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Testing problems with linear or angular inequality constraints

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1 TESTING PROBLEMS WITH LINEAR INEQUALITY CONSTRAINTS

1.0 General introduction and summary

A well-known issue in the practice of hypothesis testing is whether the formulation of the alternative should be "one-sided" or "two-sided". One-sided alternatives occur quite generally in the comparison of treatments A and B if it is known a priori that A cannot be worse than B. Here a one-sided version of the two-sample t-test or the two-sample Wilcoxon-Mann-Whitney test is often applied. Such a test has the obvious advantage of being more powerful than its two-sided counterpart.

The present study is concerned mainly with certain generalizations of one-sided hypothesis testing problems, namely problems where the alternative is restricted by at least two linear inequalities. A celebrated example in the statistical literature is that of testing homogeneity against upward trend, say in the context of a one-way analysis of variance. A secondary theme of the present study is constituted by testing problems where linear inequalities restrict the null hypothesis rather than the alternative (but not both). An example arises if the model assumption of upward trend has to be tested against the alternative that at least two samples violate the assumed order of treatment means.

The set-up of Ch. 1 and Ch. 5, which serve introductory and illustrative purposes, is outlined first. Next a general description is given of the contents of Chs. 2-4, the main body of the study. Finally, §1.0.1 contains four suggestions for cursory reading of the book.

Introduction to testing problems with linear inequality constraints (Ch. 1); case studies (Ch. 5). A broad class of testing problems with a restricted alternative is specified in §1.4 as an abstract testing problem which will be called the *general problem*. In §1.5 some linear data transformations are used to bring the general problem into a canonical form. The polyhedral cone determined by the transformed linear inequality constraints is pointed. The resulting *main problem* has a mathematically convenient form, but its explicit specification is not needed in practice. The practical relevance of the general problem is illustrated in §1.6, where potential applications are introduced in the form of specific well-known classes of testing problems. The "real world" cases in §1.3.1 and §1.3.2 serve to show

that the deliberate formulation of the (possibly restricted) alternative is as important as the construction of a reasonable test. These case studies are worked out in §5.2 and §5.3, respectively. The self-contained discussion of the "one-sided crossover trial problem" in §5.1 covers the most important concepts and results of the present study.

Analogous to the specification of the main problem in §1.5, a class of testing problems with the null hypothesis restricted by linear inequalities is specified in §1.7. In §1.3.3 an example of this *reversed main problem* is introduced in the context of "Mokken's latent trait model"; here the problem is to test certain monotonicity assumptions. The relevant procedures to be proposed in §5.4 follow essentially from certain duality properties of the reversed main problem in relation to the main problem.

For the sake of completeness, §1.2 gives a general introduction to the theory of hypothesis testing. Frequently used notations are listed in §1.1.

Towards a unified practice of inequality constrained testing (Chs. 2-4).

Two very recent publications contain reviews of the impressive statistical literature about testing problems with linear inequality constraints, namely ROBERTSON et al. (1988) and SHAPIRO (1988). In these references much attention is paid to likelihood ratio tests.

In spite of the wealth of theory about these tests and the nice general principle involved, the *Statisticus Practicus* will often refrain from applying such a procedure, even if he is completely convinced that certain linear inequalities constitute an appropriate restriction of the alternative. The reason may be that these tests are too complicated to perform. Instead, the statistician may content himself with applying his favorite (simple) test. Often this will amount either to reducing the dimension of the alternative or to ignoring the relevant inequality constraints. Chs. 2-4 provide a remedy in the form of a generally applicable approach that leads to simple tests with good power properties on the original polyhedral cone-shaped alternative.

An example. Consider the problem of testing homogeneity of the expectations $\mu_1, \dots, \mu_k$ of k normal distributions with common variance $\sigma^2=1$, on the basis of samples of size $n_1, \dots, n_k$, respectively ($k>2$). Let it be assumed that $\mu_1 \leq \dots \leq \mu_k$ (upward trend). Although the likelihood ratio test, due to BARTHOLOMEW (1959, 1961b), has good power properties, it is complicated to perform unless the n_i are equal. Alternatively, one might consider

statistics of the form $\sum_i c_i X_i$, where the c_i depend on the n_i only (the X_i represent the sample means; $\sum_i c_i = 0$). The "optimal scores" c_i^* proposed by ABELSON and TUKEY (1963) and SCHAAFSMA and SMID (1966) correspond to the minimax ray, or central half-line, of the relevant pointed polyhedral cone ($\mu: \mu_1 \leq \dots \leq \mu_k, \sum_i n_i \mu_i = 0$). The test based on $\sum_i c_i^* X_i$ is easy to carry out but its performance tends to become poor near the edges of this cone. Now the so-called *circular likelihood ratio (CLR-) tests* considered in Ch. 4 are based on test statistics that somehow combine a linear statistic, say $\sum_i c_i^* X_i$, with $\sum_i n_i (X_i - \bar{X})^2$, the ordinary χ^2 -statistic ($\bar{X}$ represents the overall mean). CLR-tests are easy to perform. A carefully chosen CLR-test will more or less mimic the original LR-test.

In a sense, the present study is the result of combining the work of SCHAAFSMA (1966) and PINCUS (1975). In working out his "most stringent somewhere most powerful tests" for various instances of the general problem, the former author explicitly described the geometry of the involved alternatives. The latter author suggested (i) to approximate the relevant pointed polyhedral cone by a circular cone, so that several linear inequalities are replaced by one angular inequality, and (ii) to apply the likelihood ratio test to the *modified problem* thus obtained. My work on the practical implementation of the involved CLR-tests consists of

- (1) tabulating the relevant critical values (v. §3.3, App. A1, App. A2);
- (2) applying and extending Schaafsma's geometrical results to answer the question what circular cone is most suitable as a substitute for the original polyhedral cone (v. §4.1, §§4.3-4.5; cf. §1.6, §4.6, §4.8);
- (3) comparing the power functions of certain CLR-tests with those of some well-known competitors (v. §4.2, §4.7, App. B; cf. §3.7);
- (4) constructing CLR-tests for the reversed main problem (v. §3.8).

Some interesting results were obtained. Apart from abstract duality properties like those described in SHAPIRO (1988), some surprising duality correspondences turned out to exist between apparently unrelated instances of the general problem (v. §4.5). The *most stringent CLR-test* has been determined as a function of the dimensionality and the so-called minimax angle of the involved polyhedral cone (for $\alpha = .05, .01$; v. §4.3).

Related topics. The well-known book by BARLOW et al. (1971) on order restricted statistical inference has obtained a voluminous successor in ROBERTSON et al. (1988). In both books the emphasis is on the likelihood

ratio principle and the theory of "isotonic regression over a quasi-ordered set". The unified treatment of problems of testing for or against an order restriction in the latter book is based on the theory of "Fenchel duality".

The "normal means" theory in Chs. 2-4 of ROBERTSON et al. (1988) applies immediately to our general problem. The practical difficulties associated with likelihood ratio tests are alleviated to a certain extent, notably for the cases of "simple order" and "simple tree order", by approximations to the null distributions of the relevant test statistics and by extensive tables of critical values. In §4.1 of ROBERTSON et al. (1988) the normal theory results are extended to tests of ordered hypotheses in exponential families. As an example we mention the result that critical values of the likelihood ratio test statistic for trend among binomial proportions can be approximated by those for the normal limiting problem. By contrast, for this binomial trend problem an "asymptotic CLR-test" is given in §5.4 of the present study. Ch. 5 in ROBERTSON et al. (1988) treats some contingency table problems of which we consider only asymptotic versions. For topics like one-sided testing by distribution-free methods and one-sided testing in multivariate analysis we refer to ROBERTSON et al. (1988), §4.5 and §4.6.

1.0.1 Various shortcuts through the study

Not every potential reader has the same interests. In the manner of CORTÁZAR (1963), four different routes through the study are suggested, depending on the presumed statistical attitude. (On any such route, I imagine we are playing hopscotch together.) Square brackets indicate a block of (sub-)section numbers that refer to closely related topics.

Route 1. (The alternative constrained by linear inequalities; an applied view.) 1.0→

[5.1→ 1.4→ 1.5]→	Introduction to the general problem
[2.0→ 2.2→ 2.3]→	Linear and LR-tests for the main problem
[3.0→ 3.2→ 3.5]→	LR-tests for the modified problem and "combination procedures" for the "reduced modified problem"
[4.0→ 4.1→ 4.3.3→ 4.6→ 4.7]→	CLR-tests for the general problem
[1.3.1→ 5.2]→ [1.3.2→ 5.3]	Two case studies

(Background material: [1.2]; [3.4]; [4.3.1]; [4.5]).