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## Controlled Invariance for Nonlinear Systems

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$$w = \tilde{f}(z, v) = \tilde{\Pi}(z, v). \tag{28}$$

We claim that the above system is a left inverse of  $\Sigma$ . Indeed, let  $u$  be an input with  $u(0) \in S(u_0)$ . Repeated differentiation of the output with respect to time and the definition of the order yields

$$y_i^{(\alpha)}(t, x_0, u) = f^{\alpha} h_i x(t, x_0, u) = \Pi(x(t, x_0, u), u(t)). \tag{29}$$

Now, if we start  $\tilde{\Sigma}x_0, u_0$  from  $x_0$  and exercise the control  $v = y^{(\alpha)}(t, x_0, u)$ , the resulting solution is  $x(t, x_0, u)$ . This follows easily from uniqueness of solutions. Furthermore, the corresponding output of  $\tilde{\Sigma}x_0, u_0$  for all  $t$  in a proper interval of  $R$  containing the origin is

$$w(t) = \tilde{\Pi}(x(t, x_0, u), \Pi(x(t, x_0, u), u(t))) = u(t)$$

and the proof is complete.

#### IV. CONCLUSION

In this paper necessary and sufficient conditions for invertibility of single-input analytic systems have been presented.

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### Comments on "Controlled Invariance for Nonlinear Systems"

S. H. MIKHAIL

**Abstract**—The sufficient conditions given in Theorem 4.12 of the above paper<sup>1</sup> for controlled invariance are a special case of more general sufficient conditions reported earlier.

In the above paper<sup>1</sup> conditions are given that are sufficient (and under certain restrictions also necessary) for "controllable invariance." In it, the authors refer to earlier work that I have published [1] on "controlled invariance" for systems of the type  $\dot{x} = F(x, u)$  as a "related, but different notion" which is misleading. It seems they have failed to detect that the sufficient conditions in [1, Theorem 2.1] are more general than those in their Theorem 4.12, and were derived to accommodate many situations where  $f_*(\Delta_e^0) \cap \dot{D}$  is not constant. It could be shown that the conditions  $\dim(f_*(\Delta_e^0) \cap \dot{D}) = \text{constant}$ , and  $\text{rank } d_2[df_\alpha F(x, u)] = l = \text{constant}$  are exactly equivalent to one another in the respective notations and settings of Theorem 4.12 of the paper<sup>1</sup> and [1, Theorem 2.1], respectively. Similarly, the conditions  $f_*(\pi_*^{-1}(D)) \subset \dot{D} + f_*(\Delta_e^0)$  and condition ii) of [1, Theorem 2.1] (as well as [2, condition (2.4)]) could be shown to be

equivalent to one another, subject to the former condition being satisfied. It is worth pointing out that conditions iii) (a) and iii) (b) of [1, Theorem 2.1] are automatically satisfied once the above two conditions hold.

The sufficient conditions given in [1, Theorem 2.1], [3, Theorem 4.2], and [4, Theorem 1.2] for "controlled invariance" are the same with minor changes in presentation and format. [3, Theorem 4.1] and [4, Theorem 1.1] give sufficient conditions that are exactly equivalent to those in Theorem 4.12 of the paper,<sup>1</sup> and are shown to be special cases of the more general conditions of Theorem 4.2 of the paper<sup>1</sup> and [4, Theorem 1.2], respectively.

There may be advantages to the setting used in the paper<sup>1</sup> when the problem global controlled invariance is investigated, but that remains to be seen.

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### Authors' Reply<sup>2</sup>

HENK NIJMEIJER AND ARJAN VAN DER SCHAFT

S. H. Mikhail incorrectly points out that a result in our paper<sup>1</sup> is already contained in his work [1]. The basic observation is that the notion of controlled invariance used in [1] is related but completely different from the one used in our paper.<sup>1</sup> By working in local coordinates we can take  $\mathbb{R}^n$  as the state space,  $\mathbb{R}^m$  as the input space, and the system is defined by  $\dot{x} = f(x, u)$ . Then in [1] an involutive distribution  $D$  of fixed dimension is controlled invariant if there exists a map  $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , such that

$$[\hat{f}, D] \subset D, \quad \text{where } \hat{f}(x) = f(x, \phi(x)).$$

In the paper<sup>1</sup> (see also the references in the paper<sup>1</sup>), however, an involutive distribution  $D$  of fixed dimension is called controlled invariant if there exists a map

$$\alpha: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m,$$

with the property that  $(*) \alpha(x, \cdot): \mathbb{R}^m \rightarrow \mathbb{R}^m$  is a diffeomorphism for each  $x \in \mathbb{R}^n$ , and such that  $[\hat{f}(\cdot, \bar{u}), D] \subset D$  for each constant  $\bar{u} \in \mathbb{R}^m$ , where

$$\hat{f}(x, u) = f(x, \phi(x, u)).$$

The condition  $(*)$  expresses the fact that  $D$  is nondegenerate controlled invariant or controlled invariant with full control; see [2]. If  $(*)$  is not satisfied, then the distribution  $D$  is degenerate controlled invariant, (see Section III, Remark 4 of the paper<sup>1</sup>); see also [3] for some further explanation. In this way the notion of controlled invariance used in [1] is a sort of degenerate controlled invariance of the paper.<sup>1</sup>

The condition ii) of [1, Theorem 2.1] is indeed equivalent to the condition of Theorem 4.12 of the paper,<sup>1</sup> provided that several rank conditions hold. The condition in Theorem 4.12 of the paper<sup>1</sup> really gives

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<sup>1</sup>H. Nijmeijer and A. van der Schaft, *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 904-914, Aug. 1982.

necessary and sufficient conditions for controlled invariance in the sense of the paper,<sup>1</sup> but this same condition in [1, Theorem 2.1] only gives a sufficient condition for controlled invariance in the sense of [1].

Although the references [3], [4] of Mikhail's comment are not accessible, and they appeared after the paper<sup>1</sup> had been submitted, it is clear that a condition of the type ii) in [1, Theorem 2.1] is far from a necessary condition for controlled invariance in the sense of [1].

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Comments on "Stability of Time-Delay Systems"

T. SASAGAWA

**Abstract**—The aim of this paper is to point out that the proofs of the results in the above paper<sup>1</sup> are not correct. A counterexample is constructed for the simplest case.

In the above paper,<sup>1</sup> the authors give necessary and sufficient conditions for the stability of time-delay systems of the form

$$\dot{x}(t) = A_1x(t) + A_2x(t-h).$$

They insist in the proofs of Lemma and Theorem 1 that  $x(t) + P_1(t) * x(t) = 0$  iff  $x(t) = 0$ , where  $*$  denotes the convolution operator. Moreover, from this insistence they conclude the positive definiteness of a Lyapunov function and the negative definiteness of the time derivative of the Lyapunov function on the space of continuous functions  $x \in C([-h, 0], R^n)$ .

More concretely, they insist that the functional defined on  $C([-h, 0], R^n)$

$$V(x_t, h) = [x(t) + P_1(t) * x(t)]^T P_0 [x(t) + P_1(t) * x(t)] \quad (1)$$

is positive definite, where

$$P_0 [A_1 + P_1(0)] + [A_1 + P_1(0)]^T P_0 = -Q, \quad (Q = Q^T > 0) \quad (2)$$

$$\dot{P}_1(\tau) = [A_1 + P_1(0)] P_1(\tau), \quad (0 \leq \tau \leq h) \quad (3)$$

$$P_1(h) = A_2. \quad (4)$$

However, this is an elementary error and, hence, the proofs of Lemma and Theorem 1 are not complete.

To make sure, we construct a counterexample.

**Counterexample:** Consider the simplest case, i.e., let the system be

$$\dot{x}(t) = -x(t-h) \quad (0 \leq t < \infty, h > 0) \quad (5)$$

with the initial function  $\phi(t) = e^{\alpha t}$  ( $-h \leq t \leq 0$ ).

Equation (5) has the solution  $x(t) = e^{\alpha t}$  ( $t \geq -h$ ) if

$$\alpha + e^{-\alpha h} = 0. \quad (6)$$

On the other hand, for (3) and (4) to be satisfied, we have the relation  $P_1(\tau) = P_1(0)e^{P_1(0)\tau}$ , where

$$1 + P_1(0)e^{P_1(0)h} = 0. \quad (7)$$

The relation (6) can be transformed to  $\alpha e^{\alpha h} + 1 = 0$  and this is the same as (7). From (7),  $P_1(0)$  must be a negative constant.

Now, we can calculate with  $\alpha = P_1(0)$

$$\begin{aligned} x(t) + P_1(t) * x(t) &= x(t) + \int_0^h P_1(\tau) x(t-\tau) d\tau \\ &= \left[ 1 + P_1(0) \int_0^h e^{(P_1(0)-\alpha)\tau} d\tau \right] e^{\alpha t} \\ &= [1 + P_1(0)h] e^{P_1(0)t}. \end{aligned}$$

Hence, if we choose  $h = e^{-1}$  ( $> 0$ ),  $P_1(0) = -e$  ( $< 0$ ), the relation (7) is valid and  $V(x_t, h) = 0$  for  $x(t) = e^{-et}$  ( $\neq 0$ ). □

As is clear from this example, Lyapunov functions of the type (1) ( $P_0 > 0$ ) are not generally positive definite. However, Sasagawa [2] proved the following lemma for the functional (1) with an additional term and applied for getting a sufficient condition for asymptotic stability for more general systems.

**Lemma:** Let a functional  $V(x_t, h)$  defined on  $C([-h, 0], R^n)$  be given as follows.

$$\begin{aligned} V(x_t, h) &= [x(t) + H(t) * x(t)]^T P [x(t) + H(t) * x(t)] \\ &\quad + \nu \int_0^h |H(\tau)| d\tau \int_0^t x^T(t-s) Q x(t-s) ds \end{aligned}$$

where  $\nu > 0$ ;  $P, Q$  are symmetric positive definite matrices and  $H(t)$  is a matrix-valued function of bounded variation on  $[-h, 0]$ .

Then there exists a positive constant  $\lambda$  such that

$$\inf_{\|x_t\| \leq |x(t)|} V(x_t, h) \geq \lambda |x(t)|^2$$

for any  $x_t \in C([-h, 0], R^n)$ . □

In the above lemma,  $|\cdot|$  denotes the square root norm of a vector or a matrix and  $\|\cdot\|$  denotes the sup norm in the space  $C([-h, 0], R^n)$ .

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Additional Comments on "Stability of Time-Delay Systems"

M. BUSLOWICZ

**Abstract**—It is proved by a counterexample that the main result of the above paper<sup>1</sup> is incorrect.

Recall that in the paper<sup>1</sup> for the system

$$\dot{x}(t) = A_1x(t) + A_2x(t-h) \quad (1)$$

where  $x(t) \in R^n$ , the following sufficient condition for the stability is given. If for any given positive definite Hermitian matrix  $Q$ , there exists a

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<sup>1</sup>T. N. Lee and S. Dianat, *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 951-953, Aug. 1981.

Manuscript received April 30, 1982.  
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<sup>1</sup>T. N. Lee and S. Dianat, *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 951-953, Aug. 1981.