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A Proof of the Isoenergetic KAM-Theorem from the "Ordinary" One

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A proof is given of the isoenergetic KAM-theorem for Hamiltonian systems, using the "ordinary" KAM-theorem and a transversality argument. © 1991 Academic Press, Inc.

1. INTRODUCTION

The "ordinary" KAM-theorem is concerned with the persistent occurrence of quasi-periodic tori in nearly integrable Hamiltonian systems. Here the tori are Lagrangian, implying that their dimension is maximal, i.e., equal to the "number of degrees of freedom." For an account of this we refer to [1–3, 9]. An important variation of this result, the so-called *isoenergetic* KAM-theorem, concerns the persistent tori for a fixed value of the Hamiltonians. An account of this is given in, e.g., [3, 2]. In [5] a strong connection (equivalence) if proven between these theorems and between the corresponding theorem for symplectic maps.

In [7, 4] a somewhat more *geometric* viewpoint is assumed, based on the setting and ideas of Pöschel [9]. In [9], and later in [7, 4], it is obtained that the persistent tori under consideration foliate smoothly over a set that is the union of closed halflines. This is a nowhere dense set of positive measure and the smoothness is understood in the sense of Whitney, compare Zehnder [10]. In fact in [7, 4], also in contexts different from the Hamiltonian one, a general *unfolding* theory of quasi-periodic tori is given, using Moser [8].

The "ordinary" KAM-theorem provides *conjugacies* between (non-degenerate) integrable systems and their perturbations. These conjugacies are defined on the union of the invariant tori under concern and they are (Whitney-) smooth. So this theorem can be phrased in terms of structural stability, in this case, in [7, 4], called *quasi-periodic stability*.

Moreover, in [7, Chap. 7c; 4, Sect. 7c)] in general the relation, concerning nondegeneracy-conditions, the necessary number of unfolding

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parameters and the relevant invariants, is studied flows and maps via Poincaré-sections resp. suspensions. Here the usual correspondence between *equivalences* and *conjugacies* occurs, when passing to sections of a flow resp. to suspensions of a map. We recall that both conjugacies and equivalences are transformations that map orbits to orbits, but that conjugacies preserve the time-parametrization, while equivalences only have to preserve the direction of this parametrization. In [7, 4] the term *weak* quasi-periodic stability is employed when instead of conjugacies equivalences, likewise (Whitney-) smooth, are used between quasi-periodic tori.

In [7, Chap. 9a], the isoenergetic KAM-theorem is formulated as a statement of weak quasi-periodic stability and proven directly from the "ordinary" KAM-theorem [9] and a straightforward *transversality* argument concerning the aforementioned halflines. Technically speaking this is a more geometric and qualitative version of [5, Chap. 1, II], where presently moreover, instead of one at the time, a whole (Whitney-) smooth foliation of quasi-periodic tori is treated at once. For a similar proof also compare [7; 4, Cor. 7.1].

This article contains a slight modification of [7, Chap. 9a], presented here for the sake of general availability. It is organized as follows: In the next section a more detailed exposition of the results of [9] will be given. Then, in the last section, the isoenergetic KAM-theorem is deduced from this.

2. PRELIMINARIES

We start by recalling the setting of the problem. Let $\mathbb{T}^n := \mathbb{R}^n/(2\pi\mathbb{Z})^n$ be the standard *n*-torus, with coordinates $x = (x_1, x_2, ..., x_n)$ mod 2π. Our dynamics lives in the phase space $M := \mathbb{T}^n \times \mathbb{R}^n$, where on $\mathbb{R}^n$ we use the coordinates $y = (y_1, y_2, ..., y_n)$. We endow M with the symplectic form $\sigma := \sum_{j=1}^n dx_j \wedge dy_j$ ($= dx \wedge dy$ for short). Given a function $H: M \to \mathbb{R}$ the associated Hamiltonian vector field X_H on M is given by $dH = \iota_{X_H}\sigma$, implying that

$$X_H(x, y) = \sum_{j=1}^n \left(\frac{\partial H}{\partial y_j}(x, y) \frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j}(x, y) \frac{\partial}{\partial y_j} \right)$$

$$\left(= \frac{\partial H}{\partial y}(x, y) \frac{\partial}{\partial x} - \frac{\partial H}{\partial x}(x, y) \frac{\partial}{\partial y} \text{ for short.} \right)$$

Then H, the Hamiltonian or energy, obviously is a first integral, implying that M is a disjoint union of the X_H-invariant level sets of H. In cases of our concern this union is a foliation (with singularities).

a. *The Real Analytic Topology*

Throughout this paper we assume real analyticity. We mention however that straightfortward adaptations exist for the C^k-case with $k \leqslant \infty$ sufficiently large. Compare [9], also see [4]. In our analytic case we consider $M = (\mathbb{R}^n/(2\pi\mathbb{Z})^n) \times \mathbb{R}^n$ as the real part of $((\mathbb{C}^n/(2\pi\mathbb{Z})^n) \times \mathbb{C}^n$, while any of our Hamiltonians H on M is assumed to have a complex analytic or holomorphic extension to a neighbourhood of M in $(\mathbb{C}^n/(2\pi\mathbb{Z})^n) \times \mathbb{C}^n$.

Perturbations of such H are taken in the compact-open topology on these complex analytic extensions, which we call the *real analytic* topology. At the end of this section we shall give an explicit example of a neighborhood in this topology.

b. *Integrability*

A Hamiltonian system as above is *integrable* if for some function $H_0 \colon \mathbb{R}^n \to \mathbb{R}$ we have $H(x, y) \equiv H_0(y)$, so if H is independent of x. In that case consider the map $f \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $f := \partial H_0/\partial y$, using the same abbreviations as before. We see that now

$$X_H(x, y) = f(y)\frac{\partial}{\partial x},$$

implying that the coordinate functions $y_1, y_2, ..., y_n$ all are first integrals. So for each fixed $y_0 = (y_{01}, y_{02}, ..., y_{0n})$ the torus $\mathbb{T}^n \times \{y_0\} \subseteq M$ is X_H-invariant, while the restriction of X_H is constant, implying parallelity of the corresponding flow. We say that that $f(y_0)$ is the frequency vector of this torus, while f is the frequency map of the integrable vector field X_H. The question then is what is the behavior of the invariant foliation $\{\mathbb{T}^n \times \{y_0\} \mid y_0 \in \mathbb{R}^n\}$ of tori, under nonintegrable perturbations.

c. *Some "Cantor Sets"*

KAM-theory deals with the persistence of those invariant tori, the frequency vector $\omega = (\omega_1, \omega_2, ..., \omega_n)$ of which satisfies so-called diophantine (or small divisor) conditions. To be precise let $\tau > n$ be a constant and $\gamma > 0$ a parameter. Then for all integer vectorts $k \in \mathbb{Z}^n \backslash \{0\}$ consider the inequality

$$|\langle \omega, k \rangle| \geqslant \gamma |k|^{-\tau},$$

where $\langle \omega, k \rangle := \sum_{j=1}^n \omega_j k_j$ and $|k| := \sum_{j=1}^n |k_j|$. By $\mathbb{R}_\gamma^n$ we denote the set of $\omega \in \mathbb{R}^n$ satisfying all these inequalities for a fixed γ. We say that $\mathbb{R}_\gamma^n$ contains the diophantine frequency vectors of order γ. Note that for $\omega \in \mathbb{R}_\gamma^n$ the numbers $\omega_1, \omega_2, ..., \omega_n$ certainly are rationally independent.

Evidently with ω also all scalar multiples $s\omega$, $s \geqslant 1$, belong to $\mathbb{R}_\gamma^n$. Moreover $\mathbb{R}_\gamma^n$ intersects the unit sphere in $\mathbb{R}^n$ in a closed set, which by the

Cantor–Bendixson theorem, cf. [6], is the union of a Cantor set and a countable set. The measure of the complement of this set in the sphere, is of order γ as $\gamma \downarrow 0$. It follows that $\mathbb{R}^n_\gamma$ is a Whitney-C^∞ foliation of closed halflines, parametrized over a Cantor set of positive measure.

If the frequency map f under b is a (local) diffeomorphism, the same can be said of the pullback $f^{-1}(\mathbb{R}^n_\gamma)$. Throughout this paper we colloquially use the expression "*Cantor set*" for any such a Whitney-smooth foliations of manifolds (with boundary).

In particular we have that the collection $\mathbb{T}^n \times f^{-1}(\mathbb{R}^n_\gamma)$ of invariant tori of the integrable system X_H, is such a "Cantor set." The fact that the corresponding frequencies are rationally independent implies that these tori are *quasi-periodic*: all of them are densely filled by each of the integral curves contained in them.

d. The "Ordinary" KAM-Theorem

We next give a qualitative description of the "ordinary" KAM-theorem, compare [9]. To this end first consider an integrable Hamiltonian vector field

$$X_H(x, y) = f(y) \frac{\partial}{\partial x}$$

as under b, so with $H(x, y) \equiv H_0(y)$ and $f := \partial H_0 / \partial y$. We say that X_H is *nondegenerate* at the invariant torus $\mathbb{T}^n \times \{y_0\}$ if

$$\det \left(\frac{\partial f}{\partial y}(y_0) \right) \neq 0.$$

This condition implies that near y_0 the frequency map f is a local diffeomorphism. Hence the invariant tori $\mathbb{T}^n \times \{y\}$ for y near y_0 can be parametrized by their frequency vectors.

According to [9], given nondegeneracy of X_H at $\mathbb{T}^n \times \{y_0\}$, there exists a neighbourhood $\mathscr{V}$ of H in the real analytic topology, depending on γ, such that for all perturbations $\tilde{H} \in \mathscr{V}$ the following holds. There exists a (local) Whitney-C^∞, symplectic conjugacy $(\xi, \eta) \mapsto (x, y)$, which puts $X_{\tilde{H}}$ on an integrable normal form

$$\psi(\eta) \frac{\partial}{\partial \xi},$$

where $(\xi, \eta) \in \mathbb{T}^n \times \psi^{-1}(\mathbb{R}^n_\gamma)$ and where y varies near y_0. So η and ξ form action-angle variables for the perturbed system, when restricting to the "Cantor set" $\mathbb{T}^n \times \psi^{-1}(\mathbb{R}^n_\gamma)$.

The fact that the map $(\xi, \eta) \mapsto (x, y)$ is Whitney-C^∞ means that it can be extended as a C^∞-diffeomorphism to a full neighbourhood in $\mathbb{T}^n \times \mathbb{R}^n$.

Outside the "Cantor set," however, this nonunique extension usually can not be a conjugacy to an integrable system. Finally we mention that the map $(\xi, \eta) \mapsto (x, y)$ is analytic for fixed values of $\eta \in \psi^{-1}(\mathbb{R}^n_\gamma)$.

Next we consider the map

$$(\xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n \mapsto (\xi, f^{-1}(\psi(\eta)) \in \mathbb{T}^n \times \mathbb{R}^n,$$

observing that it conjugates the integrable systems $\psi(\eta)\partial/\partial\xi$ and $X_H(x, y) = f(y)\partial/\partial x$. Composition of this map with the inverse of the one obtained before yields a near-identity conjugacy between appropriate sub-systems of $X_{\tilde{H}}$ and X_H, namely between "Cantor sets" of quasi-periodic tori. Here the foliations of the invariant tori can be parametrized by the corresponding frequency vectors, which are preserved by the conjugacy. Note that this conjugacy is not necessarily symplectic.

We say that the vector field X_H (near $\mathbb{T}^n \times \{y_0\}$) is quasi-periodically stable. The smoothness of the involved maps guarantees that the property of having a "Cantor set" of quasi-periodic tori is open in the real analytic topology. (Here recall that "Cantor sets" have positive measure. For another proof of this stability result see [7, 4].)

Remarks. (i) For completeness' sake and in order the clarify the rôle of the parameter γ, let us more explicitly describe a real analytic neighbourhood $\mathscr{V}$ of H, from which $\tilde{H}$ is chosen, again see [9]. To this end let O denote a compact neighbourhood of $\mathbb{T}^n \times \Gamma$ in $(\mathbb{C}^n/(2\pi\mathbb{Z})^n) \times \mathbb{C}^n$, such that H has a complex analytic extension to O. Here Γ is a neighborhood of y_0 in $\mathbb{R}^n$, such that the restriction of f to Γ is a dif-feomorphism onto its image. Then $\mathscr{V}$ is the compact-open neighbourhood of H, determined as follows by O, γ and a positive constant δ provided by [9]. In fact $\mathscr{V}$ consists of all "real" analytic functions $\tilde{H}: O \to \mathbb{C}$, such that $\sup_O |\tilde{H}(x, y) - H(x, y)| < \gamma^2\delta$. For $\gamma > 0$ sufficiently small δ is independent of γ. The conjugacy then is defined on a set $\mathbb{T}^n \times \Gamma'$ with $\Gamma' \subset \Gamma$ slightly smaller, the difference vanishing for $\gamma \downarrow 0$.

(ii) Relative to a bounded neighbourhood of $\mathbb{T}^n \times \{y_0\}$ in $M = \mathbb{T}^n \times \mathbb{R}^n$, the measure of the complement of the perturbed tori is of order γ as $\gamma \downarrow 0$.

3. THE RESULT

In this section we give a formulation of the isoenergetic KAM-theorem, e.g., compare [2] or [3], and subsequently prove this from the "ordinary" KAM-theorem as stated in [9], compare Section 2d. First, however, observe that from the result of Section 2d alone it does not follow directly that in a given level set of $\tilde{H}$ *any* $X_{\tilde{H}}$-invariant tori persist.

Quoting from [2, 3] we say that the integrable system X_H at the invariant torus $\mathbb{T}^n \times \{y_0\}$ is *isoenergetically nondegenerate* if

$$\det \begin{bmatrix} \partial f/\partial y(y_0) & f(y_0)^t \\ f(y_0) & 0 \end{bmatrix} \neq 0,$$

where the superscript t denotes transposition. This condition means that, near $\mathbb{T}^n \times \{y_0\}$, in each level set of H the X_H-invariant tori can be parametrized by their frequency ratios $[f_1(y): f_2(y): \cdots : f_n(y)]$.

The following example, due to R. Douady [5], shows that isoenergetic nondegeneracy is independent of the "ordinary" nondegeneracy of Section 2d. In fact we take $n = 2$ and consider the Hamiltonians $H_1, H_2: \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}$, respectively defined by $H_1(x, y) := y_1 + y_2 + y_1^2$ and $H_2(x, y) := y_1 + y_2 + y_1^2 - y_2^2$, which both give rise to integrable vector fields X_H and X_{H_2}. Now consider the torus $\mathbb{T}^2 \times \{0\}$, invariant for both systems. It is easily verified that here X_{H_1} is isoenergetically nondegenerate, but degenerate in the "ordinary" sense, while for X_{H_2} the converse holds.

The isoenergetic KAM-theorem roughly says that isoenergetic non-degeneracy implies that, restricted to fixed energy levels, the vector field X_H is weakly quasi-periodically stable. We recall that the adjective "weak" means that the transformation, to be found between perturbed and unperturbed tori, is not necessarily a conjugacy, but only an *equivalence*.

Now, to give a precise formulation of the isoenergetic KAM-theorem, we first need the following notation: For $A \subseteq \mathbb{R}^n$ we write $A_\gamma := \{y \in A \mid f(y) \in \mathbb{R}^n_\gamma\}$, cf. Sections 2c, d. Similarly we write Γ_γ, etc.

THEOREM. *Let $X = X_H$ be a real analytic, integrable Hamiltonian vector field on $M = \mathbb{T}^n \times \mathbb{R}^n$. Let $E \in \mathbb{R}$ and $y_0 \in \mathbb{R}^n$ be such that X is isoenergetically nondegenerate at the invariant $\mathbb{T}^n \times \{y_0\} \subset H^{-1}(E)$. Then there exists a neighborhood Λ of y_0 in $\mathbb{R}^n$, such that for all real analytic Hamiltonian vector $\tilde{X} = X_{\tilde{H}}$ on M, with $\tilde{H}$ sufficiently near H in the real analytic topology, the following holds. In the level set $\tilde{H}^{-1}(E)$ there exists an $\tilde{X}$-invariant "Cantor set," which is a C^∞-near-identity diffeomorphic image of $(\mathbb{T}^n \times \Lambda_\gamma) \cap H^{-1}(E)$. The corresponding diffeomorphism in these tori is a real analytic equivalence from X to $\tilde{X}$, preserving the frequency ratios and the (trivial) normal linear behaviour.*

Before giving a proof of this we give some further remarks.

Remarks. (i) First, if we regard the energy E as a parameter, the equivalence can be chosen in analytic dependence of it, compare [9, 7, 4].

(ii) Second, both "Cantor sets" of invariant tori, perturbed and unperturbed, are Whitney-C^∞ foliations that can be parametrized by the corresponding frequency ratios.

(iii)   Moreover, relative to a suitable bounded neighbourhood of $\mathbb{T}^n \times \{ y_0 \}$ in $M = \mathbb{T}^n \times \mathbb{R}^n$, the $(2n-1)$-dimensional measure of the complement of the perturbed tori in $\tilde{H}^{-1}(E)$ is of order γ as $\gamma \downarrow 0$.

Proof of the Theorem.   We start from the results of Pöschel [9], compare Section 2d. Also, for simplicity we take $E = 0$ and $y_0 = 0$.

First, due to the following argument of R. Douady [5], it is sufficient only to consider the case where $X = X_H$ at $\mathbb{T}^n \times \{0\}$ also is nonedegenerate in the "ordinary" sense. Indeed, if this extra condition is not fulfilled, we replace the Hamiltonian H by $H + H^2$. The new vector field then satisfies both nondegeneracy conditions. Moreover, near $\mathbb{T}^n \times \{0\}$, both vector fields are conjugate when restricted to the level sets with energy 0. Finally small perturbations of H correspond to small perturbations of $H + H^2$ and *vice versa.* So from now on we assume both nondegeneracy conditions hold.

The result mentioned in Section 2d, from the "ordinary" nondegeneracy gives us a neighbourhood Γ of 0 in $\mathbb{R}^n$, such that the frequency map $f: \mathbb{R}^n$ is a diffeomorphism onto its image and such that the following holds. For a Hamiltonian $\tilde{H}$, sufficiently near H in the real analytic topology, a C^∞-diffeomorphism $\Phi: \mathbb{T}^n \times \Gamma \to \mathbb{T}^n \times \mathbb{R}^n$ onto its image exists, where the restriction $\Phi|_{\mathbb{T}^n \times \Gamma_\gamma}$ is a conjugacy between appropriate subsystems of X and $\tilde{X} = X_{\tilde{H}}$. Moreover Φ is analytic in the $\mathbb{T}^n$-direction and near the identity in the C^∞-topology. In fact all of this holds for sufficiently small neighbourhoods Γ of 0. Of course, in order to avoid having $\Gamma_\gamma = \varnothing$, for Γ small also $\gamma > 0$ has to be small, which has its consequences for the allowed size of the perturbation $\tilde{H} - H$; compare the remarks ending Section 2d. Furthermore observe that by choosing Γ appropriate we can ensure that $f(\Gamma) \subseteq \mathbb{R}^n$ is convex.

Next we consider the restrictions of X and $\tilde{X}$ to the respective level sets $H^{-1}(0)$ and $\tilde{H}^{-1}(0)$. Note that by the "ordinary" nondegeneracy both of these level sets are $(2n-1)$-dimensional manifolds, or hypersurfaces, in $\mathbb{T}^n \times \Gamma$ with Γ sufficiently small. With help of the above map Φ and the isoenergetic nondegeneracy we shall construct an equivalence between these restrictions on the level of our quasi-periodic tori.

For this purpose we give the following geometric interpretation of isoenergetic nondegeneracy. Let $\pi: \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the projection on the second factor and for any $\omega \in \mathbb{R}^n \backslash \{0\}$ consider the open half-line $s \in (0, \infty) \mapsto sw \in \mathbb{R}^n$. One easily sees that the condition means that near $f(0)$ in $\mathbb{R}^n$ all these halflines are *transverse* to the hypersurface $(f \circ \pi)(H^{-1}(0))$. Note that the halflines are sets of constant frequency ratio.

Now consider the "Cantor set" of X-invariant, quasi-periodic tori in the set $H^{-1}(0) \cap (\mathbb{T}^n \times \Gamma_\gamma)$. We shall link a large piece of this set to a "Cantor set" of X-invariant tori in $\Phi^{-1}(\tilde{H}(0))$ by an equivalence Ψ of the form
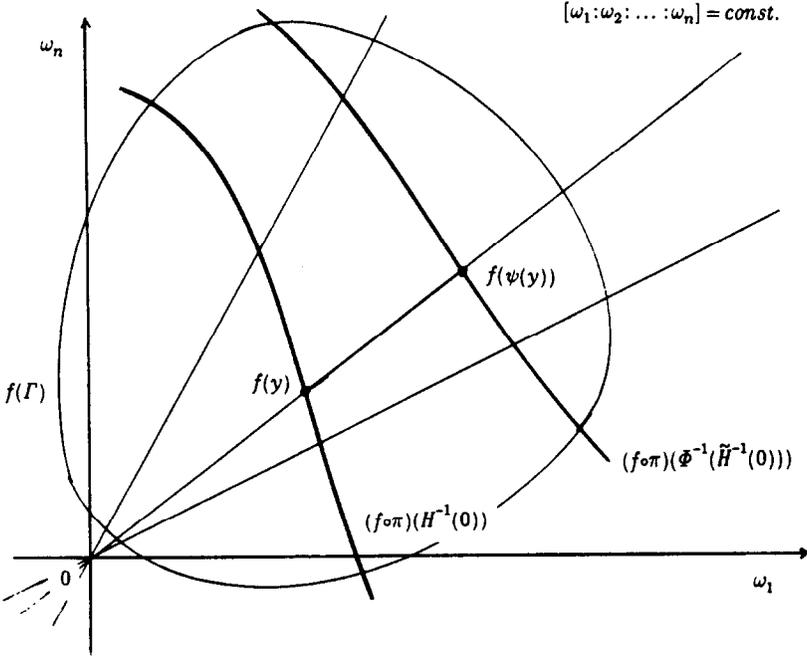
FIG. 1. Description of the map ψ in the frequency domain.

$\Psi(x, y) = (x, \psi(x, y))$. In fact, since $\tilde{H}$ is close to H and Φ near the identity, the hypersurface $(f \circ \pi)(\Phi^{-1}(\tilde{H}^{-1}(0)))$ also is transverse to the above halflines. This implies that the map ψ is completely determined by the relation $[f_1(y) : \cdots : f_n(y)] = [f_1(\psi(y)) : \cdots : f_n(\psi(y))]$ between the corresponding frequency ratios. See Fig. 1. In the frequency domain $f(\Gamma)$ this map just goes from one set to the other along the halflines described above.

Clearly this construction works for Γ sufficiently small with $f(\Gamma)$ convex. Considering the domain of ψ for $f(\Lambda)$ we then have to take the intersection of $f(\Gamma)$ and the cones on the sets $(f \circ \pi)(H^{-1}(0)) \cap f(\Gamma)$ and $(f \circ \pi)(\Phi^{-1}(\tilde{H}^{-1}(0))) \cap f(\Gamma)$ with 0 as top. This determines the neighborhood Λ in the theorem. The desired equivalence now is the composition $\Phi \circ \Psi$. Since the map Ψ is quite simple, this composition has the same regularity as Φ. ∎

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