Gibbs Measures and Phase Transitions

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Book Review: *Gibbs Measures and Phase Transitions*

Gibbs Measures and Phase Transitions (de Gruyter Studies in Mathematics 9), Hans-Otto Georgii, de Gruyter, Berlin, 1988.

This monograph treats mathematical models for infinite lattice systems of random variables with spatial interaction (in physical terminology: classical equilibrium statistical mechanics). Such models aim at describing the cooperative phenomenon of a phase transition, which is often associated with the breaking of a symmetry (i.e., the system exhibits two or more equilibrium states that do not inherit the full symmetry of the interaction). The prototype is the ferromagnetic nearest-neighbor Ising model on $\mathbb{Z}^d(d \geqslant 1)$, where spin random variables $\tau_i = \pm 1$ attached to the sites $i \in \mathbb{Z}^d$ interact via a potential $-J\tau_i\tau_j 1_{\{|i-j|=1\}}$ $(J>0)$. If $d \geqslant 2$, then there exists $0 < J_c < \infty$ such that the $\pm$ symmetry is preserved for $J \leqslant J_c$ but is broken for $J > J_c$.

The language of the book is probability theory. As the author says in his preface: "My intention is that this monograph serve as an introductory text for a general mathematical audience including advanced graduate students, as a source of rigorous results for physicists, and as a reference work for the expert." In our opinion, especially attractive aspects of the book are: (1) it gives a general exposition of the theory via rigorous step-by-step proofs and assumes no prior knowledge of the subject; (2) it presents lots of explicit examples that illustrate the main results; (3) it explains the intuitive physical background and motivates the choice of models. A point of criticism is that at some places the text is somewhat heavy going due to the high degree of generality of the framework in which the theory is presented. However, this never detracts from the main picture, and the initial effort needed to absorb the technical tools introduced at the beginning is amply rewarded as one progresses. The book is supplemented with an extensive list of references and with bibliographical notes that guide the reader to related parts of the literature not covered (e.g., history, correlation inequalities, exactly solved models, Pirogov–Sinai theory, cluster expansions).

Let us now turn to the content. Gibbs measures model the equilibrium states of a physical system. Given a space of configurations

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$E^S = \{\omega = (\omega_i): \omega_i \in E \text{ for all } i \in S\}$ with S the index set and E the state space. Typically, $S = \mathbb{Z}^d$ $(d \geqslant 1)$ and $E = \{+1, -1\}$, $\mathbb{Z}$ or $\mathbb{R}$. A Gibbs measure should be a probability measure on E^S that assigns weight $Z^{-1}e^{-H(\omega)}$ to configuration ω. Here $H(\omega)$ is the energy of ω (Hamiltonian) and Z is a normalization constant (partition function). The starting point of the theory is the observation that this definition only makes sense when S is finite, but that this problem may be overcome by looking at conditional probability measures. For infinite S one considers the finite subsets $\mathscr{S} = \{\Lambda \subset S: \Lambda \text{ finite}\}$. For each $\Lambda \in \mathscr{S}$ one fixes $(\omega_i)_{i \in S \setminus \Lambda}$ and requires that the conditional weight of $(\omega_i)_{i \in \Lambda}$ is given by $Z_\Lambda^{-1} e^{-H_\Lambda(\omega)}$. Here $H_\Lambda(\omega)$ is the energy of ω inside Λ and is of the form $H_\Lambda(\omega) = \sum_{A \in \mathscr{S}, A \cap \Lambda \neq \varnothing} \Phi_A(\omega)$, where $\Phi = (\Phi_A)_{A \in \mathscr{S}}$ is a family of (tempered) potentials carrying the interaction of the random variables in A. This defines a family of conditional probability measures $\gamma = (\gamma_\Lambda)_{\Lambda \in \mathscr{S}}$, called a specification, and each γ_Λ is viewed as a probability kernel from $S \setminus \Lambda$ to S. The set of probability measures $G(\gamma) = \{\mu: \mu\gamma_\Lambda = \mu \text{ for all } \Lambda \in \mathscr{S}\}$ are called the Gibbs measures specified by γ. The aim of the book is to study the structure of $G(\gamma)$ depending on the choice of Φ. In the given setup the specifications γ are the more natural mathematical objects, rather than the interactions Φ, which have a clearer physical interpretation.

The book has four parts. *Part I* contains the general theory and starts with the formal definition and construction of Gibbs measures. The existence problem is discussed [i.e., conditions for $G(\gamma) \neq \varnothing$], the role of symmetries is investigated [i.e., which $\mu \in G(\gamma)$ inherit what symmetries of Φ], and several examples are given for nonuniqueness [i.e., $|G(\gamma)| > 1$]. The latter is what is associated with the physical phenomenon of a phase transition. There is an extensive exposition of Dobrushin's condition for uniqueness [i.e., $|G(\gamma)| = 1$], together with examples. Examples are also given of nonexistence and of partial breaking of symmetries. Finally, it is shown that each $\mu \in G(\gamma)$ admits a unique representation as a convex combination of extreme Gibbs measures (Choquet simplex). The latter are denoted by Ex $G(\gamma)$ and represent the pure phases of the physical system.

Part II deals with Markov and with Gauss random fields as Gibbs measures. The Markov random field property arises from a nearest-neighbor potential (i.e., $\Phi_A \equiv 0$ for $|A| > 2$). First the case $S = \mathbb{Z}$ is considered. A Markov chain is a Markov random field, but the reverse need not be true: the one-sided Markov property is stronger than the two-sided one. Sufficient conditions for $\mu \in G(\gamma)$ to be a Markov chain are: (i) $\mu \in \text{Ex } G(\gamma)$; (ii) $\mu \in G_\theta(\gamma)$. The latter are the shift-invariant Gibbs measures. In fact, either $G_\theta(\gamma) = \varnothing$ or $G_\theta(\gamma) = \{\mu\}$. Three examples are discussed where Φ is shift-invariant but $G(\gamma)$ contains non-shift-invariant or even non-Markov-chain elements. Next the results are extended to Markov random fields on trees and the phase diagram of the Ising model on $S =$ (Cayley tree) is computed. Gaussian random fields are shown to be Gibbs measures (under a weak kind of Markov property) for a pair potential that can be explicitly computed in terms of the mean and covariance function. Fourier analysis for shift-invariant interaction on $S = \mathbb{Z}^d$ leads to a complete description of the corresponding $G(\gamma)$.

Part III covers the shift-invariant Gibbs measures $G_\theta(\gamma)$ and the Gibbs variational principle. After a section on ergodicity, the existence of the thermodynamic limit is shown for the various quantities needed: specific energy $\langle \mu, \Phi \rangle$, specific (relative) entropy $h(\mu)$, and (negative) specific Gibbs free energy $P(\Phi)$, also called the pressure. The Gibbs variational principle says that $-P(\Phi) = \inf_\mu [\langle \mu, \Phi \rangle - h(\mu)]$, with the infimum running over all shift-invariant probability measures μ, and says that the infimum is attained at all $\mu \in G_\theta(\gamma)$ for the given Φ. An equivalent description is that each $\mu \in G_\theta(\gamma)$ is a tangent functional to the convex functional $P(\Phi)$ on the space of (tempered) potentials. This geometric property is used to show various existence and genericity results for phase transitions.

Part IV presents the method of reflection positivity, which can be used to prove the existence of phase transitions, and applies it to a dozen or so models on $S = \mathbb{Z}^d$. A measure μ is reflection positive for a reflection τ about a hyperplane if $\mu(A\tau A) \geqslant 0$ for every observable A on one side of the hyperplane. For certain special interactions Φ every Gibbs measure obtained from periodic boundary conditions has this property. The method is highly special, but the results one can obtain along this line are rather impressive. In some cases they are the best results known: e.g., breaking of continuous symmetries, transitions between ordered and disordered states not connected with any symmetry breaking.

While Parts I and III contain general structure and properties, Parts II and IV are more specialized and reflect to a certain extent the taste of the author. Concluding, we find Georgii's book a most valuable contribution to the literature and we recommend it to any reader who is interested in rigorous statistical mechanics and is reasonably at home in probability theory.

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