$ and its cyclic permutations. Collinearity is given by equality in the respective triangle inequality. Double collisions between two particles are obviously given by the corresponding distance being zero. At the triple collision all distances are vanishing.

As will become clear below for the discussion of the Hill regions, it is useful to introduce yet another coordinate system. To this end we first define

$$ (w_1, w_2, w_3) = (\rho_1^2 - \rho_2^2, 2 \rho_1 \rho_2 \cos \phi, 2 \rho_1 \rho_2 \sin \phi), \quad (6.27) $$

where $w_1, w_2 \in \mathbb{R}$ and $w_3 \geq 0$. Equation (6.27) shows that the Jacobi coordinates are confocal parabolic coordinates in the space of the coordinates $(w_1, w_2, w_3)$. The coordinate $w_3$ is twice the area of the parallelogram spanned by the Jacobi
vectors \( s_1 \) and \( s_2 \). This implies that collinear configurations are contained in the plane \( w_3 = 0 \). This plane hence also contains the collisions. As double collisions of particles of particles 1 and 3 have \( \rho_1 = 0 \) we see from (6.27) that these are located on the negative \( w_1 \) axis. Double collisions of particles 1 and 2 occur on the line in the plane \( w_3 = 0 \) where

\[
\frac{w_2}{w_1} = \frac{2 \sqrt{m_1 m_2 m_3 \sqrt{m_1 + m_2 + m_3}}}{m_1 (m_1 + m_2 + m_3) - m_2 m_3}. \tag{6.28}
\]

Similarly collisions of particles 2 and 3 occur in this plane at

\[
\frac{w_2}{w_1} = -\frac{2 \sqrt{m_1 m_2 m_3 \sqrt{m_1 + m_2 + m_3}}}{m_3 (m_1 + m_2 + m_3) - m_1 m_2}. \tag{6.29}
\]

The triple collision is located at the origin \( w_1 = w_2 = w_3 = 0 \).

The final coordinate system we are considering is then given by spherical co-ordinates in the \((w_1, w_2, w_3)\) coordinate space which give the Dragt’s coordinates \((\omega, \chi, \psi)\) defined as (see [30] and the references therein)

\[
(w_1, w_2, w_3) = \left(\omega \cos \chi \cos \psi, \omega \cos \chi \sin \psi, \omega \sin \chi \right), \tag{6.30}
\]

where \( \omega \geq 0, \ 0 \leq \chi \leq \pi/2 \) and \( 0 \leq \psi \leq 2\pi \). Note that \( \chi \) is the latitude, not the colatitude, and \( \omega = \rho_1^2 + \rho_2^2 \).

For completeness, we also give the expressions for the inter particle distances in terms of Dragt’s coordinates:

\[
d_{12} = \frac{1}{\sqrt{2\mu_1}} \sqrt{\omega + \omega \cos \chi \cos \psi},
\]

\[
d_{13} = \sqrt{\frac{\mu_1}{2m_1^2}} (\omega + \omega \cos \chi \cos \psi) + \frac{1}{2\mu_2} (\omega - \omega \cos \chi \cos \psi) - \frac{\sqrt{\mu_1}}{m_1 \sqrt{\mu_2}} \omega \cos \chi \sin \psi,
\]

\[
d_{23} = \sqrt{\frac{\mu_1}{2m_3^2}} (\omega + \omega \cos \chi \cos \psi) + \frac{1}{2\mu_2} (\omega - \omega \cos \chi \cos \psi) + \frac{\sqrt{\mu_1}}{m_3 \sqrt{\mu_2}} \omega \cos \chi \sin \psi.
\]

By choosing the internal coordinates as the Jacobi coordinates \((\rho_1, \rho_2, \phi)\) the inertia and metric tensors become [30]

\[
M = \begin{bmatrix}
\rho_2^2 \sin^2 \phi & -\rho_2^2 \sin \phi \cos \phi & 0 \\
-\rho_2^2 \sin \phi \cos \phi & \rho_1^2 + \rho_2^2 \cos^2 \phi & 0 \\
0 & 0 & \rho_1^2 + \rho_2^2
\end{bmatrix}, \tag{6.32}
\]
6.3. Reduction of 3-body systems

Figure 6.2: The coordinate surface $\omega = 1$ of the Dragt’s coordinates in the space of the inter particle distances $d_{12}$, $d_{13}$ and $d_{23}$. This surface can be viewed as the shape space $\tilde{Q}$ defined in (6.49).
and

\[
[g_{\mu\nu}] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{\rho_1^2 \rho_2^2}{\rho_1^2 + \rho_2^2}
\end{bmatrix}.
\]  \hspace{1cm} (6.33)

The gauge potential is

\[
A_{\rho_1} = A_{\rho_2} = (0, 0, 0), \quad A_\phi = (0, 0, \frac{\rho_2^2}{\rho_1^2 + \rho_2^2}).
\]  \hspace{1cm} (6.34)

The big advantage of Dragt’s coordinates is that the inertia and metric tensors are diagonal [30]:

\[
M = \begin{bmatrix}
\omega \sin^2 \frac{\chi}{2} & 0 & 0 \\
0 & \omega \cos^2 \frac{\chi}{2} & 0 \\
0 & 0 & \omega
\end{bmatrix}
\]  \hspace{1cm} (6.35)

and

\[
[g_{\mu\nu}] = \frac{1}{4} \begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\omega} & 0 \\
0 & 0 & \omega \cos^2 \chi
\end{bmatrix},
\]  \hspace{1cm} (6.36)

In particular the principal moments of inertia \(M_1 = \omega \sin^2 \frac{\chi}{2}, \ M_2 = \omega \cos^2 \frac{\chi}{2}\) and \(M_3 = \omega\) are ordered by magnitude on the diagonal of \(M\) (note that \(0 \leq \chi \leq \pi/2\)).

The gauge potential becomes in this case

\[
A_\omega = A_\chi = (0, 0, 0), \quad A_\psi = (0, 0, -\frac{1}{2} \sin \chi).
\]  \hspace{1cm} (6.37)

For later purposes, we define \(I\) as half the trace of the inertia tensor \(M\) which in Jacobi and Dragt’s coordinates, respectively, is then given by

\[
I := \frac{1}{2} \text{tr} M = \rho_1^2 + \rho_2^2 = \omega.
\]  \hspace{1cm} (6.38)

In the celestial mechanics literature \(I\) is referred to as moment of inertia (see, e.g., [37]). More precisely it is the moment of inertia with respect to rotations about the axis that contains the centre of mass and is perpendicular to the plane in which the three bodies are lying.

Remark. From (6.35) we see the following.

1. For \(\chi = 0\), i.e. \(w_3 = 0\) and \(w_1^2 + w_2^2 = \omega^2\) which corresponds to collinear configurations, the middle and the biggest principal moments of inertia become both equal to \(\omega\) and the smallest principal moment of inertia is equal to zero.
6.4 Charged 3-body systems

2. In order to have equality of the smallest and middle principal moments of inertia we need \( \sin^2 \frac{\chi}{2} = \cos^2 \frac{\chi}{2} \) which gives \( \chi = \pi/2 \), i.e. \((w_1, w_2, w_3) = (0, 0, \omega)\).

3. The only way to have equality of all three principal moments of inertia is to have \( \omega = 0 \) which corresponds to a triple collision.

6.4 Charged 3-body systems

In this chapter we are considering 3-body systems with Hamiltonians of the form

\[
H(x, p) = \frac{1}{2} p^T M^{-1} p + V(x),
\]

where \( V = V_{\text{Newton}} + V_{\text{Coulomb}} \) with

\[
V_{\text{Newton}}(x) = - \sum_{1 \leq i < j \leq 3} G \frac{m_i m_j}{\|x_i - x_j\|},
\]

\[
V_{\text{Coulomb}}(x) = \sum_{1 \leq i < j \leq 3} \frac{1}{4\pi\varepsilon_0} \frac{Q_i Q_j}{\|x_i - x_j\|}.
\]

Here \( G \) is the gravitational constant, \( \frac{1}{4\pi\varepsilon_0} \) is the Coulomb force constant, and \( Q_i \) is the charge of the \( i \)th particle, \( i = 1, 2, 3 \).

We will use atomic units which are defined as follows:

- unit of length: \( a_0 = 5.291772192(17) \times 10^{-11} \) m (which is called ‘bohr’),
- unit of mass: \( m_e = 9.10938291(40) \times 10^{-31} \) kg (electron mass),
- unit of time: \( \hbar/E_h = 2.418884326505(16) \times 10^{-17} \) s (where \( \hbar \) is Planck’s constant divided by \( 2\pi \) and \( E_h \) is a unit of energy called ‘hartree’),
- unit of charge: \( e = 1.602176565(35) \times 10^{-19} \) C (elementary charge).

In these units

- the Coulomb force constant \( \frac{1}{4\pi\varepsilon_0} \) has the value 1,
- the gravitational constant \( G \) has the value \( 2.400446611 \times 10^{-43} \).

This means that for charged 3-body system (of reasonable mass), the gravitational interaction can safely be neglected. We still the gravitational interaction as we will also consider the example of a gravitational 3-body system (without charges) for illustration.
6.5 Dilation symmetry and the shape-orientation space

The potential \( V \) in (6.39) and the inertia tensor \( M \) are homogenous functions of the distances between the particles of degree \(-1\) and \(2\), respectively. We make this more formal by defining an \( \mathbb{R} \) action as follows.

**Definition 6.2.** The dilation transformation is the \( \mathbb{R} \) action defined as
\[
d : \mathbb{R} \times Q \to Q, \quad (l, (s_1, s_2)) \mapsto (e^l s_1, e^l s_2). \tag{6.42}
\]
We write
\[
d_\lambda (s_1, s_2) = (\lambda s_1, \lambda s_2), \tag{6.43}
\]
where \( \lambda = e^l > 0 \).

The dilation transformation defines a singular line bundle \( J \to J/\mathbb{R} \). The only point with nontrivial isotropy is the triple collision point. The bundle \( (J_2 \cup J_1) \to (J_2 \cup J_1)/\mathbb{R} \) is smooth. The dilation \( \mathbb{R} \) action commutes with the \( SO(3) \) action in (6.21), i.e. \( d_\lambda (Rs_1, Rs_2) = (\lambda Rs_1, \lambda Rs_2) = (R\lambda s_1, R\lambda s_2) \) for all \( \lambda > 0 \) and \( R \in SO(3) \). As we can identify the internal space \( Q = J/\mathbb{R} \) with the (image of the) section given by the \( xxy \) gauge the dilation transformation also ‘induces’ a map on the internal space \( Q \). From (6.25) we get
\[
d_\lambda (\bar{r}_1, \bar{r}_2) = (\lambda \rho_1 \bar{\varepsilon}_1, \lambda \rho_2 \cos \phi \bar{\varepsilon}_1 + \lambda \rho_2 \sin \phi \bar{\varepsilon}_2), \tag{6.44}
\]
i.e. in Jacobi coordinates the induced dilation map \( Q \to Q, q \mapsto d_\lambda (q) \) reads
\[
d_\lambda (\rho_1, \rho_2, \phi) = (\lambda \rho_1, \lambda \rho_2, \phi). \tag{6.45}
\]
In terms of Dragt’s coordinates the map becomes
\[
d_\lambda (\omega, \chi, \psi) = (\lambda^2 \omega, \chi, \psi). \tag{6.46}
\]
In order to avoid a cumbersome notation we here use the same symbol for the induced map as in (6.43).

Homogeneity of \( V \) and \( M \) now means that
\[
V(d_\lambda (q)) = \lambda^{-1} V(q) \quad \text{and} \quad M(d_\lambda (q)) = \lambda^2 M(q). \tag{6.47}
\]
Note that for the moment of inertia \( I \) defined in (6.38), we similarly have
\[
I(d_\lambda (q)) = \lambda^2 I(q). \tag{6.48}
\]
In fact the (induced) dilation map also defines a line bundle \( Q \to Q/\mathbb{R} \). For this bundle, we can construct a section by noting that except for the triple collision
6.5. Dilation symmetry and the shape-orientation space

point we can find for each point $q \in Q$ a $\lambda > 0$ such that for the moment of inertia defined in (6.38) we have $I(d_\lambda(q)) = 1$. We can then identify the quotient space $Q/\mathbb{R}$ with this section.

**Definition 6.3.** The shape space is the dilation reduced internal space given by

$$\tilde{Q} := \{ q \in Q : I(q) = 1 \}. \quad (6.49)$$

The points in the shape space are referred to as shapes and denoted by $\tilde{q}$.

Note that the condition $I(q) = 1$ excludes the triple collision.

We can identify the shape space $\tilde{Q}$ with the upper hemisphere in the coordinate space $(w_1, w_2, w_3)$ on which we can use the Dragt’s coordinates $(\chi, \psi)$ as coordinates. For comparison, $\hat{Q}$ is shown in the space of inter particle distances in Fig.6.2. The orbits of the dilation $\mathbb{R}$ action in the internal spaces with coordinates $(w_1, w_2, w_3)$ and $(d_{12}, d_{13}, d_{23})$ are straight line rays emanating from (but not including) the origin which corresponds to the triple collision point. In view of the definition of the shape space according to (6.49), we can view the shape space to form a submanifold of the internal space and this way also can define the action of the dilation transformation $d_\lambda$ on points $\tilde{q} \in \tilde{Q}$. For $q = \tilde{q}$ with $q \in Q$, we set $d_\lambda(\tilde{q}) := d_\lambda(q)$. Let

$$\tilde{V} := V|_{\tilde{Q}} \quad \text{and} \quad \tilde{M} := M|_{\tilde{Q}}. \quad (6.50)$$

Then for $\tilde{q} \in \tilde{Q}$,

$$V(d_\lambda(\tilde{q})) = \lambda^{-1} \tilde{V}(\tilde{q}) \quad \text{and} \quad M(d_\lambda(\tilde{q})) = \lambda^2 \tilde{M}(\tilde{q}). \quad (6.51)$$

In addition we define the normalised body angular momenta

$$\tilde{J} = \frac{1}{\|J\|} J \quad (6.52)$$

and the normalised angular momentum sphere

$$S_1^2 = \{ \tilde{J} \in \mathbb{R}^3 : \tilde{J}_1^2 + \tilde{J}_2^2 + \tilde{J}_3^2 = 1 \}. \quad (6.53)$$

Whereas the shape space determines the shape formed by the 3 bodies the normalised angular momentum sphere $S_1$ contains information about the orientation of the 3-body system in space.

We make the following definition.

**Definition 6.4.** The shape-orientation space is the product

$$\tilde{Q} \times S_1^2. \quad (6.54)$$

Points in this product are referred to as shape-orientation points and denoted as $(\tilde{q}, \tilde{J})$. 
6.6 Hill regions

For a given energy $E$, the Hill region of a Hamiltonian system whose phase space is a cotangent bundle over configuration space is in general defined as the projection of the energy surface to configuration space. As the angular momentum is conserved for $N$-body systems it is useful to not only consider the projection of the energy surface but the projection of the integral manifold where both the energy and the angular momentum are fixed. Furthermore it is useful to consider the projection of the integral manifold to a space reduced by the symmetries of translations and rotations and, for charged $N$-body systems, also by dilations. The reduced phase space is then however no longer a cotangent bundle and it needs to be defined what a Hill region should be in this case. We do this as follows.

**Definition 6.5.** For given value $E$ of the energy and magnitude $r > 0$ of the angular momentum (i.e. $r = \|L\| > 0$), we say that a shape-orientation point $(\tilde{q}, \tilde{J})$ is in the Hill region if for the Hamiltonian function $H(q,p,J)$ in (6.13), there exists a $\lambda > 0$ and $p$ such that

$$H(d_\lambda(\tilde{q}), p, r\tilde{J}) = E.$$  

(6.55)

For computations, it is useful to rephrase this definition as follows.

**Lemma 6.6.** For given values of the energy $E = H$ and magnitude of the angular momentum $r = \|L\| > 0$, a shape-orientation point $(\tilde{q}, \tilde{J})$ is in the Hill region if and only if there exists a $\lambda > 0$ such that

$$F(\lambda) := \lambda^2 E - r^2 \frac{1}{2} J \cdot [\tilde{M}(\tilde{q})]^{-1} \cdot \tilde{J} - \lambda \tilde{V}(\tilde{q}) \geq 0,$$  

(6.56)

where $\tilde{V}$ and $\tilde{M}$ are the restrictions of $V$ and $M$ to the shape space (see (6.50)).

**Proof.** As the metric $g^{\mu\nu}$ is positive definite the vibrational kinetic energy (the second term in the Hamiltonian in (6.13)) is nonnegative, we can satisfy the energy equation (6.55) for some $\lambda > 0$ and $p$ if and only if

$$r^2 \frac{1}{2} J \cdot [M(d_\lambda(\tilde{q}))]^{-1} \cdot \tilde{J} + V(d_\lambda(\tilde{q})) \leq E.$$  

(6.57)

By (6.51) this is equivalent to

$$r^2 \lambda^{-2} \frac{1}{2} J \cdot [\tilde{M}(\tilde{q})]^{-1} \cdot \tilde{J} + \lambda^{-1} \tilde{V}(\tilde{q}) \leq E.$$  

(6.58)

Now the inequality $F(\lambda) \geq 0$ in (6.56) is obtained from multiplying (6.58) by $\lambda^2$ and reordering terms. □
6.6. Hill regions

From Lemma 6.6, we see that in order to decide whether a shape orientation point \((\tilde{q}, \tilde{J})\) is in the Hill region we need to study the polynomial \(F(\lambda)\) with its coefficients being fixed by the given \((\tilde{q}, \tilde{J})\). Let

\[
E_R = \frac{1}{2} \tilde{J} \cdot \tilde{M}^{-1} \cdot \tilde{J}
\]

denote the rotational kinetic energy. For \(r > 0\), \(E_R\) is strictly positive. The discriminant of \(F(\lambda)\) is then

\[
\Delta = 4 E E_R + \tilde{V}^2.
\]

The equation of vanishing discriminant, \(\Delta = 0\), defines a cone in the space \((E, \tilde{V}, E_R)\), see Fig. 6.3a, of which only the part where \(E_R \geq 0\) is relevant. For a fixed value of \(E_R\), the zero discriminant defines a parabola in the \((E, \tilde{V})\) plane (see Fig. 6.3b). For \(E_R \to 0\), the parabolas collapse to the negative \(E_R\)-axis. For \(E_R \to \infty\), the parabolas approach the \(V\)-axis. For fixed \(E_R > 0\), the parabola together with the coordinate axes divides the \((E, \tilde{V})\)-plane into the six regions I, IIa, IIb, IIIa, IIIb and IV marked in Fig. 6.3b. In the first quadrant of the \((E, \tilde{V})\)-plane (region I), \(F\) has real roots of which one is positive and the other is negative and \(F\) has a minimum that is attained at a positive value of \(\lambda\). The second quadrant consists of regions IIa and IIb. In region IIa, \(F\) has two negative real roots and a maximum attained at a negative value. In region IIb the roots are complex but \(F\) still has a maximum attained at a negative value. The third quadrant consists of regions IIIa and IIIb where in region IIIb the roots are again complex and \(F\) still has a maximum which is however now attained at a positive \(\lambda\). In region IIIa there are two positive real roots and \(F\) has a maximum that is attained for a positive \(\lambda\). In region IV in the fourth quadrant \(F\) has two real roots of which one is negative and one is positive and \(F\) has a minimum that is attained at a negative value of \(\lambda\).

We deduce

**Theorem 6.7.** Let

\[
\lambda_{\pm} := \frac{\tilde{V}}{2E} \pm \sqrt{\frac{\Delta}{4E^2}}
\]

be the roots of \(F(\lambda)\) defined in (6.56) and \(r > 0\). Then we have:

1. For energies \(E > 0\), every shape-orientation point belongs to the Hill region.

2. For energies \(E < 0\), a shape-orientation point with \(\tilde{V} > 0\) is for no orientation in the Hill region.

3. For energies \(E < 0\), a shape-orientation point with \(\tilde{V} < 0\) is in the Hill region if and only if the roots \(\lambda_{\pm}\) are real.
Chapter 6. Bifurcation of the Hill Regions

It follows from Theorem 6.7 that the case 3 where \( E, \tilde{V} < 0 \) requires more attention. For \( E, \tilde{V} < 0 \), the condition to have real roots, \( \Delta \geq 0 \), is equivalent to

\[
\frac{\tilde{V}}{2 \sqrt{\frac{1}{2} \tilde{J} \cdot \tilde{M}^{-1} \cdot \tilde{J} + 1 \cdot \tilde{J}}} \leq -\sqrt{-E r^2}. \tag{6.62}
\]

With the left hand side of (6.62) we can define the function

\[
\tilde{Q} \times S^2_1 \rightarrow \mathbb{R}, \quad (\tilde{q}, \tilde{J}) \mapsto \frac{\tilde{V}(\tilde{q})}{2 \sqrt{\frac{1}{2} \tilde{J} \cdot (\tilde{M}(\tilde{q}))^{-1} \cdot \tilde{J}}}.
\]

The Hill region for given \( E < 0 \) and \( r > 0 \) is then the region in the shape-orientation space \( \tilde{Q} \times S^2_1 \) that is enclosed by the level set corresponding to the value \(-\sqrt{-E r^2}\) of this function. To study the bifurcations of the Hill region we need to study the critical points of the function (6.63). The critical values can be expressed in terms of

\[
\nu := -E r^2
\]

which we can hence view as the bifurcation parameter.

As \( \tilde{V} \) does not depend on \( \tilde{J} \) the \( \tilde{J} \) components of the critical points of the function defined in (6.63) are given by the critical points of the function

\[
\tilde{J} \mapsto \frac{1}{2} \tilde{J} \cdot [\tilde{M}(\tilde{q})]^{-1} \cdot \tilde{J}
\]

on the sphere \( \tilde{J} \cdot \tilde{J} = 1 \) for fixed shape \( \tilde{q} \in \tilde{Q} \). A given shape \( \tilde{q} \) can be considered to define a rigid body with moment of inertia tensor \( \tilde{M} \) (where the trace of \( \tilde{M} \) is equal to 1). If the principal moments of inertia are different then the rigid body is asymmetric. For shape coordinates for which the moment of inertia tensor is diagonal like Dragt’s coordinates, the critical values of \( \tilde{J} \) on the normalised angular momentum sphere \( S^2_1 \) are then given by \( \tilde{J} \) equal to \( (\pm 1, 0, 0), (0, \pm 1, 0) \) or \( (0, 0, \pm 1) \). The critical shapes \( \tilde{q} \) are then critical points of the functions

\[
\tilde{q} \mapsto \sqrt{\tilde{M}_k(\tilde{q}) \tilde{V}(\tilde{q})}, \quad k = 1, 2, 3, \tag{6.66}
\]

where the \( \tilde{M}_k(\tilde{q}) \) are the dilation reduced principal moments of inertia, i.e., the eigenvalues of \( \tilde{M}(\tilde{q}) \). If we use Dragt’s coordinates \( (\chi, \psi) \) as shape coordinates then the critical shape-orientations points \( (\tilde{q}, \tilde{J}) \) are given by the critical points of the functions

\[
(\chi, \psi) \mapsto \begin{cases} 
\tilde{V} \sin \frac{\chi}{2} & \text{for } \tilde{J} = (\pm 1, 0, 0) \\
\tilde{V} \cos \frac{\chi}{2} & \text{for } \tilde{J} = (0, \pm 1, 0) \\
\tilde{V} & \text{for } \tilde{J} = (0, 0, \pm 1) 
\end{cases} \tag{6.67}
\]
6.6. Hill regions

Figure 6.3: Surface of vanishing discriminant of $F(\lambda)$ defined in (6.56) in the space $(E, \tilde{V}, E_R)$ where $E_R = r^2 \frac{1}{2} \tilde{J} \cdot \mathbf{M}^{-1} \cdot \tilde{J}$ is the rotational energy (a). In panel (b) a cut for fixed $E_R = 1$ in (a) is shown. Together with the coordinate axes it divides the $(E, \tilde{V})$ plane into six regions I, IIa, IIb, IIIa, IIIb and IV. The remaining panels show the graphs of $F : \lambda \mapsto F(\lambda)$ for values of $(E, \tilde{V})$ in these different regions.
Chapter 6. Bifurcation of the Hill Regions

In the examples in Sec. 6.8 we will visualise the Hill regions defined according to Definition 6.5 in terms of their projection to the shape space \( \tilde{Q} \). These projections can be understood in terms of the contours of the functions (6.66) (or (6.67) when we use Dragt’s coordinates) on \( \tilde{Q} \) as follows. Consider a fixed negative value of the energy \( E < 0 \), a fixed value of the magnitude of the angular momentum \( r > 0 \) and a given shape \( \tilde{q} \in \tilde{Q} \). Let us rewrite the condition for a shape-orientation \((\tilde{q}, \tilde{J})\) to be in the Hill region given by the inequality (6.62) as

\[
- \frac{V(\tilde{q})}{2\sqrt{-Er^2}} \geq \tilde{E}_R, \tag{6.68}
\]

where

\[
\tilde{E}_R := \frac{1}{2} \tilde{J} \cdot [\tilde{M}(\tilde{q})]^{-1} \cdot \tilde{J}. \tag{6.69}
\]

is the ‘normalised’ rotational energy which gives the rotational energy of a rigid body with moment of inertia tensor \( \tilde{M}(\tilde{q}) \) rotating with a total angular momentum of unit magnitude. In Fig. 6.4 we show the level sets of \( \tilde{E}_R \) of this rigid body on the unit angular momentum sphere \( S^2_1 \) for the case of distinct principal moments of inertia. On \( S^2_1 \) the function \( \tilde{E}_R \) has the three critical values

\[
\tilde{E}_{R3} := \frac{1}{2M_3(\tilde{q})} < \tilde{E}_{R2} := \frac{1}{2M_2(\tilde{q})} < \tilde{E}_{R1} := \frac{1}{2M_1(\tilde{q})} \tag{6.70}
\]

corresponding to a minimum, a saddle and a maximum, respectively.

If \( -V(\tilde{q})/(2\sqrt{-Er^2}) < \tilde{E}_{R3} \) then the inequality (6.68) is not satisfied for any point \( \tilde{J} \in S^2_1 \). If \( \tilde{E}_{R3} < -V(\tilde{q})/(2\sqrt{-Er^2}) < \tilde{E}_{R2} \) then the inequality (6.68) is satisfied for points \( \tilde{J} \in S^2_1 \) in the ‘caps’ that are given by (closed) neighborhoods of the points \( \tilde{J} = (0, 0, \pm 1) \) (see Fig. 6.4). If \( \tilde{E}_{R2} < -V(\tilde{q})/(2\sqrt{-Er^2}) < \tilde{E}_{R1} \) then the inequality (6.68) is satisfied for orientations \( \tilde{J} \in S^2_1 \) in the ‘ring’ that is obtained from excluding two (open) neighbourhoods near the poles \( \tilde{J} = (\pm 1, 0, 0) \) on the unit sphere \( \tilde{J} \in S^2_1 \). If \( \tilde{E}_{R1} < -V(\tilde{q})/(2\sqrt{-Er^2}) \) then the inequality (6.68) is satisfied for any orientation \( \tilde{J} \in S^2_1 \). The latter two cases are also illustrated in Fig. 6.4.

6.7 Relative equilibria and central configurations

A natural question that arises is how the critical points discussed in the previous section are related to relative equilibria. Relative equilibria are the stationary solutions of the reduced equations of motion (6.14). They satisfy the equations
6.7. Relative equilibria and central configurations

Figure 6.4: Top panel: level sets of the function defined in (6.65) on the normalised angular momentum sphere $S^2_1$ for a fixed shape corresponding to an asymmetric Euler top (with principal moments of inertia $1/3$, $2/3$ and $1$). The function value increases as the color goes from light blue to red. Lower panel: dependence of the accessible region on the normalised angular momentum sphere on the value of $-\tilde{V}/(2\sqrt{-E_{r^2}})$ (see the discussion at the end of Sec. 6.6).
Chapter 6. Bifurcation of the Hill Regions

\[
\mathbf{J} \times (\mathbf{M}^{-1} \cdot \mathbf{J}) = 0, \quad (6.71)
\]
\[
p_\mu = \mathbf{J} \cdot \mathbf{A}_\mu, \quad (6.72)
\]
\[
\frac{\partial}{\partial q_\mu} \left( \frac{1}{2} \mathbf{J} \cdot \mathbf{M}^{-1} \cdot \mathbf{J} + V(q) \right) = 0. \quad (6.73)
\]

Equation (6.71) implies that at relative equilibria the body angular momentum \( \mathbf{J} \) is an eigenvector of the moment of inertia tensor, i.e., \( \mathbf{J} \) is parallel to a principal axis. This means that the 3-body system is rotating about one of its principal axes. Using this fact the internal coordinates of relative equilibria can be found from the critical points of the effective potential

\[
v_{\text{eff}}(q) := \frac{1}{2} \mathbf{J} \cdot [\mathbf{M}(q)]^{-1} \cdot \mathbf{J} + V(q), \quad (6.74)
\]

where \( \mathbf{J} \) is a fixed vector of a given modulus \( r \) parallel to a chosen principal axis.

For rotations about the first, second or third principal axis where \( \mathbf{J} = (\pm r, 0, 0) \), \( \mathbf{J} = (0, \pm r, 0) \) or \( \mathbf{J} = (0, 0, \pm r) \), respectively, the effective potential becomes

\[
v_{\text{eff}}(q) = \frac{1}{2} \frac{r^2}{M_k(q)} + V(q), \quad k = 1, 2, 3. \quad (6.75)
\]

The critical internal coordinates satisfy

\[
\frac{\partial V}{\partial q_\mu} = \frac{1}{2} \frac{r^2}{M_k^2} \frac{\partial M_k}{\partial q_\mu}. \quad (6.76)
\]

Filling in the critical internal coordinates into (6.72) then gives the momenta of the relative equilibria.

**Theorem 6.8.** The critical points of the function (6.63) are shape-orientation points of relative equilibria.

**Proof.** At the end of Sec. 6.6 we have seen that the \( \mathbf{J} \) components of the critical shape-orientations points of (6.63) are eigenvectors of the shape dependent moment inertia tensor \( \tilde{\mathbf{M}} \) and that the shape components are then critical points of the functions \( \sqrt{M_k} \tilde{V} \), \( k = 1, 2, 3 \), defined in (6.66), i.e.

\[
0 = \frac{\partial}{\partial q_\mu} \sqrt{M_i} \tilde{V} = \frac{1}{2} \frac{\tilde{V}}{\sqrt{M_k}} \frac{\partial \tilde{M}_k}{\partial q_\mu} + \sqrt{M_k} \frac{\partial \tilde{V}}{\partial q_\mu}, \quad \mu = 1, 2, \quad (6.77)
\]

or
6.7. Relative equilibria and central configurations

\[ \frac{\partial \tilde{V}}{\partial \tilde{q}_\mu} = -\frac{1}{2} \tilde{V} \frac{\partial \tilde{M}_k}{\partial \tilde{q}_\mu}, \quad \mu = 1, 2. \] (6.78)

We complete the shape space coordinates \( \tilde{q}_\mu, \mu = 1, 2, \) to coordinates on the internal space \( Q \) by adding the parameter \( \tilde{q}_3 = \lambda \) and noting that for each \( q \) in \( Q \), there exist a \( \lambda > 0 \) and \( \tilde{q} \in \tilde{Q} \) such that \( d_\lambda(\tilde{q}) = q \).

Dividing (6.78) by \( \lambda \) and using (6.51) we see that the equivalence of (6.76) and (6.78) requires for \( \mu = 1, 2 \) that

\[ \frac{r^2}{M_k} = -V. \] (6.79)

For \( \mu = 3 \), i.e., \( q_3 = \lambda \), the left hand side of (6.76) gives by using again (6.51)

\[ \frac{\partial V}{\partial \lambda} = \frac{1}{\lambda^2} \tilde{V} = -\frac{1}{\lambda} V. \] (6.80)

The right hand side of (6.76) gives by using again (6.51)

\[ \frac{1}{2} \frac{r^2}{M_k^2} \frac{\partial M_k}{\partial \lambda} = \frac{1}{2} \frac{r^2}{M_k^2} \lambda^2 \tilde{M}_k = \frac{1}{2} \frac{r^2}{M_k^2} 2\lambda \tilde{M}_k = \frac{r^2}{M_k^2} \lambda M_k. \] (6.81)

Equality of (6.80) and (6.81) is again equivalent to (6.79).

It hence remains to be shown that (6.79) holds. The left hand side of (6.79) can be identified with twice the kinetic energy of a relative equilibrium. Like the potential it is stationary at a relative equilibrium. The equality then follows from the Virial Theorem which says that for homogeneous potential of degree \(-1\), twice the time average of the kinetic energy equals minus the time average of the potential energy [18]. For relative equilibria, the time average is trivial as it involves averaging constant functions.

Remark. In chapter 3 we have seen that central configurations are given by the critical points of the potential \( V \) restricted to surfaces \( I = \text{const.} \) in the internal space. Given our definition of the function \( \tilde{V} \) in (6.50) the critical points of \( \tilde{V} \) are central configurations. Note that the biggest of the dilation reduced moments of inertia, \( \tilde{M}_k \), is equal to one (see (6.67)). So whereas the critical points of the functions \( \tilde{q} \mapsto \sqrt{\tilde{M}_k(\tilde{q}) \tilde{V}(\tilde{q})} \), \( k = 1, 2, 3 \), correspond according to theorem 6.8 to relative equilibria only the critical points of the function \( \tilde{q} \mapsto \sqrt{\tilde{M}_3(\tilde{q}) \tilde{V}(\tilde{q})} \) correspond to central configurations. So not all relative equilibria are related to central configurations (see chapter 3).
6.8 Examples

In the following we will give two examples of charged three-body problems: a compound of two electrons and one positron, and the helium atom consisting of a nucleus and two electrons. We start however with a gravitational three-body problem to illustrate the procedure.

6.8.1 Gravitational three-body problem

As discussed in [34] there are nine critical values of the bifurcation parameter $\nu$ for the gravitational three-body problem. They are given by

$$
\nu_2 = \frac{G^2 (m_1 m_2)^3}{2 (m_1 + m_2)}, \quad \nu_3 = \frac{G^2 (m_1 m_3)^3}{2 (m_1 + m_3)}, \quad \nu_4 = \frac{G^2 (m_2 m_3)^3}{2 (m_2 + m_3)},
$$

(6.82)

which are due to critical points at infinity,

$$
\nu_5 = \frac{G^2}{2} \left( m_1 m_2 \sqrt{\frac{m_1 m_2}{m_1 + m_2}} + m_1 m_3 \sqrt{\frac{m_1 m_3}{m_1 + m_3}} + m_2 m_3 \sqrt{\frac{m_2 m_3}{m_2 + m_3}} \right),
$$

(6.83)

corresponding to degenerate critical points resulting from restrictions on the admissible orientations (this critical value has been discussed earlier by Simo [47] and Saari [46, 44, 45] and we will get back to it below),

$$
\nu_6 = \frac{G^2}{2} \left( m_1 m_2 + m_2 m_3 + m_1 m_3 \right) \left( \frac{m_1 m_2}{d_{12}} + \frac{m_2 m_3}{d_{23}} + \frac{m_1 m_3}{d_{13}} \right)^2,
$$

(6.84)

which comes from the central configuration given by the Lagrange equilateral triangle and for $k = 7, 8, 9$,

$$
\nu_k = \frac{G^2}{2} \left( \frac{m_1 m_2 d_{12}^2 + m_2 m_3 d_{23}^2 + m_1 m_3 d_{13}^2}{m_1 + m_2 + m_3} \left( \frac{m_1 m_2}{d_{12}} + \frac{m_2 m_3}{d_{23}} + \frac{m_1 m_3}{d_{13}} \right) \right)^2,
$$

(6.85)

where the inter particle distances $d_{ij}$ are determined by the three Euler collinear central configurations.

In our example we choose the masses $m_1 = 1.6, m_2 = 1.2$ and $m_3 = 1$. Also we set $G = 1$ (which can always be achieved by a suitable scaling). The numerical values of the nontrivial critical bifurcation parameter are then

$$
\nu_2 \approx 0.3927272727, \quad \nu_3 \approx 0.7876923077, \quad \nu_4 \approx 1.263908571, \quad \nu_5 \approx 6.961348535, \quad \nu_6 \approx 13.83605894, \quad \nu_7 \approx 18.56904438, \quad \nu_8 \approx 19.12865697, \quad \nu_9 \approx 19.44296212.
$$

(6.86)
6.8. Examples

Following Sec. 6.5 we identify the shape space $\tilde{Q}$ with the upper hemisphere of the unit sphere in the $(w_1, w_2, w_3)$-space. We visualise the shape space by projecting this hemisphere to the unit disk in the $(w_1, w_2)$-plane. In Fig. 6.5 we show the contours of the functions defined in (6.66) in this projection. Recall from the discussion in Sec. 6.3 that the boundary of the unit disk in the $(w_1, w_2)$-plane corresponds to the collinear configurations. Following (6.28) and (6.29) the collision of particles 1 and 2 is located on the boundary at a polar angle of about $48^\circ$, the collision of 2 and 3 is at a polar angle of about $-71^\circ$ and the collision of 1 and 3 is at polar angle $180^\circ$. At these points the potential function $\tilde{V}$ is $-\infty$. The Euler collinear configurations are located a polar angles of approximately $117^\circ$ with particle 1 between particles 2 and 3, $-121^\circ$ with particle 3 between particles 1 and 2, and $-19^\circ$ with particle 2 between particles 1 and 3.

The critical values of $\nu$ can be identified with the following events for the contours in Fig. 6.5 upon varying $\nu$:

(i) At $\nu_2$, $\nu_3$ and $\nu_4$ the contours of $\sqrt{\tilde{M}_1 \tilde{V}}$ (see Fig. 6.5a) successively touch/detach from the boundary of the shape space.

(ii) At $\nu_5$ the contours of $\sqrt{\tilde{M}_1 \tilde{V}}$ and $\sqrt{\tilde{M}_2 \tilde{V}}$ simultaneously shrink to or emerge from, respectively, a point at the centre of the unit disk.

(iii) At $\nu_6$ the contours of $\sqrt{\tilde{M}_3 \tilde{V}}$ shrink to a point or emerge from a point close to but not at the centre of the unit disk.

(iv) At $\nu_7$, $\nu_8$ and $\nu_9$ the contours of $\sqrt{\tilde{M}_1 \tilde{V}}$ and $\sqrt{\tilde{M}_2 \tilde{V}}$ simultaneously touch/detach from the boundary of the shape space.

In addition when $\nu$ approaches $\nu_1 = 0$ from above then the contour of $\sqrt{\tilde{M}_1 \tilde{V}}$ converges to the boundary of the shape space.

In the different panels in Figs. 6.6 and 6.7 we show superpositions of the contours of the functions $\sqrt{\tilde{M}_k \tilde{V}}$, $k = 1, 2, 3$, for a representative value of $\nu$ fixed in each panel. The regions enclosed are colored according to what the accessible region on the orientation sphere is. The color code is the same as the one used in Fig. 6.4. We start with large values of $\nu$. For $\nu > \nu_9$, there is a simply connected dark grey shaded region corresponding to points for which the accessible region on the orientation sphere is empty, i.e. these shapes do not belong to the Hill region. This dark shaded region ‘spills out’ of the shape space at three places. Attached to the dark grey shaded region are three blue strips where the accessible regions on the orientation sphere are two caps for every point inside. Attached to these regions are the three red regions where for each point inside this region the accessible region on the orientation sphere is a ring. Each of these red regions contains one double collision on its boundary.
Chapter 6. Bifurcation of the Hill Regions

Figure 6.5: Gravitational three-body problem: contours $-\sqrt{2n}, n = 1, \ldots, 20$, of the functions $\sqrt{M_k V}$ defined in (6.66) with $k = 1$ (a), $k = 2$ (b) and $k = 3$ (c). The contours are shown on the shape space $\tilde{Q}$ represented as the projection of the upper hemisphere of the unit sphere in the $(w_1, w_2, w_3)$-space projected to the unit disk in the $(w_1, w_2)$-plane. The ticks on the boundary mark the double collisions (see the main text).

When $\nu$ decreases below the values $\nu_0, \nu_8$ and $\nu_7$ the dark grey shaded region successfully detaches from the boundary of the shape space such that for $\nu_0 < \nu < \nu_7$ the dark shaded region is completely contained in the interior of the shape space and the red region has become connected. At all instances a blue region is located between the dark grey region and the red region. Recall that on the boundary of the shape space which corresponds to collinear configurations the principal moments of inertia $\tilde{M}_2$ and $\tilde{M}_3$ are equal (and $\tilde{M}_1 = 0$). This is why the red and the blue contours detach from the boundary of the shape space simultaneously in Fig. 6.5 and why the blue strips in Fig. 6.6 join at $\nu_0, \nu_8$ and $\nu_7$ in the observed way.

When $\nu$ decreases to $\nu_0$ from above the dark grey region shrinks to a point which for the present choice of masses is close to but not at $(w_1, w_2, w_3) = (0, 0, 1)$ and for $\nu < \nu_0$ it has vanished. The Hill region (projected to the shape space) is hence simply connected for $\nu < \nu_6$.

The blue region is simply connected for $\nu_5 < \nu < \nu_6$. When $\nu$ decreases to $\nu_5$ the blue region shrinks to a point at $(w_1, w_2, w_3) = (0, 0, 1)$, and when $\nu$ decreases below $\nu_5$ a simply connected green region grows out of this point. For every point in the green region any point on the orientation sphere is accessible.

Recall that at $(w_1, w_2, w_3) = (0, 0, 1)$, the moments of inertia $\tilde{M}_1$ and $\tilde{M}_2$ are equal (see Remark ). In this case the full circle $\tilde{J}_1^2 + \tilde{J}_2^2 = 1, \tilde{J}_3 = 0$ consists of critical points of the function (6.65). The critical value $\nu_5$ is degenerate in this sense.
6.8. Examples

Figure 6.6: Gravitational three-body problem: superposition of the contours of the functions $\sqrt{M_k} \hat{V}$, $k = 1, 2, 3$, defined in (6.66) for specific values of $\nu$: $\nu_9 < \nu$ (a), $\nu = \nu_9$ (b), $\nu_8 < \nu < \nu_9$ (c), $\nu = \nu_8$ (d), $\nu_7 < \nu < \nu_8$ (e), $\nu = \nu_7$ (f), $\nu_6 < \nu < \nu_7$ (g), $\nu = \nu_6$ (h), and $\nu_5 < \nu < \nu_6$ (i). The contours are shown on the shape space $\mathcal{Q}$ represented in the same way as in Fig. 6.5.
Chapter 6. Bifurcation of the Hill Regions

Figure 6.7: Gravitational three-body problem: continuation of Fig. 6.6 with values of $\nu$ chosen according to $\nu = \nu_5$ (j), $\nu_5 < \nu < \nu_4$ (k), $\nu = \nu_4$ (l), $\nu_4 < \nu < \nu_3$ (m), $\nu = \nu_3$ (n), $\nu_3 < \nu < \nu_2$ (o), $\nu = \nu_2$ (p), $\nu_2 < \nu < \nu_1$ (q), and $\nu \leq \nu_1 = 0$ (r).
6.8. Examples

For \( \nu_4 < \nu < \nu_5 \), the green region is completely contained in the interior of the shape space. When \( \nu \) takes the values \( \nu_4, \nu_3 \) and \( \nu_2 \) the green region successively starts to touch the boundary of the shape space at the double collision points. At \( \nu = \nu_1 = 0 \) the green region fills the full shape space. This remains to be the case for any \( \nu \leq 0 \).

As the bifurcation scenario is partly difficult to see in Fig. 6.7 we show in Fig. 6.8 the corresponding contours of the function \( \sqrt{\tilde{M}_1 \tilde{V}} \) in the \((\chi, \psi)\)-plane, \( 0 \leq \chi \leq 2\pi, \ 0 \leq \psi \leq \pi/2 \).

We note that the results above agree with the results stated in [34].

6.8.2 Compound of two electrons and one positron

In the following we consider a system of two electrons \( e^- \) and one positron \( e^+ \). In atomic units, \( e^- \) has charge -1 and mass 1. The positron \( e^+ \) has the same mass but opposite charge. The two electrons are labeled 1 and 2. The positron is assigned the label 3.

In Fig. 6.9 we show the contours of the functions \( \sqrt{\tilde{M}_1 \tilde{V}} \), \( k = 1, 2, 3 \), defined in (6.66) analogously to Fig. 6.5 for the gravitational case. For equal masses, the double collision of particles 1 and 2 (two electrons) occurs at a polar angle of 60° and of particles 2 and 3 (electron and positron) at angle \(-60^\circ\) (see (6.28) and (6.29)). The collision of particles 1 and 3 (again electron and positron) is located at 180°. The potential \( \tilde{V} \) is \(-\infty\) at the collisions of either of the electrons with the positron (polar angles \(-60^\circ\) and \(180^\circ\)) and \(+\infty\) at the collisions of the two electrons (polar angle \(60^\circ\)).

From the contours in Fig. 6.9 we conclude that together with the critical value \( \nu_1 = 0 \) (\( \nu_1 = 0 \) is further commented on in the discussion of Fig. 6.10 below) there are four critical values for \( \nu \) caused by the following events.

(i) The contours of \( \sqrt{\tilde{M}_1 \tilde{V}} \) touch/detach from the boundary of the shape space (see Fig. 6.9(a)). This happens simultaneously at both of the symmetric double collisions points of the electrons with the positron.

(ii) The function \( \sqrt{\tilde{M}_1 \tilde{V}} \) has a critical point in the interior of the the shape space (see Fig. 6.9(a)).

(iii) The contours of \( \sqrt{\tilde{M}_2 \tilde{V}} \) and \( \sqrt{\tilde{M}_3 \tilde{V}} \) touch/detach from the boundary of the shape space at polar angle \(-120^\circ\) (see Figs. 6.9(b) and (c)). This happens for the same value of \( \nu \) as \( \sqrt{\tilde{M}_2 \tilde{V}} \) and \( \sqrt{\tilde{M}_3 \tilde{V}} \) agree on the boundary of the shape space (i.e. for collinear configurations).

From the comparison with the gravitational case we expect event (i) to be related to critical points at infinity. The critical points at infinity result from two
Figure 6.8: Gravitational three-body problem: contours of the function $\sqrt{\tilde{M}_1 \tilde{V}}$ in the $(\chi, \psi)$-plane, $0 \leq \chi \leq 2\pi$, $0 \leq \psi \leq \pi/2$, for the same values of $\nu$ as in the corresponding panels in Fig. 6.7.
6.8. Examples

a)  

b)  

c)  

Figure 6.9: Compound of two electrons and one positron: contours $-n/8$, $n = 1, \ldots, 20$, of the functions $\sqrt{M_k \tilde{V}}$ defined in (6.66) with $k = 1$ (a), $k = 2$ (b) and $k = 3$ (c). The presentation is analogously to Fig. 6.5.

charges $Q_1$ and $Q_2$ of opposite sign and masses $m_1$ and $m_2$ corotating infinitely far apart from the third particle. The resulting value for $\nu$ is easily calculated to give

$$\nu = \frac{1}{2} \mu Q_1^2 Q_2^2,$$

(6.87)

where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass. Filling in $Q_1 = 1$ and $Q_2 = +1$ and $m_1 = m_2 = 1$ gives

$$\nu_2 = \frac{1}{4}. \quad (6.88)$$

The event (ii) results from a non-collinear relative equilibrium involving rotation about the first principal axis. The corresponding value of $\nu$ is calculated to be

$$\nu_3 \approx 0.2925594730. \quad (6.89)$$

From what we learned from the gravitational case we expect event (iii) to be related to a collinear central configuration. The candidate for this configuration is the one where the positron is located right between the two electrons on a line. The corresponding value of $\nu$ can be calculated to be

$$\nu_4 = \frac{9}{4} = 2.25. \quad (6.90)$$

In the different panels in Fig. 6.10 we again show superpositions of the contours of the functions $\sqrt{M_k \tilde{V}}$, $k = 1, 2, 3$, for a representative value of $\nu$ fixed
Figure 6.10: Compound of two electrons and one positron: superposition of the contours of the functions $\sqrt{M_k \bar{V}}$, $k = 1, 2, 3$, defined in (6.66) for specific values of $\nu$: $\nu_4 < \nu$ (a), $\nu = \nu_4$ (b), $\nu_3 < \nu < \nu_4$ (c), $\nu = \nu_3$ (d), $\nu_2 < \nu < \nu_3$ (e), $\nu = \nu_2$ (f), $\nu_1 = 0 < \nu < \nu_2$ (g), $\nu = \nu_1 \leq 0$ (h), and $\nu < \nu_1 = 0$ (i). The contours are shown on the shape space $\mathcal{Q}$ represented in the same way as in Fig. 6.9.
in each panel. The regions enclosed are colored according to what the accessible region on the orientation sphere is. The presentation and color code is the same as in Figs. 6.6 and 6.7 for the gravitational case. Starting with large values of $\nu$ there is for $\nu > \nu_4$ a grey region not belonging to the Hill region that separates the accessible part of the shape space into two disconnected components. For points in the two red components, the accessible region on the orientation sphere is a ring. These two components contain on their boundary the double collisions where one of the electrons collides with the positron. Between the grey region and the red regions there are blue strips where the accessible region on the orientation sphere consists of two caps. At $\nu = \nu_4$ the boundary of the grey regions and the blue region are tangential to the boundary of the shape space at the point corresponding to the collinear central configuration which has polar angle $-120^\circ$ in the $(w_1, w_2)$-plane. For $\nu < \nu_4$ the Hill region is simply connected. For $\nu_3 < \nu < \nu_4$, the blue and the red regions are both simply connected. At $\nu = \nu_3$ a green dot emerges in the red region because of which the red region is no longer simply connected for $\nu < \nu_3$. This green dot results from a non-collinear relative equilibrium involving rotation about the first principal axes. When $\nu$ decreases from $\nu_3$ to $\nu_2$ the green region grows until it touches the boundary of the shape space for the first time at $\nu = \nu_2$. At the same value of $\nu$ the blue and the green region touch at the center of the unit disk in the $(w_1, w_2)$-plane where the smallest and the middle principal moments of inertia are equal. When $\nu$ decreases between $\nu_2$ and $\nu_1 = 0$ the red and the blue regions are shrinking due to the growing green region until at $\nu = \nu_1 = 0$ both the red region and the blue regions have vanished. Unlike the gravitational case there remains a forbidden region near the point of the electron-electron collision at polar angle $60^\circ$. For $\nu < \nu_1 = 0$, the full shape space belongs to the Hill region and, like in the gravitational case, for every shape, every orientation is possible.

### 6.8.3 Helium

We here consider the helium atom as a charged three-body system consisting of two electrons and a nucleus. The electrons have masses $m_1 = m_2 = 1$, the nucleus has mass $m_3 = 7.289.56$. The electron both have charge $-1$. The nucleus has charge $+2$.

In Fig. 6.11 we show again the contours of the functions $\sqrt{\tilde{M}_k \tilde{V}}$, $k = 1, 2, 3$, defined in (6.66) analogously to Fig. 6.5 for the gravitational case. The collision of particles 1 and 2 (two electrons) occurs at a polar angle of about $89.99214109^\circ$ and of particles 2 and 3 (electron and nucleus) at a polar angle of about $-0.01571780034^\circ$ (see (6.28) and (6.29)). The collision of particles 1 and 3 (again electron and nucleus) is located at $180^\circ$. As the double collision are close to the coordinate axes in Fig. 6.11 we do not mark them by special ticks. The
potential $\tilde{V}$ is $-\infty$ at the collisions of either of the electrons with the nucleus (polar angles $-0.01571780034^\circ$ and $180^\circ$) and $+\infty$ at the collisions of the two electrons (polar angle 89.99214109).

From the contours in Fig. 6.11 we conclude that there are together with the critical value $\nu_1 = 0$ in total 5 critical values of $\nu$ resulting from the following events.

(i) The contours of $\sqrt{M_1\tilde{V}}$ touch/detach from the boundary of the shape space (see Fig. 6.11(a)). This happens simultaneously at both of the double collision points of the electrons with the nucleus.

(ii) $\sqrt{M_1\tilde{V}}$ and $\sqrt{M_2\tilde{V}}$ simultaneously have a critical point at the center of the unit disk in the $(w_1, w_2)$-plane (see Fig. 6.11(a) and (b)).

(iii) $\sqrt{M_1\tilde{V}}$ has a critical point close to the positive $w_2$-axis away from the center of the unit disk in the $(w_1, w_2)$-plane (see Fig. 6.11(b)).

(iv) The contours of $\sqrt{M_2\tilde{V}}$ and $\sqrt{M_3\tilde{V}}$ touch/detach from the boundary of the shape space at polar angle $-120^\circ$ (see Figs. 6.9(b) and (c)). This again happens for the same value of $\nu$ as $\sqrt{M_2\tilde{V}}$ and $\sqrt{M_3\tilde{V}}$ agree on the boundary of the shape space (i.e. for collinear configurations).

In the different panels in Fig. 6.12 we again show superpositions of the contours of the functions $\sqrt{M_k\tilde{V}}$, $k = 1, 2, 3$, for a representative value of $\nu$ fixed in each panel. The presentation and color code is again the same as in Figs. 6.6 and 6.7 for the gravitational case. We again start with large values of $\nu$. The
6.8. Examples

Figure 6.12: Helium: superposition of the contours of the functions $\sqrt{M_k \tilde{V}}$, $k = 1, 2, 3$, defined in (6.66) for specific values of $\nu$: $\nu_5 < \nu$ (a), $\nu = \nu_5$ (b), $\nu_4 < \nu < \nu_5$ (c), $\nu = \nu_4$ (d), $\nu_3 < \nu < \nu_4$ (e), $\nu = \nu_3$ (f), $\nu_2 = 0 < \nu < \nu_3$ (g), $\nu = \nu_2 \leq 0$ (h), $\nu_1 = 0 < \nu < \nu_2 = 0$ (i), $\nu = \nu_1 = 0$ (j), and $\nu < \nu_1 = 0$. The contours are shown on the shape space $\tilde{Q}$ represented in the same way as in Fig. 6.9.
bifurcation of the Hill region at $\nu = \nu_5$ due to the collinear central configuration is very similar to the example of the compound of two electrons and a positron. When $\nu$ crosses the value $\nu_4$ of the relative equilibrium a blob detaches from the blue region. At $\nu_3$ the blue blob shrinks to a point and similar to the analogous scenario in the gravitational case a green region grows out of this region when $\nu$ decreases below $\nu_3$. The green region then grows and reaches the boundary of the shape space for the first time when $\nu = \nu_2$. When $\nu$ decreases further to $\nu_1 = 0$ the green region grows until the red and blue regions have vanished at $\nu_1 = 0$. Similar to the case of the compound of two electrons and one positron a grey forbidden region near the double collision of the electrons remains at $\nu_1 = 0$. Like in any charged three-body system for every shape any orientation is possible for $\nu < \nu_1 = 0$.

### 6.9 Conclusion and outlook

In this chapter we discussed the configurations and orientations that are admissible for given values of the conserved total energy and angular momentum for charged 3-body systems. The admissible configurations and orientations were discussed on a configuration space that is reduced by the translational, rotational and dilation symmetries of charged 3-body systems. We considered the examples of the charged 3-body systems given by the compound of two electrons and one positron and the helium atom (two electrons and a nucleus). For comparison, the well known example of the gravitational 3-body system was discussed for the scheme presented in this chapter first.
6.9. Conclusion and outlook