Controlled Invariance for Nonlinear Systems

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has the partitioning induced by the splitting of the eigenvalues of A according to (1) and (2). The block M_{11} is $r \times r$ and the other blocks have appropriate dimensions. With the above, we can now give the following.

Theorem: The $r \times n$ matrix K is defined by (8). The equation $KN = SK$ is consistent if and only if $M_{12} = 0$ where M is defined in (15).

Proof: From (15), we have $N = VMU$. Consider each side of $KN = SK$, multiplied on the right by V:

$$K(VMU)V = KVM = D[M_{11} \quad M_{12}] \tag{16}$$

and

$$SKV = [SD \quad 0]. \tag{17}$$

When (16) and (17) are equated, we see that $KN = SK$ is consistent if and only if $M_{12} = 0$.

Remark 1: The theorem shows that an aggregated reduced-order model exists for (10), provided that each matrix $V^{-1}N_iV$ is block lower triangular, matching the partitioning induced by the splitting of the eigenvalues of A (see Section II). Brockett [3] defines a realization of (10) with $B = 0$ as reducible if there exists a nonsingular matrix L such that $L^{-1}AL$ and $L^{-1}N_iL$ are all block lower triangular with the same partitioning. Clearly, if (10) admits an aggregated reduced-order model, it is reducible.

Remark 2: The rows of K are left eigenvectors of A and N_i.

Corollary: If $KN = SK$ is consistent, then $S = DM_{11}D^{-1}$ is a solution.

Remark 3: There may be bilinear systems whose aggregated reduced-order model is linear; for example, M_{11} could be zero.

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Controlled Invariance for Nonlinear Systems: Two Worked Examples

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***Abstract*—In this note, we present two worked examples of disturbance decoupling for nonlinear systems, using the concept of controlled invariance, which was recently generalized to nonlinear systems.**

In the first example, we explicitly construct a feedback which decouples a disturbance from the vertical components of the axes of a rotating rigid body, while the second example deals with a particle in a potential field subject to a disturbance.

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I. INTRODUCTION

The concept of (A, B)-invariant or *controlled invariant* subspaces turns out to be a corner stone in the solution of various synthesis problems in linear systems theory [11]. Very recently, one has obtained, from a theoretical point of view, a rather satisfying generalization of this concept to nonlinear systems, beginning with the papers of Isidori *et al.* [4] and Hirschorn [3] and continued in [5]–[8]. The derived concept of (C, A, B) invariance or *measured controlled invariance* has also been successfully treated for nonlinear systems [4], [9].

The essence of this theory is that a specific synthesis problem, for instance disturbance decoupling, for a nonlinear system can be dealt with in an *intrinsically nonlinear* way. Hence, no linearizations or approximations have to be made, and an exact solution is generated. Of course, the disadvantage is that one needs more sophisticated mathematical tools, and that sometimes the actual calculation and implementation of the solution seem to be hard.

This motivated us to write two examples of perhaps the easiest application of controlled invariance for nonlinear systems, namely disturbance decoupling. The first example deals with the dynamics of a rigid body controlled by two inputs and influenced by a disturbance. We will show how we can decouple, for instance, the vertical components of the axes of the rigid body from the disturbance. The second example is of a more pedagogical nature, dealing with (measured) controlled invariance for a particle in a potential field, subject to a disturbance.

II. EXAMPLE: THE RIGID BODY

We can describe the position of a rigid body with respect to an inertial set of axes $e_1, e_2, e_3 \in \mathbb{R}^3$ by a matrix

$$R = \begin{pmatrix} r_1 & s_2 & t_1 \\ r_2 & s_2 & t_2 \\ r_3 & s_3 & t_3 \end{pmatrix} \in SO(3).$$

Here the unit vector $r = (r_1, r_2, r_3)'$ denotes the direction of the first axis of the rigid body: r_1 is the component in the e_1 direction, r_2 is the component in the e_2 direction, and r_3 is the component in the e_3 direction. Similarly, the unit vectors $s = (s_1, s_2, s_3)'$ and $t = (t_1, t_2, t_3)'$ give the directions of the second and third axes of the rigid body. The dynamics of a rigid body with no external influences are described by (see [2], [6], [10])

$$\begin{cases} \dot{R} = S(\omega)R \\ J\dot{\omega} = S(\omega)J\omega \end{cases} \tag{1.1}$$

where $\omega = (\omega_1, \omega_2, \omega_3)'$ is the angular velocity with respect to the axes of the rigid body, J is a symmetric positive definite (3,3) matrix, and $S(\omega)$ is the anti-symmetric matrix defined by

$$S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

J is called the *inertia matrix*, the eigenvectors of J are called the *principal* axes, and we will, for simplicity, assume that the axes r, s, and t are already the principal axes; hence,

$$J = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \qquad a_i > 0, \quad i = 1,2,3.$$

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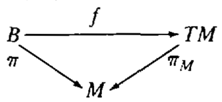
With (1.1), we associate a control system of the form (see [2] and [6])

$$\begin{cases} \dot{R} = S(\omega)R \\ J\dot{\omega} = S(\omega)J\omega + m_1u_1 + m_2u_2 + nd \end{cases} \tag{1.2}$$

where m_1, m_2, and n are vectors in $\mathbb{R}^3$, $u_1, u_2 \in \mathbb{R}$ are the controls, and $d \in \mathbb{R}$ is a disturbance (unknown input) working on the system. To be more specific, we will henceforth consider the equations

$$\begin{cases} \dot{R} = S(\omega)R \\ \begin{pmatrix} a_1\dot{\omega}_1 \\ a_2\dot{\omega}_2 \\ a_3\dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} a_1\omega_1 \\ a_2\omega_2 \\ a_3\omega_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} d. \end{cases} \tag{1.3}$$

Some nice results concerning *controllability* of (1.2) have been obtained in [2] and [12]; also see the earlier work of Baillieul and Brockett [13], [14]. For instance, (1.3) is controllable with the inputs u_1, u_2 if and only if $a_1 \neq a_2$ (also notice that (1.3) is *not* controllable with respect to the disturbance d). Finally, we mention that (1.1) and (1.2) can be elegantly described in a coordinate-free way (see [1]). Because R is an element of the Lie group $SO(3)$, ω is an element of the Lie algebra $so(3) \simeq \mathbb{R}^3$. Define the left invariant Lagrange function L on $TSO(3) \simeq SO(3)\times\mathbb{R}^3$ by $L(R,\omega) = \frac{1}{2}\omega^TJ\omega$. Then $J\omega$ can be naturally considered as an element of $so^*(3) \simeq \mathbb{R}^3$. Therefore, (1.1) is a Hamiltonian system on the phase space $T^*SO(3) \simeq SO(3)\times\mathbb{R}^3$ with Hamilton function L. Adopting the coordinate-free description of a control system used in [8] (see the references cited there), we obtain for (1.2) (without the disturbance)



with $M = T^*SO(3)$ (state space), $B = T^*SO(3)\times\mathbb{R}^2$ (input bundle), π and π_M are the obvious projections, and f is given by (1.2) (without the disturbances).

We now come to the formulation of the disturbance decoupling problem. First we will pose and solve it for the following system derived from (1.2). Let r be the first column of R (sometimes called a Poisson vector [1]). Equation (1.2) gives

$$\begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} \omega_3r_2 - \omega_2r_3 \\ -\omega_3r_1 + \omega_1r_3 \\ \omega_2r_1 - \omega_1r_2 \\ b_1\omega_2\omega_3 \\ b_2\omega_1\omega_3 \\ b_3\omega_1\omega_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_1^{-1} \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_2^{-1} \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_3^{-1} \end{pmatrix} d \tag{1.4}$$

where

$$b_1 := \frac{a_2 - a_3}{a_1}, \quad b_2 := \frac{a_3 - a_1}{a_2}, \quad b_3 := \frac{a_1 - a_2}{a_3}.$$

Notice that because $|r| = 1$, this system actually lives on $S^2\times\mathbb{R}^3$. Define the input vector fields $B_1 := (0\ 0\ 0\ a_1^{-1}\ 0\ 0)^T$, $B_2 := (0\ 0\ 0\ 0\ a_2^{-1}\ 0)^T$. Introduce z (the to-be-controlled variable) by $z := r_3$. We will study the following disturbance decoupling problem.

Construct, if possible, a state feedback for (1.4) such that after feedback, the disturbance d does not influence the function z.

Remark: In [12], an exposition of disturbance decoupling problems for the rigid body also can be found. However, explicit results are not obtained.

Following the theory mentioned in the Introduction, we have to find a controlled invariant distribution D in the kernel of the function z which contains the disturbance vector field $(0\ 0\ 0\ 0\ 0\ a_3^{-1})^T$. It can be rather easily seen that the distribution $D := \operatorname{span}\{X_1, X_2\}$ where

$$X_1(r,\omega) := (0,0,0,0,0,1)',$$

$$X_2(r,\omega) := (r_2, -r_1, 0, \omega_2, -\omega_1, 0)'$$

does the job. In fact, a tedious calculation following the algorithm in [7] shows that this D is the largest controlled invariant distribution contained in $\operatorname{Ker} dz$. Hence, at least locally (see [5], [7], [8]), we can construct the required feedback. (Notice also that D has no constant dimension; see some comments later on.)

How do we construct this feedback? First we will modify the input vector field B_1 and B_2 to vector fields $\tilde{B}_1$ and $\tilde{B}_2$ such that the system after this modification is *input insensitive* [7], i.e., $[\tilde{B}_i, D] \subset D$. Notice that

$$[B_1, X_1] = \mathbf{0}, \qquad [B_2, X_1] = \mathbf{0} \tag{1.5a}$$

$$[B_1, X_2] = -\frac{a_2}{a_1}B_2, \qquad [B_2, X_2] = \frac{a_1}{a_2}B_1. \tag{1.5b}$$

It is easy to see that possible $\tilde{B}_i$ are given by

$$\tilde{B}_1(r,\omega) = (0,0,0,\omega_2,-\omega_1,0)',$$

$$\tilde{B}_2(r,\omega) = (0,0,0,\omega_1,\omega_2,0)' \tag{1.6}$$

(see also the remark at the end). Notice that $[\tilde{B}_i, X_j] = 0$, $i = 1,2$, $j = 1,2$.

As a second step for computing the decoupling state feedback, we will first compute the feedback with respect to these modified input vector fields. Hence, we are looking for functions $\alpha(r,\omega)$, $\beta(r,\omega)$ such that

$$\left[\begin{pmatrix} \omega_3r_2 - \omega_2r_3 \\ -\omega_3r_1 - \omega_1r_3 \\ \omega_2r_1 - \omega_1r_2 \\ b_1\omega_2\omega_3 \\ b_2\omega_1\omega_3 \\ b_3\omega_1\omega_2 \end{pmatrix} + \alpha(r,\omega)\begin{pmatrix} 0 \\ 0 \\ 0 \\ \omega_2 \\ -\omega_1 \\ 0 \end{pmatrix} + \beta(r,\omega)\begin{pmatrix} 0 \\ 0 \\ 0 \\ \omega_1 \\ \omega_2 \\ 0 \end{pmatrix}, D\right] \subset D. \tag{1.7}$$

With respect to the basis $\{X_1, X_2\}$ of D, this leads to the following two equations:

$$\left[\begin{pmatrix} \omega_3r_2 - \omega_2r_3 \\ -\omega_3r_1 + \omega_1r_3 \\ \omega_2r_1 - \omega_1r_2 \\ b_1\omega_2\omega_3 \\ b_2\omega_1\omega_3 \\ b_3\omega_1\omega_2 \end{pmatrix} + \alpha(r,\omega)\begin{pmatrix} 0 \\ 0 \\ 0 \\ \omega_2 \\ -\omega_1 \\ 0 \end{pmatrix} + \beta(r,\omega)\begin{pmatrix} 0 \\ 0 \\ 0 \\ \omega_1 \\ \omega_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

$$= -\begin{pmatrix} r_2 \\ -r_1 \\ 0 \\ b_1\omega_2 \\ b_2\omega_1 \\ 0 \end{pmatrix} - \frac{\partial\alpha}{\partial\omega_3}(r,\omega)\begin{pmatrix} 0 \\ 0 \\ 0 \\ \omega_2 \\ -\omega_1 \\ 0 \end{pmatrix} - \frac{\partial\beta}{\partial\omega_3}(r,\omega)\begin{pmatrix} 0 \\ 0 \\ 0 \\ \omega_1 \\ \omega_2 \\ 0 \end{pmatrix} \in D. \tag{1.8}$$

Therefore,

$$\begin{cases} \omega_2\dfrac{\partial\alpha}{\partial\omega_3}(r,\omega) + \omega_1\dfrac{\partial\beta}{\partial\omega_3}(r,\omega) = -(b_1 - 1)\omega_2 \\ -\omega_1\dfrac{\partial\alpha}{\partial\omega_3}(r,\omega) + \omega_2\dfrac{\partial\beta}{\partial\omega_3}(r,\omega) = -(b_2 + 1)\omega_1 \end{cases} \tag{1.9}$$

or

$$\frac{\partial\alpha}{\partial\omega_3}(r,\omega) = (+1 - b_1)\frac{\omega_2^2}{\omega_1^2 + \omega_2^2} + (1 + b_2)\frac{\omega_1^2}{\omega_1^2 + \omega_2^2} \tag{1.10a}$$

$$\frac{\partial\beta}{\partial\omega_3}(r,\omega) = -\frac{(b_1 + b_2)\omega_1\omega_2}{\omega_1^2 + \omega_2^2}. \tag{1.10b}$$

And also,

$$\left[\begin{pmatrix} \omega_3 r_2 - \omega_2 r_3 \\ -\omega_3 r_1 + \omega_1 r_3 \\ \omega_2 r_1 - \omega_1 r_2 \\ b_1\omega_2\omega_3 \\ b_2\omega_1\omega_3 \\ b_3\omega_1\omega_2 \end{pmatrix} + \alpha(r,\omega)\begin{pmatrix} 0\\0\\0\\ \omega_2 \\ -\omega_1 \\ 0\end{pmatrix} + \beta(r,\omega)\begin{pmatrix}0\\0\\0\\ \omega_1\\ \omega_2\\ 0\end{pmatrix}, \begin{pmatrix} r_2 \\ -r_1 \\ 0 \\ \omega_2 \\ -\omega_1 \\ 0\end{pmatrix}\right]$$

$$= \begin{pmatrix} 0\\0\\0\\ (b_1+b_2)\omega_1\omega_3 \\ -(b_1+b_2)\omega_2\omega_3 \\ b_3(\omega_1^2-\omega_2^2)\end{pmatrix} - X_2(\alpha(r,\omega))\begin{pmatrix}0\\0\\0\\ \omega_2\\ -\omega_1\\ 0\end{pmatrix} - X_2(\beta(r,\omega))\begin{pmatrix}0\\0\\0\\ \omega_1 \\ \omega_2 \\ 0\end{pmatrix} \in D.$$

Therefore,

$$\begin{cases} (b_1+b_2)\omega_1\omega_3 - X_2(\alpha(r,\omega))\omega_2 - X_2(\beta(r,\omega))\omega_1 = 0 \\ -(b_1+b_2)\omega_2\omega_3 + X_2(\alpha(r,\omega))\omega_1 - X_2(\beta(r,\omega))\omega_2 = 0. \end{cases} \tag{1.12}$$

Thus,

$$X_2(\alpha(r,\omega)) = \omega_2 \frac{\partial \alpha}{\partial \omega_1}(r,\omega) - \omega_1 \frac{\partial \alpha}{\partial \omega_2}(r,\omega) = \frac{2(b_1+b_2)\omega_1\omega_2\omega_3}{\omega_1^2+\omega_2^2} \tag{1.13a}$$

$$X_2(\beta(r,\omega)) = \omega_2 \frac{\partial \beta}{\partial \omega_1}(r,\omega) - \omega_1 \frac{\partial \beta}{\partial \omega_2}(r,\omega) = \frac{(b_1+b_2)\omega_3(\omega_1^2-\omega_2^2)}{\omega_1^2+\omega_2^2}. \tag{1.13b}$$

Now an easy integrating procedure, as described in [8], leads to the following (not unique) solutions:

$$\alpha(r,\omega) = \frac{(1-b_1)\omega_2^2\omega_3 + (1+b_2)\omega_1^2\omega_3}{\omega_1^2+\omega_2^2} \tag{1.14a}$$

$$\beta(r,\omega) = \frac{-(b_1+b_2)\omega_1\omega_2\omega_3}{\omega_1^2+\omega_2^2}. \tag{1.14b}$$

The feedback given by (1.14) is expressed with respect to the vector fields $\tilde{B}_1$ and $\tilde{B}_2$. The feedback $\bar{\alpha}$ and $\bar{\beta}$ with respect to the original input vector field B_1 and B_2, respectively, can be computed by using the relations $\tilde{B}_1 = a_1\omega_2 B_1 - a_2\omega_1 B_2$ and $\tilde{B}_2 = a_1\omega_1 B_1 + a_2\omega_2 B_2$:

$$\bar{\alpha}(r,\omega) = a_1\omega_2\alpha(r,\omega) + a_1\omega_1\beta(r,\omega) = a_1(1-b_1)\omega_2\omega_3 \tag{1.15a}$$

$$\bar{\beta}(r,\omega) = -a_2\omega_1\alpha(r,\omega) + a_2\omega_2\beta(r,\omega) = -a_2(1+b_2)\omega_1\omega_3. \tag{1.15b}$$

We see that the state feedback $u_1 = \bar{\alpha}(r,\omega)$, $u_2 = \bar{\beta}(r,\omega)$ defined by (1.15) is globally well defined, although the modification of the input vector fields [see (1.6)] is not of full rank everywhere. So for open-loop feedback, we can apply

$$\begin{cases} u_1 = \bar{\alpha}(r,\omega) + \omega_2 v_1 + \omega_1 v_2 \\ u_2 = \bar{\beta}(r,\omega) - \omega_1 v_1 + \omega_2 v_2 \end{cases} \tag{1.16}$$

where v_1 and v_2 denote the new inputs. With this feedback, we obtain from (1.4) the following system in decoupled form:

$$\begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} \omega_3 r_2 - \omega_2 r_3 \\ -\omega_3 r_1 + \omega_1 r_3 \\ \omega_2 r_1 - \omega_1 r_2 \\ \omega_2\omega_3 \\ -\omega_1\omega_3 \\ b_3\omega_1\omega_2 \end{pmatrix} + \begin{pmatrix} 0\\0\\0\\ \omega_2 a_1^{-1} \\ -\omega_1 a_2^{-1} \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} 0\\0\\0\\ \omega_1 a_1^{-1} \\ \omega_2 a_2^{-1} \\ 0\end{pmatrix} v_2 + \begin{pmatrix}0\\0\\0\\0\\0\\ a_3^{-1}\end{pmatrix} d. \tag{1.17}$$

Notice that the input vector fields of (1.17) are zero at points where $\omega_1 = \omega_2 = 0$. Once more we emphasize that the singularities in (1.10), (1.13), and (1.14) do not affect the global feedback of (1.15) and (1.16).

It is interesting to see if there are cases for which (1.4) is already in disturbance decoupled form, and therefore we do not have to apply feedback. From (1.15), it follows that this happens if $1-b_1=0$ and $1+b_2=0$. Using the definition of b_1 and b_2, this gives $a_3=0$. So our rigid body reduces to a rigid plane!

Following [8], there exists an integrable connection in the input bundle of (1.4), i.e., $S^2\times\mathbb{R}^3\times\mathbb{R}^2$, which corresponds to the feedback (1.16). Actually, this connection is only uniquely determined above the distribution D (corresponding to the nonuniqueness of the decoupling feedback). Following the notation of [8], the connection above D is given by

$$X_i(r,\omega) + K_i(r,\omega)v\frac{\partial}{\partial v} + h_i(r,\omega)\frac{\partial}{\partial v}, \qquad i=1,2$$

where $v = \begin{pmatrix} v_1 \\ v_2\end{pmatrix}$ denotes the input space $\mathbb{R}^2$. From (1.5), it follows that

$$K_1(r,\omega) = \begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}, \qquad K_2(r,\omega) = \begin{pmatrix} 0 & \frac{a_1}{a_2} \\ -\frac{a_2}{a_1} & 0 \end{pmatrix}.$$

From (1.9) and (1.2), it follows that

$$h_1(r,\omega) = \begin{pmatrix} -a_1 b_1\omega_2 \\ -a_2 b_2 \omega_1 \end{pmatrix} \quad \text{and} \quad h_2(r,\omega) = \begin{pmatrix} a_1(b_1+b_2)\omega_1\omega_3 \\ -a_2(b_1+b_2)\omega_2\omega_3 \end{pmatrix}.$$

In conclusion, the feedback (1.16) solves the disturbance decoupling problem for (1.4). We now deliver the coup de grâce.

Instead of using r in (1.4), we could also have used the two other axes of the rigid body s and t. Posing for s and t the same disturbance decoupling problem (with $z=s_3$, respectively, $z=t_3$) gives the same feedback (1.16) because this feedback only depends on ω_1, ω_2, and ω_3!

Therefore, feedback (1.16) is a decoupling feedback for the full system (1.3), which decouples the whole last row (r_3, s_3, t_3) of the matrix R of the disturbance.

Conclusion: Consider the system (1.3). The feedback defined by (1.16)

$$\begin{cases} u_1 = a_1(1-b_1)\omega_2\omega_3 + \omega_2 v_1 + \omega_1 v_2 \\ u_2 = -a_2(1+b_2)\omega_1\omega_3 - \omega_1 v_1 + \omega_2 v_2 \end{cases}$$

decouples the last row (r_3, s_3, t_3) of R (i.e., the components of the axes of the body in the e_3 direction) from the disturbance.

III. Example: A Particle in a Potential Field

The following example will serve as a mathematical illustration of the notion of measured controlled invariance (cf. [9]). Consider the following mechanical model:

$$\begin{cases} \dot{q}_1 = p_1 \\ \dot{q}_2 = p_2 \\ \dot{p}_1 = \frac{\partial V}{\partial q_1}(q_1,q_2) + u \\ \dot{p}_2 = \frac{\partial V}{\partial q_2}(q_1,q_2) + d \end{cases} \tag{2.1}$$

where $(q_1,q_2,p_1,p_2)\in T^*(S^1\times\mathbb{R})$, $V\colon S^1\times\mathbb{R}\to\mathbb{R}$ is a smooth function, and u and d represent the input and the disturbance, respectively. So we are dealing with a particle (of unit mass) moving on a cylinder according to a potential force given by the potential function V.

Together with (2.1), we consider the two "output" functions

$$y = C(q_1, q_2, p_1, p_2) = q_2, \qquad C: T^*(S^1 \times \mathbb{R}) \to \mathbb{R} \tag{2.2}$$

and

$$z = \tilde{C}(q_1, q_2, p_1, p_2) = q_1, \qquad \tilde{C}: T^*(S^1 \times \mathbb{R}) \to S^1. \tag{2.3}$$

The variable y represents the measurement or output of the system and z is the so-called to-be-controlled variable. With the bundle approach of [9] (see also [8]), we obtain the following diagrams:

$$\begin{array}{ccc} T^*(S^1 \times \mathbb{R}) \times \mathbb{R} & \xrightarrow{(C, id)} & \mathbb{R} \times \mathbb{R} \\ \pi \downarrow & & \downarrow \pi \\ T^*(S^1 \times \mathbb{R}) & \xrightarrow[C]{} & \mathbb{R} \end{array} \tag{2.4a}$$

and

$$\begin{array}{ccc} T^*(S^1 \times \mathbb{R}) \times \mathbb{R} & \xrightarrow{f} & T(T^*(S^1 \times \mathbb{R})) \\ & \searrow^{\pi} & \downarrow p \\ & & T^*(S^1 \times \mathbb{R}) \end{array} \tag{2.4b}$$

where f is given by (2.1) and p is the canonical projection.

We will study the following problem.

Disturbance Decoupling with Measurements: Is it possible to construct an output feedback, i.e., a state feedback which only depends on the output y, such that the disturbance d is isolated from the to-be-controlled variable z?

Following [9], we will first solve the easier DDP and afterwards investigate DDPM. We notice that

$$\ker d\tilde{C} = \operatorname{span}\left\{ \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2} \right\}, \tag{2.5}$$

and a straightforward calculation shows that (cf. [4], [7])

$$D := V^*_{\ker d\tilde{C}} = \operatorname{span}\left\{ \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_2} \right\}. \tag{2.6}$$

Now the disturbance enters via the vector field $\partial/\partial p_2$, so we see that DDP is solvable (see [3] and [4]).

From the bundle description given by (2.4a), (2.4b), it follows that for DDPM, we need to check conditions ii) and iii) of [9, Theorem 3.2]. Notice that

$$D \cap \ker dC = \operatorname{span}\left\{ \frac{\partial}{\partial p_2} \right\}, \tag{2.7}$$

and so we have

$$\left[\begin{pmatrix} p_1 \\ p_2 \\ \dfrac{\partial V}{\partial q_1}(q_1, q_2) \\ \dfrac{\partial V}{\partial q_2}(q_1, q_2) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\partial}{\partial q_2} \in D \tag{2.8a}$$

and

$$\left[\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] = 0 \in D. \tag{2.8b}$$

Therefore, condition ii) of [9, Theorem 3.2] is satisfied. Finally, we see that the last condition of this theorem is satisfied if and only if there exists a function $k: \mathbb{R} \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial q_1}(q_1, q_2) + k(q_2) \tag{2.9}$$

only depends on q_1. This leads to the following representation for the potential function V:

$$V(q_1, q_2) = f(q_1) + g(q_2) q_1 + h(q_2) \tag{2.10}$$

for some functions $g, h: \mathbb{R} \to \mathbb{R}$ and $f: S^1 \to \mathbb{R}$.

Conclusion: DDPM is solvable if the potential function can be represented as in (2.10).

Remark: For this example, we have shown that the distribution $D = V^*_{\ker C}$ satisfies the properties for measured controlled invariance. In principle, it might be necessary to shrink the distribution D such that it becomes measured controlled invariant. It is not necessarily true that there exists a supremal measured controlled invariant distribution, as already can be illustrated by a linear control system.

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Stability Improvement of Nonlinear Systems by Feedback

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***Abstract*—This paper is concerned with the property of stabilizing a nonlinear system to a specified equilibrium point arbitrarily fast by appropriate smooth feedback. Two closely related forms of this property are explored. One refers to the asymptotic transfer to the critical point with exponential decay, and it is shown that the systems possessing the above**

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