Chapter 4

Critical Points of The Map of Integrals

4.1 Introduction

The integral manifolds are defined as the level sets of the map $\mathcal{F}$ of integrals introduced in section 2.6. We are interested in their topology and in the respective critical points, where the topology possibly changes. In this chapter we concentrate on the latter. We should keep in mind that a critical point is a notion in phase space whereas a central configuration is a notion in configuration space.

The number of central configurations depends on the number of bodies and of course on the potential. For the gravitational three-body problem, Euler and Lagrange found three collinear and two triangular central configurations respectively. However, already for the gravitational four-body problem this number is unknown [37].

According to Euler-Moulton, there are exactly $\frac{N!}{2}$ collinear central configurations in the gravitational $N$-body problem, one of each ordering of the masses on the line [39, 35]. For the collinear central configurations in the $N$-body problem with general homogeneous potential, see the work of Woodlin and Xie [52]. For $N \geq 4$, there are only partial results. In case of the gravitational four-body problem for equal masses an exact count of the number of non-collinear relative equilibria is known by Albouy [3, 4]. Xia [53] also found an exact count of the number of non-collinear relative equilibria in the gravitational four-body problem for sufficiently small mass. For $N \geq 5$ even the finiteness of the number of central configurations is an open question. This is sometimes called the Chazy-Wintner-Smale problem. Chazy [16] and Wintner [51] proposed this question in the form
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of conjecture and listed by Smale as problem number 6 on his list of problems for the 21st century [50]. For the planar gravitational four-body problem Hampton and Moeckel [24] showed that the number of central configurations is finite.


By changing forces from gravitational force to electrostatic force, the number of central configurations could be change. Depending on the signs of the charges Pérez-Chavela et al [41] found maximum number of five collinear central configurations for the charged three particles system, and depending on the values of the parameter (masses and charges) he found the total number of non-collinear central configurations to be 0 or 2. The purpose of this chapter is to find the number of collinear and non-collinear central configurations for the charged three particles system.

This chapter is organized as follows. Section 4.1 starts with the introduction. In the subsections 4.1.1, 4.1.2, 4.1.3 we describe central configurations, a class of potentials and outline of the methods respectively. We divide central configurations into collinear and non-collinear central configurations. Section 4.2 describe collinear central configurations. Furthermore, in subsections 4.2.1, 4.2.2 we first identify the space of collinear configurations and the reduced space of collinear configuration. In subsections 4.2.3, 4.2.4 we try to find polynomial equations for central configurations on the reduced space. Furthermore we take the intersection of the discriminant set of the polynomial with parameter family of $f$ consists of three curves including the curve $\Gamma$, on which we find special points. In subsections 4.2.5, 4.2.6 we find the number of zeros of the polynomial and action of the permutation groups. In subsections 4.2.7, 4.2.8 we determined number of collinear central configurations and give miscellaneous remarks. The last section describes non-collinear central configurations.

4.1.1 Central configurations

We start with a slightly more general definition of central configurations. We consider a system of $N$ bodies in the phase space $M = T^* (\mathbb{R}^{3N})$. In the phase space $M$ we take coordinates $(q,p)$ and $\Pi : T^* (\mathbb{R}^{3N}) \to \mathbb{R}^{3N} : (q,p) \mapsto q$ defines the projection to configuration space. The dynamics of the system is described by the Hamiltonian function $H$ of the form $\dot{H}(p,q) = T(p) + V(q)$. The physical interpretation is that $T$ is the kinetic energy and $V$ is the potential energy. In fact, the interaction of the bodies is encoded in $V$. Now a configuration $q$ is called
central if the following equations hold

\[
\frac{\partial}{\partial q_i} V(q) = \lambda m_i (q_i - Q(q)) 
\] (4.1)

for \( i \in \{1, \ldots, N\} \). Here \( Q \) is the centre of mass of the bodies. This equation turns up when we are looking for critical points of the integral map of the \( N \)-body system. That is, when this system is symmetric with respect to spatial rotations and translations. This puts constraints on the potential function \( V \).

### 4.1.2 Class of potentials

The potential we use in this chapter comes from a larger class \( \mathcal{V} \) defined in chapter 2. But to perform actual computations we need to be more specific. The general form of the potential reads

\[
V(q) = \sum_{i<j} \gamma_{ij} f(||q_i - q_j||) 
\]

where \( f \) is a real function and \( \gamma_{ij} \) contains properties of the bodies. Here we will take \( f(x) = \frac{1}{x} \) and \( \gamma_{ij} = -Q_i Q_j \) where \( Q_i \) is the electrical charge of body \( i \) and \( i \in \{1,2,3\} \). In our case of three bodies we may replace the combinations \( Q_i Q_j \) by three parameters \( a_k \) with \( i,j,k \in \{1,2,3\} \) and different. Then we have

\[
V(q) = -\frac{a_3}{||q_1 - q_2||} - \frac{a_1}{||q_2 - q_3||} - \frac{a_2}{||q_3 - q_1||}. 
\] (4.2)

However we should keep in mind that the map taking the \( Q_i \) to \( a_i \) is not onto. Indeed the map \( \mathbb{R}^3 \to \mathbb{R}^3 : x \mapsto y = (-x_2 x_3, -x_1 x_3, -x_1 x_2) \) has image \( y_1 y_2 y_3 \leq 0 \), a collection of four octants.

### 4.1.3 Outline of the method

Our approach differs from that in the literature, in particular in treating the collinear central configurations. We start with identifying the space of collinear configurations. On this space the translation, rotation, dilation and permutation groups act. Then we first find the reduced space of collinear configurations by reducing with respect to translations, rotations and dilations. In case of three bodies the reduced space is one dimensional and the equation for central configurations reduces to a single equation. However, due to the fact that the masses are undetermined and the potential contains unknown coefficients we have in fact a parameter family of equations. When we use the potential from equation (4.2) the reduced equation is a fifth degree polynomial. In general it will be impossible
to find explicit solutions of the (reduced) equation for central configurations. But these (unknown) solutions depend on the parameters and by using failure of the implicit function theorem we are able to find regions in parameter space with at least a constant number of solutions. We will take a more geometric point of view and consider the equation for central configurations on the product of the parameter space and the reduced space. Then the equation defines a hypersurface. Projecting the latter on the parameter space we obtain the just mentioned regions by locating the projection singularities. (As it turns out, the surface itself has no singularities.) The simplest singularities are folds and where a fold is parallel to the projection direction we get more complicated singularities. We carry out this program in some detail for the potential in equation (4.2). After some simplifications, for example we split the parameters into two sets and consider a family (parametrised by the masses) of parameter families (parametrised by the coefficients of the potential) and fix the masses. Then we find fold curves with cusps in a two dimensional parameter space. At the very end we look at the action of the permutation group to find the number of solutions to the equation for collinear central configurations. This method is admittedly not the simplest or quickest to solve the problem of finding the collinear central configurations of the three-body problem with the aforementioned potential. But it lends itself to generalisations to systems with more bodies and different potentials.

4.2 Collinear central configurations

In this section we consider collinear central configurations for a three-body problem with potential of the form defined in 4.2. However, we formulate our results as general as possible. To find the collinear central configurations we proceed as follows. First we identify the space of collinear configurations. Using the fact that the equation for central configurations is invariant with respect to rotations, translations and dilations we reduce with respect to these symmetries. On the reduced space the equation for central configurations reduces to a polynomial of degree five. Furthermore this polynomial depends on parameters, namely the masses of the bodies, but also the parameters in the potential, see equation (4.3). So a priori we have a six parameter family of polynomials. On the space of collinear configurations we also have the action of the group $S_3$, permuting the bodies. The equation for collinear central configurations is not invariant with respect to permutations unless we also include the parameters. Therefore we are still able to exploit this property by considering its action on the reduced space. We cannot hope to explicitly solve the quintic equation. But by intersecting the discriminant set of a general quintic equation with our parameter family we are able to find regions in parameter space with the same number of solutions cor-
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responding to collinear central configurations. The main result for three bodies is the following.

**Theorem 4.1.** Consider a three-body problem with potential:

$$V(q) = -\frac{a_3}{||q_1 - q_2||} - \frac{a_1}{||q_2 - q_3||} - \frac{a_2}{||q_3 - q_1||}.$$  \hspace{1cm} (4.3)

If we take the following values for the masses and coefficients of the potential $m_1 := m$, $m_2 := m$ and $m_3 := 1$ and furthermore $a_3 := 1$, then there are 13 different regions in the $(a_1, a_2)$ parameter plane with a constant number of collinear central configurations. Up to permutations this number can be 0, 1, 2 or 3.

In figure 4.1 together with table I the number of real zeros of the reduced equation on each of the intervals $I_i$ (defined below) on the reduced space are indicated. Each of these intervals corresponds to a certain order of the bodies.

**Figure 4.1:** Regions in the $(a_1, a_2)$-parameter plane with different numbers of real zeros of $f$ in the intervals $I_i$. The regions are numbered 1 to 13 and in table I the numbers of real zeros are listed. When compactifying the parameter plane by adding one point at infinity, the branches in the left picture meet at infinity as shown in the right picture where the ‘origin’ represents the added point at infinity.
4.2. Collinear central configurations

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Table I: The number of zeros of $f$ in each region in the intervals $I_1$, $I_2$ and $I_3$.

4.2.1 The space of collinear configurations

Let us first identify the space of collinear configurations for an $N$-body system. The point $q \in \mathbb{R}^{3N}$ is a collinear configuration if the differences $q_i - q_j$ are multiples of the same vector. If we assume that the center of mass is at the origin, then the vectors $q_i$ are multiples of the same vector. In other words a unit vector $v \in \mathbb{R}^3$ exists such that $q_i = r_i v$, with $r_i \in \mathbb{R}$ and $|r_i| = ||q_i||$ for $i \in \{1, \ldots, N\}$. Thus the space of collinear configurations, that is $N$ parallel vectors in $\mathbb{R}^3$, is topologically equivalent to $\mathbb{R}^N \times S^2$. On this space we have four group actions, namely actions of rotations ($SO(3)$), translations, dilations and permutations. The first acts in the standard way on the $S^2$ part while the translations, dilations and permutations act on the $\mathbb{R}^N$ part.

Lemma 4.2. Equation (4.1) for collinear central configurations is invariant under rotations, translations and dilations.

Proof. Since $\nabla V(q)$ transforms like $q$ under rotations equation (4.1) is invariant see proposition 3.6. Clearly equation (4.1) is translation invariant. If we define the action of dilations as $(q, \lambda) \mapsto (tq, t^{-3}\lambda)$ for $t \in \mathbb{R} > 0$, equation (4.1) is dilation invariant. 

4.2.2 The reduced space of collinear configurations

From this observation we infer that the problem of finding collinear central configurations will become simpler on the reduced space of collinear configurations.

Lemma 4.3. The reduced space of collinear configurations, reduction with respect to rotations, translations and dilations, is topologically equivalent to $\mathbb{R}P^{N-2} \times e_1$. 

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Proof. The space of collinear configurations is \( \mathbb{R}^N \times S^2 \). First we reduce with respect to rotations. Every two points on \( S^2 \) are \( SO(3) \)-equivalent, or put differently there is only one \( SO(3) \)-orbit on \( S^2 \). Therefore the reduced space is a single point which we choose to be \( e_1 \). The translations act on \( \mathbb{R}^N \) as follows \((r_1, r_2, \ldots, r_N) \mapsto (r_1, r_2, \ldots, r_N) + t(1, 1, \ldots, 1)\) for \( t \in \mathbb{R} \). Thus the orbits of the translation group are parallel lines with direction vector \((1, 1, \ldots, 1) \in \mathbb{R}^N \). Every plane transverse to the vector \((1, 1, \ldots, 1)\) may serve as a reduced phase space. A natural choice in this context is the plane

\[
V_N := \{ x \in \mathbb{R}^N | \sum_{i=1}^{N} m_i x_i = 0 \}.
\]

Note that \( V_N \) as a vector space is equivalent to \( \mathbb{R}^{N-1} \). On \( V_N \) the dilations act as \( x \mapsto tx \) for \( t \in \mathbb{R}^* > 0 \). The orbits are lines through the origin. Then the reduced space is topologically equivalent to \( \mathbb{R}P^{N-2} \).

Let us now restrict to three body systems. This case is somewhat special because there are also three relative distances. If we take for example \( x := r_2 - r_3 \), \( y := r_3 - r_1 \) and \( z := r_1 - r_2 \), then another natural choice is \( V_3 := \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\} \). Now \( V_3 \) as a vector space is equivalent to \( \mathbb{R}^2 \) and thus the reduced space with respect to dilations is topologically equivalent \( \mathbb{R}P^1 \). Let us take \( u \), defined by \( y = ux \), as a coordinate on this space. Then we are effectively working in \( \mathbb{R} \cup \{ \infty \} \) which is again topologically equivalent to \( \mathbb{R}P^1 \) (and to the circle).

Thus we define three intervals for \( u \).

Definition 4.4. Let \( I_1 \), \( I_2 \) and \( I_3 \) be three intervals in \( \mathbb{R} \cup \{ \infty \} \), defined as: \( I_1 := (-1, 0) \), \( I_2 := (\infty, -1) \) and \( I_3 := (0, \infty) \).

Figure 4.2: Left: The reduced space \( \mathbb{R} \cup \{ \infty \} \) of collinear configurations with the intervals \( I_1 \), \( I_2 \) and \( I_3 \). Inside the circle collisions of pairs are indicated at the special values of \( u \). Right: A collinear configuration with two relative distances \( x \) and \( y \).
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Then \( u \in I_1 \) corresponds to the orders \((2, 1, 3)\) or \((3, 1, 2)\), \( u \in I_2 \) corresponds to the orders \((1, 2, 3)\) or \((3, 2, 1)\) and \( u \in I_3 \) corresponds to the orders \((1, 3, 2)\) or \((2, 3, 1)\), see figure 4.2.

4.2.3 Polynomial equation for collinear central configurations on reduced space

Now that we know the reduced space of collinear configurations we derive a reduced equation for the collinear central configurations. Using the same coordinates as in the previous section we find a single equation for \( u \). In deriving this equation we also encounter the intermediate coordinates \( x, y \) and \( z \). Their signs have a relation with the order of the bodies as is implicit in the discussion of the previous section. Here we will need them again, therefore we define them separately.

Definition 4.5. The signs of \( x, y \) and \( z \) are called \( \sigma, \tau \) and \( \rho \) respectively.

Note that not every combination of signs is allowed since \( x + y + z \equiv 0 \). In view of the definition of \( x, y \) and \( z \) and figure 4.2 a natural choice to begin with is \( \sigma = 1, \tau = 1 \) and \( \rho = -1 \). Then we have the following.

Proposition 4.6. On the reduced space of collinear configurations the equation for central configurations reads \( f(u) = 0 \), where

\[
\begin{align*}
\mathcal{f}_1(u) &= a_1 \mathcal{f}_1(u) + a_2 \mathcal{f}_2(u) + a_3 \mathcal{f}_3(u) \\
\mathcal{f}_1(u) &= m_1 u^2 (1 + u)^2 (2m_2 + m_2 u + m_3 u) \\
\mathcal{f}_2(u) &= -m_2 (1 + u)^2 (m_1 + m_3 + m_1 u) \\
\mathcal{f}_3(u) &= -m_3 u^2 (m_2 - m_1 u)
\end{align*}
\] (4.4)

The polynomial \( f \) depends on six parameters: \( a_1, a_2, a_3, m_1, m_2 \) and \( m_3 \).

Proof. We start the proof with the components of \( \frac{\partial V(r)}{\partial r_i} = -m_i \lambda r_i \). We will eventually eliminate \( \lambda \), then the resulting equations will be linear in \( a_i \). Taking linear combinations of the above equations with coefficients \((-m_2, -m_1, 0), (0, m_3, -m_2)\) and \((-m_3, 0, m_1)\), we get equations depending on differences \( r_i - r_j \) only. Following the reduction steps in the proof of lemma 4.3, we switch to variables \( x, y \) and \( z \) and replace for example \( \frac{(r_1 - r_2)}{(r_1 - r_2)^2 + r_1^2} \) by \( \frac{r_1}{2z} \), but we immediately use the
The sum of these equations with coefficients \((m_3, m_1, m_2)\) yields \(\lambda m_1 m_2 m_3 (x + y + z)\) which is zero because \(x + y + z\) is identical to zero. However this implies that the equations are linearly dependent (which is not surprising). Eliminating \(\lambda\), we will be left with a single equation. Now if \((x, y, z)\) is a solution, then \((tx, ty, tz)\) for \(t \in \mathbb{R}^*\) is also a solution (dilation symmetry). Like in lemma 4.3 we set \(y = ux\) and \(z = -x - ux\), then we get an equation for \(u\) only, namely \(f(u) = 0\), depending on the parameters \(a\) and \(m\).

Note that the polynomial \(f\) is homogeneous of degree one in \(a\) and homogeneous of degree two in \(m\). This in turn implies that scaling the parameters \(a \mapsto ta\) and \(m \mapsto sm\) with \(t, s \in \mathbb{R}^*\) does not change the solutions of \(f(u) = 0\).

Our ultimate goal is to obtain the number of collinear central configurations for each value of the parameters \(a\) and \(m\). In order to do that we have to know how the permutation group of the bodies acts on the reduced space. This will be postponed. For the moment our aim is to divide the parameters space into regions with a constant number of real solutions of \(f(u) = 0\). Since the parameter space is six dimensional we make some simplifications. First we fix the values of the masses \(m\) at arbitrary values. Then we scale the parameters \(a\) as follows \((a_1, a_2, a_3) \mapsto (\mu_1, \mu_2, 1) = \frac{1}{a_3} (a_1, a_2, a_3)\) assuming \(a_3 \neq 0\). Thus we are left with a two parameter family of polynomials. The discriminant set of \(f\) is now a collection of curves in the parameter plane. On these curves \(f\) has double zeros, so by crossing one of these curves the number of real zeros jumps by two. Using a more detailed analysis we will be able to tell how the number of zeros in each interval \(I_1, I_2\) and \(I_3\) changes.

**Proposition 4.7.** The intersection of the discriminant set of a fifth degree polynomial and the two parameter family \(f\) consists of three curves: both coordinate axes \(a_1 = 0\) and \(a_2 = 0\) and a curve \(\Gamma\). These curves separate regions in the \((a_1, a_2)\) parameter plane, on each region the number of solutions of \(f(u) = 0\) in the intervals \(I_1, I_2\) and \(I_3\) is constant. The curve \(\Gamma\) is parametrized by \(u \in \mathbb{R} \cup \{\infty\}\) and depends on \(m\). The parametrization of \(\Gamma\), called \(c : \mathbb{R} \cup \{\infty\} \to \mathbb{R}^2\), is given by

\[
c(u) = \left( \frac{f_2(u) f_3'(u) - f_2'(u) f_3(u)}{f_2(u) f_1'(u) - f_1'(u) f_2(u)}, \frac{f_1(u) f_3'(u) - f_3(u) f_1'(u)}{f_2(u) f_1'(u) - f_1'(u) f_2(u)} \right)
\]  

(4.5)
where \( f_1, f_2 \) and \( f_3 \) are defined in equation \((4.4)\). The curves and regions are shown in figures \(4.3\) and \(4.1\) for the special case \( m_1 = m_2 \) and \( m_3 = 1 \).

**Proof.** We first consider two special cases. Expanding the polynomial \( f \) using the definition in equation \((4.4)\) we have:

\[
f(u) = a_1 m_1 (m_2 + m_3) u^5 + a_1 m_1 (3m_2 + 2m_3) u^4 \\
+ (a_1 m_1 (3m_2 + m_3) - a_2 m_1 m_3 + a_3 m_1 m_3) u^3 \\
+ (a_1 m_1 m_2 - a_2 m_2 (3m_1 + m_3) - a_3 m_2 m_3) u^2 \\
- a_2 m_2 (3m_1 + 2m_3) u - a_2 m_2 (m_1 + m_3).
\]

From this expression we infer that if \( a_1 \) tends to zero, two zeros of \( f \) tend to infinity and if \( a_2 \) tends to zero, two zeros of \( f \) tend to zero. Therefore on the lines \( a_1 = 0 \) and \( a_2 = 0 \), \( f \) has double zeros. In general \( f \) has double zeros on the discriminant set defined by the equations:

\[
\begin{align*}
  f(u) &= 0 \\
  f'(u) &= 0
\end{align*}
\]

The equations are linear in \( a \), we solve them for \( a_1 \) and \( a_2 \) by using Cramer’s rule and the decomposition of \( f \) defined in equation \((4.4)\). Thus we find the parametrization \( c \) of \( \Gamma \) in equation \((4.5)\). \( \square \)

**4.2.4 Special points of the curve \( \Gamma \)**

On the curve \( \Gamma \) the polynomial \( f \) has double zeros. Therefore \( \Gamma \) is a fold curve so we may expect to find cusps. Moreover the parametrization of \( \Gamma \) is by rational functions, so we may also expect points at infinity on \( \Gamma \). For the next exposition we write the parametrization explicitly (cf equation \((4.5)\))

\[
c(u) = \left( \frac{m_3 g_1(u; m)}{m_1 g_3(u; m)}, \frac{m_3 g_2(u; m)}{m_2 g_3(u; m)} \right) \text{ for } u \in \mathbb{R}, \quad (4.6)
\]

where

\[
\begin{align*}
g_1(u) &:= m_1 (3m_1 + m_2 + m_3) u^2 - m_1 (-3m_1 + m_2 - 3m_3) u - 2m_2 (m_1 + m_3) \\
g_2(u) &:= u^3 \left[ -2m_1 (m_2 + m_3) u^2 + m_2 (-m_1 + 3(m_2 + m_3)) u \\
&\quad + m_2 (m_1 + 3m_2 + m_3) \right] \\
g_3(u) &:= (1 + u)^3 \left[ 2m_1 (m_2 + m_3) u^2 + (3m_3(m_2 + m_3) + m_1 (4m_2 + 3m_3)) u \\
&\quad + 2m_2 (m_1 + m_3) \right].
\end{align*}
\]

To keep the computations tractable we now set \( m_1 := m, m_2 := m \) and \( m_3 := 1 \). The consequences of this choice are:
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\[ \begin{align*}
\mu_1 & = \xi - u \\
\mu_2 & = \xi + u \\
u & = \eta - u \\
-1 & = \eta + u \\
0 & = \xi_0 \\
1 & = \eta_0 \\
\infty & = \xi_+ \\
\end{align*} \]

Figure 4.3: Left: Schematic picture of the curve \( \Gamma \). The arrows indicate how the three branches are connected at infinity. For \( u \in \{ \xi-, \xi_0, \xi+ \} \), \( \Gamma \) has points at infinity, for \( u \in \{ \eta-, \eta_0, \eta+ \} \), \( \Gamma \) has cusp points. Right: Special values of \( u \) on the reduced space.

1. When \( m_1 = m_2 = m \) the curve \( \Gamma \) has a symmetry property. Let \( T(a_1, a_2) = (a_2, a_1) \) then \( Tc(u) = c(\frac{1}{u}) \). This will be explained in more detail in section 4.2.6.

2. A consequence of the previous is that if \( \eta \) is a value such that \( c(\eta) \) is a singular point, then \( c(\eta^{-1}) \) is also a singular point. If \( \Gamma \) has an odd number of singular points, one must occur for \( u = 1 \) or \( u = -1 \). The singular points of \( \Gamma \) are found by solving \( c'(u) = 0 \).

3. Similarly, if \( \xi \) is a value such that \( c(\xi) \) is a point at infinity (taking an appropriate limit), then \( c(\xi^{-1}) \) is also a point at infinity. In case \( \Gamma \) has an odd number of such points, one must occur for \( u = 1 \) or \( u = -1 \).

With these choices for the parameters \( m \) we have the following result.

**Lemma 4.8.** The curve \( \Gamma \) defined in equation (4.6), has six special points.

(i) There are three points at infinity defined by the zeros of \( g_3 \) (see (4.6)) namely at \( u \) equal to one of the following

\[ \xi \pm = \frac{-3 - 6m - 4m^2 \pm \sqrt{9 + 36m + 44m^2 + 16m^3}}{4m(1 + m)}; \]
\[ \xi_0 = -1. \]
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For $\xi_{\pm}$ the corresponding points on $\Gamma$ are regular points at infinity. Note that $\xi_-\xi_+ = 1$. For $\xi_0$ the corresponding point is a degenerate cusp at infinity, with equivalent local parametrisation $t \mapsto (t^3 + O(t^4), t^4 + O(t^5))$. The curve $\Gamma$ has three branches, namely $c((\xi_-, \xi_0)), c((\xi_0, \xi_+)$ and $c((\xi_+, \infty) \cup (\infty, \xi_-)$, see figure 4.3.

(ii) There are three singular points at $u$ taking one of the values:

$$\eta_\pm = \frac{-5 - 12m - 8m^2 \pm \sqrt{21 + 80m + 92m^2 + 32m^3}}{2(1 + 5m + 4m^2)}$$

$$\eta_0 = 1$$

Each singular point is a regular cusp with an equivalent local parametrisation $t \mapsto (t^2 + O(t^3), t^3 + O(t^4))$. Note that $\eta_-\eta_+ = 1$.

The inequalities $\xi_- < \eta_- < \xi_0 < \eta_+ < \xi_+$, valid for all $m > 0$, imply that the six special points on $\Gamma$ are alternately cusps and points at infinity.

Proof. Points at infinity are found from solving $g_3(u; m) = 0$. Since $g_3$ has an easy factorisation, see equation (4.4), we immediately find the solutions presented in the lemma. For each solution we have $g_1(\xi, m) \neq 0$ and $g_2(\xi, m) \neq 0$ so that we find indeed points at infinity. The singular points are found from solving $c'(\eta) = (0, 0)$. After some computations we find that $\eta$ must be a zero of the polynomial

$$f_3 f_2 f_1'' - f_2 f_3 f_1'' - f_3 f_1 f_2'' + f_1 f_3 f_2'' - f_2 f_1 f_3'' - f_1 f_2 f_3''$$

where the $f_i$ are defined in equation (4.4). For our choice $m_1 = m_2 = m$ and $m_3 = 1$ this polynomial factorises as

$$(u - 1)u^2(u + 1)(1 + 5m + 4m^2 + 5u + 12m u + 8m^2 u + u^2 + 5m^2 u + 4m^2 u^2)$$

from which we readily find the values of $\eta$. For a local study of the singular points we look at the Taylor series of $A[c(\eta + t) - c(\eta)]$, where $A$ is a linear transformation that locally puts $\Gamma$ in a standard form, namely $A = mat(Jv, v)$ where $v$ is the tangent vector of $\Gamma$ at the singularity: $v := \frac{dt}{dt}(c(\eta + t) - c(\eta))|_{t=0}$ and $J$ is the matrix of a rotation over $\frac{\pi}{2}$. For each value of $\eta$ we find

$$A[c(\eta + t) - c(\eta)] = (\gamma_1(m)t^3 + O(t^4), \gamma_2(m)t^2 + O(t^3))$$

where the $\gamma_i$ are functions of $m$. In general they are non-zero for positive values of $m$. Their explicit expressions are quite involved except for $\eta = 1$ then we have $\gamma_1(m) = \frac{7 + 8m}{21 + 80m + 64m^2}$ and $\gamma_2(m) = \frac{3(7 + 8m)}{8(3 + 4m)^2}$. For a local study of points at
infinity we look at the Taylor series of $\frac{c}{||c||^2}$. Let $\xi$ be a value such that $c(\xi)$ is a point at infinity, then we find
\[
\frac{c(\xi + t)}{||c(\xi + t)||^2} = (\gamma_1(m) t + O(t^2), \gamma_2(m) t + O(t^2))
\]
for $\xi = \xi_-$ and $\xi = \xi_+$, so these are regular points since the $\gamma_i$ are positive for $m$ positive. For $\xi = -1$ however, we find after a linear transformation
\[
A \frac{c(\xi + t)}{||c(\xi + t)||^2} = \left( \frac{3 + 2m}{4} t^3 + O(t^4), \frac{3}{16} (3 + 2m)^2 t^4 + O(t^5) \right).
\]
Even for general values of $m_1, m_2$ and $m_3$ we find
\[
A(m_1, m_2, m_3) \frac{c(\xi + t)}{||c(\xi + t)||^2} = (\beta_1(m_1, m_2, m_3) t^3 + O(t^4), \beta(m_1, m_2, m_3) t^4 + O(t^5)),
\]
where the $\beta_i$ are positive. Therefore at $\xi = -1$ we have a singular point at infinity.

We present a schematic picture of the curve $\Gamma$ in figure 4.3. In particular the curve is not to scale near the cusps. This picture however serves to show the global structure and the alternation of singular points and points at infinity. Moreover from this picture we get regions in the parameter plane with a constant number of real solutions of $f$.

### 4.2.5 The number of zeros of the polynomial $f$

In the previous paragraph we found regions in the $(a_1, a_2)$-parameter plane with a constant number of real solutions of $f(u) = 0$. Since these regions are bounded by curves of double zeros of $f$, the number of zeros changes by two upon crossing one of them. As soon as we know the number of real solutions in one region we can find the numbers in the other regions by the following lemma.

**Lemma 4.9.** On crossing the $\mu_1$ axis the number of zeros in $I_1$ and $I_3$ both increase or decrease by one. But on crossing the $\mu_2$ axis the number of zeros in both $I_2$ and $I_3$ increase or decrease by one. The curve $\Gamma$ is the union of images of the intervals $I_i$ by the parametrisation $c$. On crossing the image of $I_i$, the number of real zeros in $I_i$ changes by two.

**Proof.** On the $a_1$ axis the polynomial $f$ has a double zero at $u = 0$, but $a_2$ axis the polynomial $f$ has a double zero at $u = \infty$. On a part of $\Gamma$ which is the image of $I_i$ the polynomial $f$ has a double zero on $I_i$. 

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To use lemma 4.9 we determine the number of real solutions in one region.

**Lemma 4.10.** If we take the masses as $m_1 := m$, $m_2 := m$ and $m_3 := 1$, then for $a_1 = a_2 = 1$, the polynomial $f$ has one real zero in the interval $I_3$.

**Proof.** With the masses and parameters as stated the polynomial factorizes as
$$f(u) = m(1 + m)(u - 1)(1 + 3u + 5u^2 + 3u^3 + u^4 + m(1 + u)^4).$$ Then $f$ has a real zero at $u = 1 \in I_3$ and the other zeros of $f$ are complex (in fact two reciprocal complex conjugate pairs).

Using these two lemmas we can easily construct the numbers of zeros as indicated in figure 4.1. Starting in region 1, $f$ has only one real zero which lies in $I_3$. Crossing the $a_2$-axis into region 11, the number of zeros in both $I_2$ and $I_3$ increases or decreases by one. Since in region 1 there are no zeros of $f$ in $I_2$, the number of zeros increases and the number of zeros of $f$ for each interval $I_i$ changes from $(0, 0, 1)$ to $(0, 1, 2)$. Starting again in region 1 but now crossing the $a_1$-axis into region 12, the number of zeros of $f$ changes from $(0, 0, 1)$ to $(1, 0, 2)$. Let us now start in region 12, crossing the curve $\Gamma$ occurs on the image of $I_3$ and the number of zeros on $I_3$ changes by two. However we can not decide whether it is an increase or a decrease. Therefore we start again in region 12 but now cross the $a_2$-axis into region 13. Then the number of zeros in both $I_2$ and $I_3$ changes by one. Since in region 12 there are no zeros of $f$ in interval $I_2$, the change must be an increase. Reasoning in this manner we obtain all triples.

### 4.2.6 The action of the permutation group

We still have to consider the action of the permutation group on the reduced space of collinear configurations. The permutation group $S_3$ of three bodies is generated by $\pi_1 = (1 \ 2)$ and $\pi_2 = (2 \ 3)$. Then the other elements are $\pi_0 = \text{id}$, $\pi_3 = \pi_1 \circ \pi_2$, $\pi_4 = \pi_2 \circ \pi_1$ and $\pi_5 = \pi_1 \circ \pi_2 \circ \pi_1$. Consider the product space of collinear configurations and the space of parameters $(r, m, a) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. Let $S_3$ act on each factor in the same standard way, then we immediately have the following.

**Lemma 4.11.** The equation for (collinear) central configurations (4.1) with potential $V$ from equation (4.3) is invariant with respect to the action of $S_3$ defined above.

Now we consider the reduced space of collinear configurations, then there is an induced $S_3$-action on this space. Using the intermediate variables $x = r_2 - r_3$, $y = r_3 - r_1$ and $z = r_1 - r_2$ and the coordinate $u$ on the reduced space, defined by $y = ux$ we readily obtain the following result.

**Lemma 4.12.** The induced $S_3$-action on the reduced space of collinear configurations is determined by the homomorphisms $h_i$ with: $h_0 = \text{id}$, $h_1(u) = \frac{1}{u}$, $h_2(u) = -(1 + u)$, $h_3 = h_1 \circ h_2$, $h_4 = h_2 \circ h_1$ and $h_5 = h_1 \circ h_2 \circ h_1$. 

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Note that the three disjoint intervals $I_1$, $I_2$ and $I_3$ in $\mathbb{R} \cup \{\infty\}$, see [4.4], are permuted by the $h_i$, see table II.

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$I_1$</th>
<th>$I_2$</th>
<th>$I_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_2$</td>
<td>$I_2$</td>
<td>$I_1$</td>
<td>$I_3$</td>
</tr>
<tr>
<td>$h_3$</td>
<td>$I_3$</td>
<td>$I_1$</td>
<td>$I_2$</td>
</tr>
</tbody>
</table>

Table II: Permutation of the intervals $I_1$, $I_2$ and $I_3$ by $h_1$ and $h_2$.

On the reduced space of collinear configurations equation $f(u) = 0$, see (4.4), determines the collinear central configurations. The polynomial $f$ depends on parameters $m$ and $\alpha$ and we explicitly write $f(u; \alpha, m) = \mu(\alpha)$ to stress that fact. The $\alpha$ are up to a sign equal to the parameters $a$ in the potential. In deriving $f$ we made a choice for the signs of the intermediate variables $x$, $y$ and $z$. The $\alpha$ contain these signs. Therefore the $S_3$-action on $\alpha$ is non-standard, in fact we have $\phi_0 = id$, $\phi_1(\alpha) = (\alpha_2, \alpha_1, \alpha_3)$, $\phi_2(\alpha) = (-\alpha_1, \alpha_3, \alpha_2)$, $\phi_3 = \phi_1 \circ \phi_2$, $\phi_4 = \phi_2 \circ \phi_1$ and $\phi_5 = \phi_1 \circ \phi_2 \circ \phi_1$. A short computation immediately reveals the following.

**Lemma 4.13.** The polynomial $f$ transforms under the generators of the $S_3$ action on $(r, \alpha, m)$ according to

$$-u^5 f(h_1(u); \phi_1(\alpha), \pi_1(m)) = f(u; \alpha, m),$$

$$f(h_2(u); \phi_2(\alpha), \pi_2(m)) = f(u; \alpha, m).$$

This means that the collinear central configurations for permuted bodies follow from the single equation $f(u, \alpha, m) = 0$ with the particular sign choice.

Before finding the discriminant set of $f$ we simplified the problem by using the fact that $f$ is homogeneous of degree 1 in $\alpha$ and assuming $\alpha_3 \neq 0$ we scaled $(\alpha_1, \alpha_2, \alpha_3) \mapsto \frac{1}{\alpha_3}(\alpha_1, \alpha_2, \alpha_3)$. Then we get an $S_3$-action $\psi$ on the two parameter space with coordinates $(\mu_1, \mu_2) = \mu(\alpha) = (\frac{\alpha_1}{\alpha_3}, \frac{\alpha_2}{\alpha_3})$ defined by $\psi_i(\mu(\alpha)) = \mu(\phi_i(\alpha))$. In particular we have

$$\psi_1(\mu_1, \mu_2) = (\mu_2, \mu_1),$$

$$\psi_2(\mu_1, \mu_2) = (-\frac{\mu_1}{\mu_2}, \frac{1}{\mu_2}).$$

One of the components of the discriminant set of $f$ is the curve $\Gamma$. Using the lemmas and definitions above we arrive at the final result.

**Corollary 4.14.** Under permutations of the bodies, the curve $\Gamma$ transforms via its parametrisation $c$ as

$$\psi_i \left( c(h_i(u); \pi_i(m)) \right) = c(u; m)$$

for $i \in \{0, 1, 2, 3, 4, 5\}$. 

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4.2.7 The number of collinear central configurations

Now that we know the number of solutions of $f(u) = 0$ in each interval $I_1$, $I_2$ and $I_3$ together with the action of the permutation group on the reduced space of collinear configurations and the parameter space we count the number of collinear configurations. Fixing the order of the bodies such that the differences $x = r_2 - r_3$ and $y = r_3 - r_1$ have equal sign is equivalent to requiring $u \in I_3$ on the reduced space. This means that only solutions of $f(u) = 0$ in $I_3$ correspond to collinear central configurations. This is the choice we made in section 4.2.3.

Figure 4.4: The surface defined by $f(u,m) = 0$ for a three particle system with gravitational interaction. The fold lines (red) are projected on the $(m_1,m_2)$-parameter plane.
Lemma 4.15. If the bodies are ordered \((1, 3, 2)\) or \((2, 3, 1)\) the collinear central configurations correspond to solutions of \(f(u) = 0\) in the interval \(I_3\) of the reduced space of collinear configurations. Depending on the values of the parameters there are 0, 1, 2 or 3 central configurations.

This follows immediately from table I. If we permute the bodies, we know from the action of the permutation group that the central configurations are determined by a similar equation. However by the action of the symmetry group we know how the regions in the parameter plane with of this new equation are related to regions of the equation \(f(u) = 0\). Thus we have the following corollary.

Corollary 4.16. The number of collinear central configurations in the three-body problem with potential \(V\) from equation (4.2) is up to permutations of the bodies equal to 0, 1, 2 or 3 depending on the values of the parameters.

4.2.8 Miscellaneous remarks

In the following we give several remarks on the previous analysis

Remark.
1. Although we allow electrically charged bodies we only include Coulomb interaction. That is magnetic interaction because of moving electrical charges is ignored. Including this interaction would break rotation symmetry.

2. The permutation group consists of six elements but we see only three different orders in the reduced space of collinear configurations. The reason is that for example the permutation \((1, 2, 3) \mapsto (3, 2, 1)\) can be regarded as a rotation over \(\pi\) in configuration space.

3. Using the same method for the classical gravitational three-body problem we find a single equation for the collinear central configurations on the reduced space of collinear configurations. In fact we find a three parameter family of equations where the masses of the bodies are the parameters, namely

\[
f(u; m) = (m_2 + m_3)u^5 + (3m_2 + 2m_3)u^4 + (3m_2 + m_3)u^3
- (3m_1 + m_3)u^2 - (3m_1 + 2m_3)u - (m_1 + m_2).
\]

Using the fact that \(f\) is homogeneous in \(m\) we may divide by for example \(m_3\) (at least one of the masses will be non zero) or equivalently set \(m_3 = 1\). Thus we consider a two parameter family. The discriminant set of \(f\) is a set of curves in the \((m_1, m_2)\)-parameter plane. These can be viewed as fold lines under the projection of the surface defined by \(f(u; m) = 0\) in the \(u\)-direction, see figure 4.4. In this case however it is not hard to show that the curves constituting the discriminant set stay outside the first quadrant in the \((m_1, m_2)\)-parameter plane. Since the masses
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are positive there is a fixed number of collinear central configurations. To find this number we only need to check one example, taking $m_1 = m_2 = 1$ we see that there is only one collinear central configuration for $u = 1$.

4.3 Non-collinear central configurations

In section we compute non-collinear central configurations for gravitational as well for the charged three-body problem. To find the number of non collinear central configurations we recall the definition of central configurations that is

$$\frac{\partial V}{\partial q_i} = -m_i \lambda q_i.$$ 

Before actually solving this equation we look at the actions of the symmetry groups. Since we assumed that the center of mass is at the origin we skip the translations. The equation for a central configuration is rotation invariant so they come in families which are $SO(3)$-orbits. Dilations, however, do not leave the equation invariant. But a dilation acting on a central configuration yields another central configuration, in particular with a different value of $\lambda$. The latter can be interpreted as a Lagrange multiplier when we introduce the moment of inertia, see 3.5. Using the moment of inertia the equation to be solved can be rewritten as

$$\frac{\partial V}{\partial q_i} = -\lambda \frac{\partial I}{\partial q_i}.$$ 

We interpret this equation as defining critical points of $V$ restricted to a level set of $I$. Once we have found a relation between the values of $V$, $I$ and $\lambda$. From the homogeneity of $V$ and $I$, namely degree $-1$ and $2$, we get

$$\sum_i \langle \frac{\partial V}{\partial q_i}, q_i \rangle = -V \quad \text{and} \quad \sum_i \langle \frac{\partial I}{q_i}, q_i \rangle = 2I.$$ 

Thus we have $V = 2\lambda I$, for the proof we refer to 3.8. Let us now try to actually find central configurations. We first change coordinates. Three particles inevitably lie in a plane and since the center of mass is at the origin this plane passes through zero. But if the particles happen to be collinear this plane is not usually defined. Now $V$ is a function of mutual distances $\rho_{ij} = ||q_i - q_j||$ only:

$$V = -\sum_{i<j} \frac{Gm_im_j}{||q_i - q_j||} = -\sum_{i<j} \frac{Gm_im_j}{\rho_{ij}}.$$ 

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The same turns out for $I$:

$$\sum_{i<j} m_i m_j \rho_{ij}^2 = \sum_{i<j} m_i m_j \|q_i - q_j\|^2$$

$$= \sum_{i<j} m_i m_j (\|q_i\|^2 + \|q_j\|^2) - 2 \sum_{i<j} m_i m_j q_i q_j$$

$$= 4MI - 2 \sum_j m_j \sum_i m_i q_i q_j$$

$$= 2MI$$

with $M = \sum_i m_i$. Thus $I$ is also a function of the $\rho_{ij}$. Disregarding the (change of) coordinates, the central configurations are found where $V$ restricted to $I$ has a critical point. Therefore we have to solve

$$\frac{\partial V}{\partial \rho_{ij}} = -\lambda \frac{\partial I}{\partial \rho_{ij}}$$

which now reads:

$$\frac{m_i m_j}{\rho_{ij}^2} = \lambda m_i m_j \rho_{ij},$$

and the solution is $\rho_{ij} = \sqrt[3]{\frac{1}{\lambda}}$.

**Remark.** For the system of three bodies if the distance between the bodies are equal then in case of non-collinear configuration these bodies make equilateral triangle. In planar case we get two equilateral triangular central configurations and if we consider rotation in $\mathbb{R}^3$ we get one non-collinear central configuration.

To find the number of non-collinear central configuration for the charged three-body problem. We use the potential defined in the first section of this chapter, i.e.,

$$V(q) = -\frac{a_3}{\sqrt{(q_1 - q_2)^2}} - \frac{a_1}{\sqrt{(q_2 - q_3)^2}} - \frac{a_2}{\sqrt{(q_3 - q_1)^2}},$$

where $\gamma_{12} = a_3 = m_1 m_2 - Q_1 Q_2$, $\gamma_{23} = a_1 = m_2 m_3 - Q_2 Q_3$ and $\gamma_{13} = a_2 = m_1 m_3 - Q_1 Q_3$. With the change of coordinates the potential and moment of inertia can be written as $V = \sum_{i<j} \frac{\gamma_{ij}}{\rho_{ij}}$ and $I = m_i m_j \rho_{ij}^2$. Using the definition

$$\frac{\partial V}{\partial \rho_{ij}} = -\lambda \frac{\partial I}{\partial \rho_{ij}}$$

$$\frac{\gamma_{ij}}{\rho_{ij}^2} = \lambda 2m_i m_j \rho_{ij},$$

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we get the following solution

$$\rho_{ij}^3 = \frac{\gamma_{ij}}{\lambda m_i m_j}.$$ 

**Constraint for triangular configurations:** The above relationship will present a unique central configurations solutions, if $\gamma_{12}, \gamma_{23}$ and $\gamma_{13}$ have the same signs. Furthermore, this solution will present triangular central configurations solution, if the above relationship satisfy the condition of triangle inequality in such way that each particle is at the vertices of the triangle i.e.,

$$(m_i \gamma_{ik})^{\frac{1}{3}} + (m_j \gamma_{jk})^{\frac{1}{3}} > (m_k \gamma_{ki})^{\frac{1}{3}}.$$ 

where $(i, j, k)$ permute cyclically in $(1, 2, 3)$.

**Remark.** In the following we mention important remarks

(i) Non-collinear central configurations exits only if the charges have the same signs, that is if $a_3, a_2$ and $a_1$ have the same sign. This is because if there are charges of different signs not all of the triangle inequality can be satisfied simultaneously.

(ii) In case if in $a_3, a_1, a_2$ the charges are zero, then $a_3, a_1, a_2$ are equal and we get the Lagrange equilateral triangular configurations.

(iii) The total number of non-collinear central configurations in the planar case are either 0 or 2. But in case of $\mathbb{R}^3$, if the signs of the parameter satisfy the triangle inequality then the total number of non-collinear central configurations is one.

### 4.4 Conclusion and outlook

The integral manifolds of an $N$-body system may change topology at the critical values of the map of integrals. Some critical values result from central configurations. In this chapter we studied the central configurations of the charged three-body system. We distinguished between collinear and non-collinear central configurations. For the collinear case, we presented a comprehensive study by reducing the space of collinear configurations by the symmetries. This results in a single equation for the central configurations which however depends on several parameters. We found that the number of solutions is 0, 1, 2 or 3 depending on the values of these parameters. For the non-collinear case, there is at most one central configuration.