A geometric approach to multi-modal and multi-agent systems
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Document Version
Publisher's PDF, also known as Version of record

Publication date:
2016

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

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DISTURBANCE DECOUPLING FOR PIECEWISE AFFINE SYSTEMS

ABSTRACT: In this chapter we study the disturbance decoupling problem for continuous piecewise affine systems. We establish a set of necessary conditions and a set of sufficient conditions, both geometric in nature, for such systems to be disturbance decoupled. Furthermore, we investigate mode-independent state feedback controllers for piecewise affine systems and provide sufficient conditions for the solvability of the disturbance decoupling problem by state feedback. This chapter is based on the conference paper [Everts and Camlibel, 2014a].

3.1 INTRODUCTION

In this chapter we continue studying the disturbance decoupling problem for linear multi-modal systems. We extend the bimodal systems studied in Chapter 2 to more general continuous piecewise affine systems. Piecewise affine systems are a class of hybrid systems; they are a combination of continuous-time linear systems, the modes, together with the discrete dynamics of switching between these modes.

The switching between the several modes of a piecewise affine system is state-dependent. As discussed in more detail in Sections 1.1, 1.2 and 2.1, this state-dependent switching behavior of piecewise affine systems calls for a different approach than for switched linear systems, for which the switching is state-independent.

In this chapter, we develop a new approach that takes into account the state-dependent switching behavior of piecewise affine systems. This approach allows us to provide a set of necessary conditions and a set of sufficient conditions under which a continuous piecewise affine system is disturbance decoupled. Although these conditions do not coincide in general, we point out some special cases in which they do coincide. Furthermore, we present conditions for the existence of mode-independent static feedback controllers that render the closed-loop system disturbance decoupled. All conditions we present are geometric in nature and easily verifiable.

The following section introduces the class of continuous piecewise affine systems. For this class of systems, we define
the disturbance decoupling problem in Section 3.3 and give a set of necessary conditions and a set of sufficient conditions for such a system to be disturbance decoupled. In Section 3.4, we provide conditions under which the necessary conditions and the sufficient conditions coincide. The problem of disturbance decoupling by state feedback is discussed in Section 3.5. Finally, Section 3.6 contains the main conclusions of this chapter.

3.2 PIECEWISE AFFINE SYSTEMS

Before we can define the class of continuous piecewise affine systems, we need the notions of affine functions and piecewise affine functions. An affine function is a function $\theta : \mathbb{R}^n \to \mathbb{R}^m$ of the form $\theta(x) = Qx + q$, with $Q$ a $m \times n$ matrix and $q$ an $m$-vector. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called piecewise affine if there exists a finite set of affine functions $f_i : \mathbb{R}^n \to \mathbb{R}^m, i \in \mathcal{I}_p$, such that for each $x \in \mathbb{R}^n$ we have

$$f(x) \in \{f_1(x), f_2(x), \ldots, f_p(x)\}.$$  

The domain of a continuous piecewise affine function can be divided into a set of polyhedral regions in such a way that the restriction of the function $f$ to any of the regions is given by an affine function [Scholtes, 2012, Prop. 2.2.3]. To make this statement more precise, we quickly review some definitions and results about polyhedral sets.

A polyhedron (or polyhedral set) in $\mathbb{R}^n$ is the intersection of a finite number of closed half-spaces. Therefore, a polyhedron $P$ can be represented as $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A$ is a $p \times n$ matrix and $b$ is a $p$-vector. The dimension of a polyhedron is equal to the dimension of its affine hull. We call a polyhedron solid if it has dimension $n$. A subset $F$ of a polyhedron $P$ is called a face of $P$ if there is a vector $y \in \mathbb{R}^n$ such that

$$F = \{x \in P \mid y^T x \geq y^T z \text{ for every } z \in P\}.$$  

A face of a polyhedron $P$ is also a polyhedron and is called proper if its dimension is strictly less than that of $P$. If a face is $(n - 1)$-dimensional we call it a facet.

A finite collection $\Xi = \{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N\}$ of polyhedral sets in $\mathbb{R}^n$ is a polyhedral subdivision of $\mathbb{R}^n$ if every polyhedron in $\Xi$ is solid, the union of all polyhedra in $\Xi$ equals $\mathbb{R}^n$, and the intersection of any two polyhedra in $\Xi$ is either empty or a common proper face of both polyhedra (see Figure 3.1).
3.2 PIECEWISE AFFINE SYSTEMS

Figure 3.1: (a) Example of a polyhedral subdivision of $\mathbb{R}^2$. (b) This is not a polyhedral subdivision of $\mathbb{R}^2$, since the intersection of $\mathcal{X}_1$ and $\mathcal{X}_2$ is a proper face of $\mathcal{X}_2$ but not of $\mathcal{X}_1$.

As shown in [Scholtes, 2012, Prop. 2.2.3], for a given continuous piecewise affine function $f$ there are a finite number of polyhedral sets $\mathcal{X}_k$ with corresponding matrices $Q_k \in \mathbb{R}^{m \times n}$ and vectors $q_k \in \mathbb{R}^m$ for all $k \in \mathcal{I}_N$, such that $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N\}$ is a polyhedral subdivision of $\mathbb{R}^n$ and $f$ satisfies

$$f(x) = Q_k x + q_k \quad \forall x \in \mathcal{X}_k.$$  

A continuous piecewise affine system is a system of the form

$$\begin{align*}
\dot{x}(t) &= f(x(t)) + Ed(t) \\
z(t) &= Hx(t),
\end{align*}$$

(3.1a)

(3.1b)

where $x \in \mathbb{R}^{n_x}$ is the state, $d \in \mathbb{R}^{n_d}$ is the unknown disturbance, $z \in \mathbb{R}^{n_z}$ is the output to be controlled, $E$ and $H$ are matrices of appropriate sizes, and $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is a continuous piecewise affine function.

Since the right-hand side of (3.1a) is Lipschitz continuous in the variable $x$, for each $x_0$ and locally integrable disturbance $d$ there exists a unique absolutely continuous function $x^{x_0,d}(t)$ satisfying $x(0) = x_0$ and (3.1a) for almost all $t$. We denote the corresponding output by $z^{x_0,d}(t)$.

As stated above, the function $f$ admits a polyhedral subdivision. So there are polyhedral regions $\mathcal{X}_k$, matrices $A_k$ and vectors $g_k$, for each $k \in \mathcal{I}_N$, such that $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N\}$ is a polyhedral subdivision of $\mathbb{R}^{n_x}$ and

$$f(x) = A_k x + g_k \quad \forall x \in \mathcal{X}_k.$$
Hence, we can write system (3.1) as
\[ \dot{x}(t) = A_k x(t) + g_k + Ed(t) \quad \forall x \in \mathcal{X}_k \quad (3.2a) \]
\[ z(t) = Hx(t). \quad (3.2b) \]

We can exploit the continuity of \( f(x) \) to obtain relations between the matrices \( A_k \). Note that for any two polyhedral regions \( \mathcal{X}_k \) and \( \mathcal{X}_\ell \) sharing a facet \( F_{k\ell} := \mathcal{X}_k \cap \mathcal{X}_\ell \), we can choose a vector \( c_{k\ell} \) and a scalar \( f_{k\ell} \) such that the affine hull of \( F_{k\ell} \) is given by the hyperplane
\[ H_{k\ell} := \{ x \in \mathbb{R}^n \mid c_{k\ell}^T x + f_{k\ell} = 0 \}. \]

The continuity of \( f \) implies that for all \( x \in F_{k\ell} \subseteq H_{k\ell} \) we have
\[ A_k x + g_k = A_\ell x + g_\ell, \]
or equivalently
\[ (A_k - A_\ell)x + g_k - g_\ell = 0. \quad (3.3) \]

Since \( F_{k\ell} \) is \((n_x - 1)\)-dimensional, it follows that \( \ker c_{k\ell}^T \subseteq \ker(A_k - A_\ell) \). Hence, there is a vector \( h_{k\ell} \in \mathbb{R}^{n_x} \) such that
\[ A_k - A_\ell = h_{k\ell} c_{k\ell}^T. \quad (3.4) \]

By combining this with (3.3) and the fact that \( c_{k\ell}^T x + f_{k\ell} = 0 \) for all \( x \in F_{k\ell} \), we find that \( g_k \) and \( g_\ell \) satisfy
\[ g_k - g_\ell = h_{k\ell} f_{k\ell}. \quad (3.5) \]

Notice that, since facet \( F_{k\ell} \) is equal to facet \( F_{\ell k} \), we can assume that \( c_{k\ell} = c_{\ell k}, f_{k\ell} = f_{\ell k} \) and \( h_{k\ell} = -h_{\ell k} \). If the state \( x \) passes from one polyhedral region to another, it will always cross one or more facets, which is why facets will play an important role in this chapter.

**Example 3.1** Conewise linear systems are a special class of piecewise linear systems. In such systems, the polyhedral regions are convex cones and the corresponding subsystems are linear. As an example, consider the piecewise linear system with four modes, given by
\[
A_1 = \begin{bmatrix} 14 & 11 \\ 10.25 & 8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 6 & 7 \\ 0.25 & 3 \end{bmatrix}, \\
A_3 = \begin{bmatrix} 6 \\ 0.25 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 14 & 8 \\ 10.25 & 10 \end{bmatrix}, \quad E = 0,
\]
\[ f_{12} = f_{23} = f_{34} = f_{41} = g_1 = g_2 = g_3 = g_4 = 0, \]
\[ c_{23}^T = [2 \quad 1], \quad c_{34}^T = [1 \quad 1], \quad c_{12}^T = c_{41}^T = [1 \quad 0]. \]

See Figure 3.2 for a sketch of the corresponding vector field.
Figure 3.2: Sketch of the vector field corresponding to the continuous piecewise affine system with four modes (without disturbances) in Example 3.1. The line segments denote the facets between the polyhedral regions.

For continuous piecewise affine systems, it is sometimes more convenient to write system (3.2) in the following alternative way:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Ed(t) + g(y) \\
y(t) &=Cx(t) \\
z(t) &= Hx(t)
\end{align*}
\] (3.6a)

where \(x, z, E\) and \(H\) are as before, \(y \in \mathbb{R}^{n_y}\) is the measured output, \(A\) and \(C\) are matrices of appropriate sizes, and \(g : \mathbb{R}^{n_y} \to \mathbb{R}^{n_x}\) is a continuous piecewise affine function. In this representation, the domain \(\mathbb{R}^{n_y}\) of \(g\) admits a polyhedral subdivision: there are solid (i.e., \(n_y\)-dimensional) polyhedral regions \(\mathcal{Y}_k\), matrices \(F_k \in \mathbb{R}^{n_x \times n_y}\) and vectors \(g_k \in \mathbb{R}^{n_x}\), for all \(k \in \mathcal{I}_N\), such that \(\{\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_N\}\) is a polyhedral subdivision of \(\mathbb{R}^{n_y}\) and

\[
g(y) = F_k y + g_k, \quad \text{if} \ y \in \mathcal{Y}_k. \tag{3.6d}
\]

If \(\mathcal{Y}_k\) and \(\mathcal{Y}_\ell\) share a facet \(\bar{F}_{k\ell}\), there is a vector \(\bar{c}_{k\ell} \in \mathbb{R}^{n_y}\) and scalar \(\bar{f}_{k\ell}\) such that

\[
\bar{F}_{k\ell} \subseteq \{y \in \mathbb{R}^{n_y} \mid \bar{c}_{k\ell}^T y + \bar{f}_{k\ell} = 0\}.
\]

Since \(g\) is continuous, we can employ the same reasoning as above to see that there is a vector \(\bar{h}_{k\ell}\) such that

\[
F_k - F_\ell = \bar{h}_{k\ell} \bar{c}_{k\ell}^T, \quad g_k - g_\ell = \bar{h}_{k\ell} \bar{f}_{k\ell}.
\]
To see the equivalence between the two representations, notice that we can write system (3.2) in the form of system (3.6) by taking $C = I$, $A = A_1$, $Y_k = X_k$ and $F_k = A_k - A_1$ for $k \in \mathcal{I}_N$. On the other hand, we can write system (3.6) in the form of system (3.2) by letting $A_k = A + F_k C$ and $\lambda_k = C^{-1} Y_k$ for $k \in \mathcal{I}_N$, and using the same $g_k$, $E$ and $H$. It can be shown that resulting set $\{X_1, X_2, \ldots, X_N\}$ is a polyhedral subdivision of $\mathbb{R}^n$. The corresponding facets $F_k^{\ell}$ are equal to $C^{-1} \tilde{F}_k^{\ell}$, with $c_k^{\ell} = \tilde{c}_k^{\ell} C$ and $f_k^{\ell} = \tilde{f}_k^{\ell}$.

Combinations of linear systems and static (piecewise linear) nonlinearities, such as saturation, dead-zone and backlash, lead naturally to piecewise affine systems. A concrete example of a continuous piecewise affine system is given next.

**Example 3.2** ([Thuan and Camlibel, 2014, Example 2.2]) In high-accuracy motion control of a DC servo system, one has to deal with deadzone-type nonlinear relations between the motor torque $T$ and the current $i$ through the motor windings (see e.g. [Zhonghua et al., 2006]). This can be modeled by the continuous piecewise affine function

$$g(y) = \begin{cases} k_T y - T_- - T_\ell & \text{if } k_T y \leq T_- \\ -T_\ell & \text{if } T_- \leq k_T y \leq T_+ \\ k_T y - T_+ - T_\ell & \text{if } T_+ \leq k_T y, \end{cases}$$  

(3.7)

with $k_T$ the torque constant, $T_\ell$ the torque applied to the rotor, and $T_-$ and $T_+$ constant values. If we assume that $T_\ell$ is constant, we can describe the dynamics of the current $i$ through the motor windings and the angular position $\theta$ of the rotor with the following piecewise affine system:

$$\begin{align*}
\frac{d}{dt} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} &= \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{k_b}{L} \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{B_f}{J} \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{g}(y) \end{bmatrix} \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix},
\end{align*}$$

(3.8a, 3.8b)

where $\omega = \dot{\theta}$, $J$ is the moment of inertia of the rotor, and $L$, $R$, $k_b$, and $B_f$ are some constants (see [Thuan and Camlibel, 2014] for details).
3.3 THE DISTURBANCE DECOUPLING PROBLEM

We say that a piecewise affine system, given by (3.1), (3.2) or (3.6), is disturbance decoupled if for all initial states \( x_0 \in \mathbb{R}^{n_x} \) and all locally integrable disturbances \( d_1 \) and \( d_2 \) we have

\[
z^{x_0,d_1}(t) = z^{x_0,d_2}(t), \quad \forall t \geq 0.
\]

In this section, we give a necessary condition for a piecewise affine system to be disturbance decoupled, as well as a sufficient condition. Both conditions are geometric in nature.

**Theorem 3.3** If the system (3.2) is disturbance decoupled, then

\[
\sum_{k=1}^{N} \langle A_k \mid \text{im } E \rangle \subseteq \ker H. \tag{3.9}
\]

**Proof.** Let \( k \in \mathcal{I}_N \), and let \( d_1(t) = d \in \mathbb{R}^{n_d} \) and \( d_2 = 0 \) be two distinct constant disturbances. Since \( X_k \) is solid, we can choose an interior point \( x_0 \) of \( X_k \). Let \( x_i(t) \) denote the trajectory \( x^{x_0,d_i}(t) \) and let \( z_i(t) \) denote the corresponding output, for \( i = 1, 2 \). Since \( x_1 \) and \( x_2 \) are continuous, there exists \( \varepsilon > 0 \) such that \( x_1(t) \) and \( x_2(t) \) stay in \( X_k \) for \( t \in [0, \varepsilon) \). Thus, the trajectories \( x_1 \) and \( x_2 \) satisfy

\[
\dot{x}_i(t) = A_k x_i(t) + g_k + E d_i(t), \quad \text{for } t \in [0, \varepsilon), \ i = 1, 2.
\]

As the system is disturbance decoupled, we have that \( z_1(t) = z_2(t) \) and hence

\[
H x_1(t) = H x_2(t) \tag{3.10}
\]

for all \( t \geq 0 \). Since \( d_1 \) and \( d_2 \) are constant, we can differentiate equation (3.10) \( p \geq 1 \) times and obtain

\[
HA_k^p x_1(t) + HA_k^{p-1} Ed = HA_k^p x_2(t)
\]

for all \( t \in [0, \varepsilon) \). Using \( t = 0 \) and \( x_1(0) = x_2(0) \) we get

\[
HA_k^p Ed = 0 \quad \forall p \geq 0.
\]

Since this holds for any vector \( d \in \mathbb{R}^{n_d} \), we conclude that \( HA_k^p E = 0 \) for all \( p \geq 0 \), and hence by (1.2) we have \( \langle A_k \mid \text{im } E \rangle \subseteq \ker H \). By letting \( k \) vary over \( \{1, \ldots, N\} \), we see that (3.9) holds.

For the alternative representation of the system, given by (3.6), we have the following corollary.
Corollary 3.4 If the system (3.6) is disturbance decoupled, then

$$\sum_{k=1}^{N} \langle A + F_k C \mid \text{im} E \rangle \subseteq \ker H. \quad (3.11)$$

In general, the subspace $$\sum_{k=1}^{N} \langle A_k \mid \text{im} E \rangle$$ that appears in Theorem 3.3 is not necessarily invariant under $$A_k$$ for all $$k \in \mathcal{I}_N$$. The following theorem shows that such joint invariance relations lead to a sufficient condition.

Theorem 3.5 The system (3.2) is disturbance decoupled if there is a subspace $$\mathcal{V}$$ of $$\ker H$$ that contains $$\text{im} E$$ and that is invariant under $$A_k$$ for all $$k \in \mathcal{I}_N$$.

Proof. Let $$r$$ be the dimension of $$\mathcal{V}$$ and write $$x = \text{col}(v, w)$$, where $$v$$ consists of the first $$r$$ entries of $$x$$. Note that a piecewise affine function is still piecewise affine after a basis transformation. Moreover, the property of disturbance decoupling is invariant under basis transformations as well. Therefore, we can assume without loss of generality that the vectors $$\text{col}(v, 0)$$ correspond to the subspace $$\mathcal{V}$$.

Since $$\mathcal{V}$$ is invariant under each $$A_k$$ and satisfies $$\text{im} E \subseteq \mathcal{V} \subseteq \ker H$$, the system matrices are of the form

$$E = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & H_2 \end{bmatrix},$$

$$A_k = \begin{bmatrix} A_{11}^k & A_{12}^k \\ 0 & A_{22}^k \end{bmatrix}, \quad g_k = \begin{bmatrix} g_1^k \\ g_2^k \end{bmatrix}, \quad k \in \mathcal{I}_N,$$

where $$A_{11}^k \in \mathbb{R}^{r \times r}$$, $$g_1^k \in \mathbb{R}^r$$, $$E_1 \in \mathbb{R}^{r \times n_d}$$ and $$H_2 \in \mathbb{R}^{n_z \times (n_x - r)}$$. Hence, we can write system (3.2) as

$$\dot{v} = A_{11}^k v + A_{12}^k w + E_1 d + g_1^k \quad \forall \text{col}(v, w) \in \mathcal{X}_k$$

$$\dot{w} = A_{22}^k w + g_2^k \quad \forall \text{col}(v, w) \in \mathcal{X}_k$$

$$z = H_2 w.$$

Notice that the output $$z$$ depends only on $$w$$ and that $$w$$ does not directly depend on the disturbance $$d$$. However, the disturbance might influence the switching behavior of $$x$$ and in this way the disturbance might still influence the evolution of $$w$$. In the rest of the proof, we will show that this is not the case.

Since the subspace $$\mathcal{V}$$ is invariant under all $$A_k$$, we also have

$$(A_i - A_j) \mathcal{V} \subseteq \mathcal{V}$$
for each $i$ and $j$. In particular, when the polyhedral regions $X_i$ and $X_j$ share a facet $F_{ij}$ we can use equation (3.4) to find that

$$
h_{ij}c_{ij}^T \mathcal{V} \subseteq \mathcal{V}.
$$

It follows that we have $h_{ij} \in \mathcal{V}$ or $\mathcal{V} \subseteq \ker c_{ij}^T$. We write $h_{ij} = \text{col}(h_{ij,1}, h_{ij,2})$ and $c_{ij} = \text{col}(c_{ij,1}, c_{ij,2})$, where $h_{ij,1}$ and $c_{ij,1}$ are $r$-dimensional. Note that if $h_{ij} \in \mathcal{V}$, then $h_{ij,2} = 0$. Consequently, using equations (3.4) and (3.5), we see that in this case $A_{22}^i = A_{22}^j$ and $g_2^i = g_2^j$. Next, we will use this observation to define clusters of modes.

We partition $I_N$ into equivalence classes as follows: $i$ and $j$ are in the same equivalence class if $A_{22}^i = A_{22}^j$ and $g_2^i = g_2^j$. Let $I_1, I_2, \ldots, I_p$ denote the resulting equivalence classes and define clusters $C_1, C_2, \ldots, C_p$ as

$$
C_\ell = \bigcup_{k \in I_\ell} X_k
$$

for all $\ell \in I_p$. Although the union of the clusters is equal to $\mathbb{R}^{nx}$, $\{C_1, C_2, \ldots, C_p\}$ is not necessarily a polyhedral subdivision of $\mathbb{R}^{nx}$, since a cluster is not necessarily convex. However, within the cluster $C_\ell$, $w$ satisfies

$$
w = A_{22}^\ell w + g_2^\ell \quad \forall x \in C_\ell.
$$

Note that in the case that there is just one distinct cluster, equal to $\mathbb{R}^{nx}$, $w$ satisfies an autonomous affine system, which implies that the system (3.2) is disturbance decoupled.

If there are two or more clusters, then we have $A_{22}^i \neq A_{22}^j$ or $g_2^i \neq g_2^j$ for any facet $F_{ij}$ for which $i$ and $j$ are not in the same equivalence class. Both inequalities imply that $h_{ij,2} \neq 0$, which means that $h_{ij}$ is not an element of $\mathcal{V}$. Consequently, the normal $c_{ij}$ of the facet $F_{ij}$ satisfies $\mathcal{V} \subseteq \ker c_{ij}^T$, and hence $c_{ij}$ must be of the form $c_{ij} = \text{col}(0, c_{ij,2})$. Since there are at least two clusters, there is at least one such cluster-separating facet.

Suppose that a point $a$ is in cluster $C_\ell$, but another point $b$ is not. Then the line segment between $a$ and $b$ must intersect a cluster-separating facet $F_{ij}$ for some $i$ and $j$. For this facet, $c_{ij}^Ta + f_{ij}$ and $c_{ij}^Tb + f_{ij}$ have different signs. Hence, although the clusters might not be convex, we can determine for each $x \in \mathbb{R}^{nx}$ in which cluster it resides by only checking the values of $c_{ij}^Tx + f_{ij}$ for each cluster-separating facet $F_{ij}$. For such cluster-separating facets, we have $c_{ij}^Tx + f_{ij} = c_{ij,2}^Tw + f_{ij}$. Consequently,
the value of \( w \) is enough to completely determine the cluster that \( x \) is in, so \( w \) determines the switching between modes. Hence, we have

\[
\dot{w} = A_{22}^f w + g_{2}^f \quad \text{for} \quad \begin{bmatrix} 0 \\ w \end{bmatrix} \in \mathcal{C}_f.
\] (3.12)

Thus we see that \( w \) satisfies an autonomous piecewise affine differential equation.

We are now in a position to prove that system (3.1) is disturbance decoupled. Let \( x_0 \) be any initial condition and let \( d_1 \) and \( d_2 \) be two locally integrable disturbances. Denote the two corresponding trajectories by \( x_i(t) = x_{x0,d_i}(t) \), \( i = 1, 2 \), and write \( x_i = \text{col}(v_i, w_i) \). From (3.12) we see that \( w_1(t) = w_2(t) \) for all \( t \geq 0 \). Consequently, we have \( z_1(t) = z_2(t) \) for all \( t \geq 0 \), and hence the system is disturbance decoupled.

For the alternative representation of the system, given by (3.6), we have the following corollary.

**Corollary 3.6** The system (3.6) is disturbance decoupled if there is a subspace \( \mathcal{V} \subseteq \ker H \) that contains \( \text{im} E \) and that is invariant under \( A + F_kC \) for all \( k \in \mathcal{I}_N \).

### 3.4 NECESSARY AND SUFFICIENT CONDITIONS

The sufficient conditions for system (3.2) to be disturbance decoupled, as given by Theorem 3.5, do not coincide in general with the necessary conditions provided by Theorem 3.3, because \( \sum_{i=1}^{N} \langle A_i \mid \text{im} E \rangle \) is not necessarily invariant under each \( A_i \). In this section, we identify a number of particular cases for which the conditions do coincide.

**Corollary 3.7** If \( \sum_{i=1}^{N} \langle A_i \mid \text{im} E \rangle \) is invariant under \( A_i \) for all \( i \in \mathcal{I}_N \), then system (3.2) is disturbance decoupled if and only if

\[
\sum_{i=1}^{N} \langle A_i \mid \text{im} E \rangle \subseteq \ker H.
\]

**Proof.** Theorem 3.3 implies the necessity of the condition. For the sufficiency, let \( \mathcal{V} = \sum_{i=1}^{N} \langle A_i \mid \text{im} E \rangle \). Then \( \text{im} E \subseteq \mathcal{V} \), and by assumption we have \( \mathcal{V} \subseteq \ker H \) and \( A_i \mathcal{V} \subseteq \mathcal{V} \) for all \( i \in \mathcal{I}_N \). Hence, using Theorem 3.5, we see that system (3.2) is disturbance decoupled. ■
To investigate when $\sum_{i=1}^{N} \langle A_i \mid \text{im } E \rangle$ is invariant under $A_1, A_2, \ldots, A_N$, we first look at the case that $N = 2$, which corresponds to a bimodal linear system, as discussed in Chapter 2.

**Lemma 3.8** For two square matrices $A_1$ and $A_2$ satisfying $A_1 - A_2 = hc^T$, the subspace $\langle A_1 \mid \text{im } E \rangle + \langle A_2 \mid \text{im } E \rangle$ is invariant under both $A_1$ and $A_2$. Furthermore, we have $h \in \langle A_1 \mid \text{im } E \rangle + \langle A_2 \mid \text{im } E \rangle$, or $\langle A_1 \mid \text{im } E \rangle + \langle A_2 \mid \text{im } E \rangle \subseteq \ker c^T$.

**Proof.** Let $\mathcal{V} = \langle A_1 \mid \text{im } E \rangle + \langle A_2 \mid \text{im } E \rangle$. Since $\text{im } A_1 E$ and $\text{im } A_2 E$ are both in $\mathcal{V}$, we see that
\[
\text{im } hc^T E \subseteq \mathcal{V}.
\]
This implies that either $h \in \mathcal{V}$, or $c^T E = 0$. Suppose that $h \not\in \mathcal{V}$, then we must have $c^T E = 0$, which gives us $A_1 E = (A_2 + hc^T) E = A_2 E$. Since $\text{im } A_1^2 E$ and $\text{im } A_2^2 E$ are both contained in $\mathcal{V}$, we see that $\text{im } (A_1^2 - A_2^2) E \subseteq \mathcal{V}$, so
\[
\text{im } hc^T A_2 E = \text{im } (A_1 - A_2) A_2 E = \text{im } (A_1^2 - A_2^2) E \subseteq \mathcal{V}.
\]
Hence, since $h \not\in \mathcal{V}$, we have $c^T A_2 E = 0$, which implies
\[
A_2^2 E = A_1 A_2 E = (A_2 + hc^T) A_2 E = A_2^2 E.
\]
By continuing this argument, we see that $c^T A_2^k E = c^T A_1^k E = 0$ and $A_2^k E = A_1^k E$ for all $k \geq 0$. This implies that $\mathcal{V} = \langle A_1 \mid \text{im } E \rangle = \langle A_2 \mid \text{im } E \rangle$ and $\mathcal{V} \subseteq \ker c^T$. Hence, we have $h \in \mathcal{V}$ or $\mathcal{V} \subseteq \ker c^T$. This means that $hc^T v \in \mathcal{V}$ for any $v \in \mathcal{V}$. Since any $v \in \mathcal{V}$ can be written as $v = v_1 + v_2$, with $v_i \in \langle A_i \mid \text{im } E \rangle$, $i = 1, 2$, we have $A_1 v = A_1 v_1 + A_1 v_2 = A_1 v_1 + A_2 v_2 + hc^T v_2 \in \mathcal{V}$ for all $v \in \mathcal{V}$. Similarly, we get $A_2 v \in \mathcal{V}$ for all $v \in \mathcal{V}$. Hence, $\mathcal{V}$ is invariant under both $A_1$ and $A_2$. \(\blacksquare\)

Next, we find a sufficient condition for $\sum_{i=1}^{N} \langle A_i \mid \text{im } E \rangle$ to be invariant under all $A_i$.

**Lemma 3.9** Consider system (3.2). If $h_{k\ell} \in \sum_{i=1}^{N} \langle A_i \mid \text{im } E \rangle$ for all facets $F_{k\ell}$, then $\sum_{i=1}^{N} \langle A_i \mid \text{im } E \rangle$ is invariant under $A_i$ for all $i \in \mathcal{I}_N$.

**Proof.** From Theorem 2 in [Shen, 2014] we know that for every $i, j \in \mathcal{I}_N$ there is a finite sequence of indices $k_1, k_2, \ldots, k_{r+1}$
such that \( k_1 = i, k_{r+1} = j \) and such that \( X_{k_s} \) and \( X_{k_{s+1}} \) share a facet for each \( s \in \mathcal{I}_r \). Hence, we can write \( A_i \) as

\[
A_i = A_j + \sum_{s=1}^{r} h_{k_sk_{s+1}} c_{k_sk_{s+1}}^T.
\]

Therefore, for any element \( v_j \in \langle A_j \ | \ \text{im} \ E \rangle \) we have

\[
A_i v_j = A_j v_j + \sum_{s=1}^{r} h_{k_sk_{s+1}} c_{k_sk_{s+1}}^T v_j \in \mathcal{V},
\]

since \( h_{k_sk_{s+1}} \in \mathcal{V} \) for all \( s \in \mathcal{I}_r \). Hence, we have \( A_i \langle A_j \ | \ \text{im} \ E \rangle \subseteq \mathcal{V} \) for every \( i \) and \( j \) and we can conclude that \( \mathcal{V} \) is invariant under each \( A_i \) for all \( i \in \mathcal{I}_N \). \( \blacksquare \)

We now investigate two special cases of systems for which the necessary conditions and sufficient conditions for disturbance decoupling coincide.

**Corollary 3.10** Consider system (3.6). If \( C(sI - A)^{-1} E \) is right-invertible as a rational matrix, then system (3.6) is disturbance decoupled if and only if

\[
\sum_{k=1}^{N} \langle A + F_k C \ | \ \text{im} \ E \rangle \subseteq \ker H.
\]

**Proof.** We begin by proving the following claim: if \( C(sI - A)^{-1} E \) is right-invertible, then so is \( C(sI - A - FC)^{-1} E \) for any matrix \( F \in \mathbb{R}^{n_y \times n_x} \). For this we use the well-known property

\[
(sI - B)^{-1} - (sI - A)^{-1} = (sI - B)^{-1}(B - A)(sI - A)^{-1}.
\]

We take \( B = A + FC \) and multiply both sides with \( C \) from the left and with \( E \) from the right. Rearranging the terms then gives us

\[
C(sI - A - FC)^{-1} E = (I + C(sI - A - FC)^{-1} F) C(sI - A)^{-1} E.
\]

Since \( I + C(sI - A - FC)^{-1} F \) and \( C(sI - A)^{-1} E \) are both right-invertible as a rational matrices, the claim follows.

Let \( \mathcal{V} = \sum_{k=1}^{N} \langle A + F_k C \ | \ \text{im} \ E \rangle \). From the claim above it follows that \( C(sI - A - F_i C)^{-1} E \neq 0 \) for each \( i \), so for any facet \( \tilde{F}_{k\ell} \) we have \( \tilde{c}_{k\ell}^T C(sI - A - F_i C)^{-1} E \neq 0 \) since \( \tilde{c}_{k\ell}^T \neq 0 \). Equivalently, \( \tilde{c}_{k\ell}^T C \langle A + F_i C \ | \ \text{im} \ E \rangle \neq \{0\} \). Hence, by Lemma
we see that \( h_{k\ell} \in \mathcal{V} \) for all facets \( \tilde{F}_{k\ell} \). Then, by Lemma 3.9, \( \mathcal{V} \) is invariant under all \( A + F_k C \). From Corollaries 3.6 and 3.7 we see that system (3.6) is disturbance decoupled if and only if \( \mathcal{V} \subseteq \ker H \).

\[ \]

**Corollary 3.11** Consider system (3.2). If all normals \( c_{ij} \) to facets \( F_{ij} \) are parallel, then system (3.2) is disturbance decoupled if and only if

\[ \sum_{i=1}^{N} \langle A_i \mid \text{im}E \rangle \subseteq \ker H. \]

**Proof.** Let \( \mathcal{V} = \sum_{i=1}^{N} \langle A_i \mid \text{im}E \rangle \). If all normals \( c_{ij}^{T} \) to facets \( F_{ij} \) are parallel, then all the facets are parallel. This means that the state space is sliced up into parallel regions, each of which shares a facet with at most two other regions.

Suppose that there is a facet \( F_{ij} \) for which we have

\[ c_{ij}^{T} \langle A_i \mid \text{im}E \rangle = \{0\}, \]

then this implies that \( c_{ij}^{T} A_{i}^{p} E = 0 \) for all \( p \geq 0 \). It follows that \( A_{i}^{p} E = A_{j}^{p} E \) for all \( p \geq 0 \), which we will prove by mathematical induction. Clearly it holds for \( p = 0 \). Suppose that it holds for some value of \( p \), then for \( p + 1 \) we have

\[ A_{j}^{p+1} E = A_{j} A_{j}^{p} E = (A_{i} - h_{ij} c_{ij}^{T}) A_{i}^{p} E = A_{i}^{p+1} E, \]

which proves the claim. Consequently, we have that \( \langle A_i \mid \text{im}E \rangle = \langle A_j \mid \text{im}E \rangle \), so we also have that

\[ c_{ij}^{T} \langle A_j \mid \text{im}E \rangle = \{0\}. \]

Moreover, if \( X_j \) shares a facet \( F_{jk} \) with some other region \( X_k \) as well, then we see that

\[ c_{jk}^{T} \langle A_j \mid \text{im}E \rangle = \{0\}, \]

since \( c_{jk} \) is a multiple of \( c_{ij} \). By the same reasoning as above, we see that \( \langle A_k \mid \text{im}E \rangle = \langle A_j \mid \text{im}E \rangle \). By continuing this argument from region to region, we find that for all facets \( F_{ij} \) we have \( c_{ij}^{T} \langle A_i \mid \text{im}E \rangle = \{0\} \) and \( \langle A_j \mid \text{im}E \rangle = \langle A_i \mid \text{im}E \rangle \). Hence, we conclude that \( \mathcal{V} = \langle A_i \mid \text{im}E \rangle \) for any \( i \in \mathcal{I}_N \). As a consequence, \( \mathcal{V} \) is invariant under \( A_i \) for all \( i \in \mathcal{I}_N \).

On the other hand, suppose that \( c_{ij}^{T} \langle A_i \mid \text{im}E \rangle \neq \{0\} \) for all facets \( F_{ij} \). From Lemma 3.8 we know that \( h_{ij} \in \mathcal{V} \) for all facets.
Using Lemma 3.9, we see that also in this case $V$ is invariant under $A_i$ for all $i \in \mathcal{I}_N$.

Hence, in both cases $V$ is invariant under all $A_i$. By Corollary 3.7, system (3.6) is disturbance decoupled if and only if $V \subseteq \ker H$. $\blacksquare$

**Example 3.12** We consider the system as given in Example 3.2 and add a disturbance $d$:

$$\frac{d}{dt} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{k_L}{L} \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{B_f}{L} \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g(y) \end{bmatrix} + Ed \quad (3.13a)$$

$$y = z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i & \theta & \omega \end{bmatrix}^T, \quad (3.13b)$$

with $E \in \mathbb{R}^{n_x \times 1}$ and $g(y)$ as in (3.7). To illustrate the theory developed in this section, we discuss whether this system is disturbance decoupled for three choices for $E$: $E_1$, $E_2$ and $E_3$, where $E_i$ is the $i$th column of the $3 \times 3$ identity matrix. First note that the system satisfies the conditions in Corollary 3.11. Hence, we only have to check if $\sum_{k=1}^3 \langle A_k \mid \text{im} \rangle \subseteq \ker H$, where $A_k = A + F_k C$. For $E_2$, we have $\langle A_k \mid \text{im} \rangle = \text{im} E \subseteq \ker H$ for $k = 1, 2, 3$. Therefore, $\sum_{k=1}^3 \langle A_k \mid \text{im} \rangle$ equals $\ker H$, implying that the system is disturbance decoupled. For $E_1$, we see that $\text{im} E_1 \not\subseteq \ker H$, hence $\sum_{k=1}^3 \langle A_k \mid \text{im} \rangle \not\subseteq \ker H$. For $E_3$, we have $\text{im} E \subseteq \ker H$, but $\sum_{k=1}^3 \langle A_k \mid \text{im} \rangle = \mathbb{R}^{n_x} \not\subseteq \ker H$. Consequently, the system is not disturbance decoupled for both $E = E_1$ and $E = E_3$.

### 3.5 State Feedback

In this section, we discuss the problem of finding a state feedback law that renders a given piecewise affine system disturbance decoupled. We consider the continuous piecewise affine system

$$\dot{x}(t) = f(x(t)) + Ed(t) + Bu(t) \quad (3.14a)$$

$$z(t) = Hx(t) \quad (3.14b)$$

with $x, d, z, E$ and $H$ as before, $u \in \mathbb{R}^{n_u}$ the input, $B \in \mathbb{R}^{n_x \times n_u}$, and $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ a piecewise affine function. Like before, the function $f$ admits a polyhedral subdivision $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N\}$ of $\mathbb{R}^{n_x}$. For each region $\mathcal{X}_k$, there are matrices $A_k$ and vectors...
state feedback $g_k$ such that $f(x) = A_kx + g_k$ for all $x \in \mathcal{X}_k$. Using this, we can write system (3.14) as
\[
\dot{x}(t) = A_kx(t) + g_k + Ed(t) + Bu(t) \quad \forall x \in \mathcal{X}_k 
\] (3.15a)
\[z(t) = Hx(t). \] (3.15b)

In the rest of this section we will investigate conditions for the existence of a mode-independent state feedback law that renders system (3.15) disturbance decoupled.

We consider a mode-independent feedback law $u = Kx$, for some matrix $K \in \mathbb{R}^{n_u \times n_x}$. Applying such a feedback law to system (3.15) results in the following closed-loop system
\[
\dot{x}(t) = (A_k + BK)x(t) + g_k + Ed(t) \quad \forall x \in \mathcal{X}_k 
\] (3.16a)
\[z(t) = Hx(t). \] (3.16b)

In view of Theorem 3.5 we see that system (3.16) is disturbance decoupled if there is a subspace $\mathcal{V}$ that satisfies
\[(A_i + BK)\mathcal{V} \subseteq \mathcal{V}, \quad \forall i \in \mathcal{I}_N \] (3.17)
\[\text{im } E \subseteq \mathcal{V} \subseteq \text{ker } H. \] (3.18)

To investigate whether such a subspace with a corresponding matrix $K$ exists, we define the following set of subspaces:
\[V(H, \{A_k\}_{k=1}^N, B) = \{ \mathcal{V} \subseteq \text{ker } H \mid \exists K \in \mathbb{R}^{n_u \times n_x} \text{ s.t. } (A_k + BK)\mathcal{V} \subseteq \mathcal{V} \text{ for all } k \in \mathcal{I}_N \}. \]

It is easy to see that $\mathcal{V} \in V(H, \{A_k\}_{k=1}^N, B)$ if and only if $A_1\mathcal{V} \subseteq \mathcal{V} + \text{im } B$ and $(A_i - A_j)\mathcal{V} \subseteq \mathcal{V}$ for all $i, j \in \mathcal{I}_N$. It follows that the set $V(H, \{A_k\}_{k=1}^N, B)$ is closed under subspace addition. Thus we can define $V^*(H, \{A_k\}_{k=1}^N) = \text{im } E \subseteq V^*(H, \{A_k\}_{k=1}^N, B)$ if and only if there is a subspace $\mathcal{V}$ satisfying (3.17)-(3.18). Hence, we arrive at the following theorem.

**Theorem 3.13** There exists a feedback law $u = Kx$ that renders the system (3.16) disturbance decoupled if
\[\text{im } E \subseteq V^*(H, \{A_k\}_{k=1}^N, B). \]

**Remark 3.14** To compute $V^*(H, \{A_k\}_{k=1}^N, B)$, we refer to [Yurtseven et al., 2012, Algorithm 5.3].
Example 3.15 We extend on Example 1 and 2, by adding state feedback in the form of applying a voltage $v$ to the motor. This results in the system

$$\frac{d}{dt} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{k_b}{L} \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{B_f}{J} \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g(y) \end{bmatrix} + Ed + Bu$$

(3.19a)

$$y = z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i & \theta & \omega \end{bmatrix}^T,$$  \hspace{1cm} (3.19b)

where we choose $B = [1 \ 0 \ 0]^T$ and $E = [0 \ 0 \ 1]^T$. Then for $K = [0 \ 0 \ k_b/L]$, we have that $(A_i + BK) \ker H \subseteq \ker H$ for $i = 1, 2, 3$, hence $\ker H \subseteq \mathcal{V}^*(H, \{A_1, A_2, A_3\}, B)$. On the other hand, $\mathcal{V}^*(H, \{A_1, A_2, A_3\}, B)$ is contained in $\ker H$. Therefore, we have $\mathcal{V}^*(H, \{A_1, A_2, A_3\}, B) = \ker H$. Since $\text{im } E \subseteq \ker H$, we conclude that the feedback $u = Kx$ renders the system disturbance decoupled.

3.6 CONCLUSIONS

In this paper, we established necessary conditions as well as sufficient conditions for a continuous piecewise affine system to be disturbance decoupled. These conditions do not coincide in general. However, we identified a number of particular cases for which they do coincide. Furthermore, we provided sufficient conditions for the existence of a mode-independent static feedback controller that renders a given piecewise affine system disturbance decoupled. All presented conditions are geometric in nature and can be easily verified by utilizing extensions of the well-known subspace algorithms.

Further research possibilities include investigating the gap between the necessary conditions and sufficient conditions as well as studying mode-dependent state feedback for disturbance decoupling.

In the next chapter we will study the disturbance decoupling problem for a particular class of linear complementarity problems, which are closely related to the piecewise affine systems in this chapter.