The small window we can have access to via CMB measurements does not allow us to probe the full inflationary trajectory. This implies that disparate models of inflation can yield the same observational predictions and be organized in terms of universality classes, as long as they agree on the CMB window. We present a description of the inflationary background dynamics fully in terms of the number of e-folds $N$. This becomes a very useful language in order to describe universality properties of inflation. Then, we examine the properties of the inflaton range, a variable depending on the whole trajectory. We show its degeneracy in the super-Planckian regime; namely, its value is not uniquely determined by the inflationary observables. On the other hand, we provide strong evidence for its universality properties when the value is sub-Planckian. Both the tensor-to-scalar ratio and the spectral tilt are essential for the field range. Remarkably, this results into strengthening the usual Lyth bound by two orders of magnitudes. The novel results of this Chapter are based on the publications [iv] and [v].
4.1 The \textit{N}-formalism of inflation

In the previous chapters, we have expressed the time evolution of the system mainly in terms of the cosmic time \( t \) or the conformal time \( \tau \). However, it is possible to make other choices which may appear more natural depending on the specific context. Eventually they may lead to some simplifications in the description of the physics.

A famous example is provided by the \textit{Hamilton-Jacobi formalism} \cite{56,57}. This turns out to be a very natural choice in a Universe dominated by a scalar field. The fundamental idea is that the field \( \phi \) itself plays the role of an internal clock. This is possible as long as \( \phi \) evolves monotonically in time, that is \( \dot{\phi} \geq 0 \). Then, the relevant cosmological quantities can be regarded as a function of \( \phi \) rather than \( t \). This formalism is very useful for describing the background dynamics of inflation. However, it proves not to be suited for reheating where the field oscillates around the minimum and \( \dot{\phi} \) changes sign.

Another very natural choice for describing the time evolution of an inflationary Universe is the number of e-folds \( N \). This variable is directly related to the scale factor by means of Eq. (3.6), thus representing a concrete measure of time. In an expanding Universe, observers can simply use the physical growth of space as a universal clock.

The inflationary phase can then be specified fully in terms of \( N \) \cite{58}. This has the direct advantage to go beyond an explicit description of the microscopic mechanism generating the accelerated expansion and the deviation from a scale invariant spectrum.

The Hubble flow functions, defined by Eq. (3.9), provide the ideal tool to extract the relevant information from a description in terms of \( N \) \cite{36,37,59,60}. The inflationary observables, in particular the spectral index and the tensor-to-scalar ratio, can then be compactly expressed as

\[
    n_s = 1 + \epsilon_2 - 2\epsilon_1, \quad r = 16\epsilon_1. \tag{4.1}
\]

In order to connect these to CMB observations, one needs to evaluate these quantities at horizon crossing, denoted by \( N_\star \).

It is easy to check that it is possible to recover the expressions for \( n_s \) Eq. (3.51) and \( r \) Eq. (3.53), given in the previous chapter in terms of the slow-roll parameters \( \epsilon \) and \( \eta \), simply by substituting Eq. (3.26) into Eq. (4.1).

The link between the formulation in terms of \( \phi \) and the one in terms of \( N \) is provided by the important relation

\[
    \frac{d\phi}{dN} = \sqrt{2\epsilon_1}, \tag{4.2}
\]
which is easily obtained by combining Eq. (3.5), Eq. (3.6) and Eq (3.20). Note, however, that this last expression is exact and it is valid beyond the slow-roll approximation.

This last equation can be interpreted as a background field redefinition from $\phi$, with canonical kinetic terms, to the field $N$ with Lagrangian

$$\mathcal{L} = \sqrt{-g} \left[ \frac{1}{2} R - \epsilon_1(N) (\partial N)^2 - V(N) \right],$$

(4.3)

Generically, the functional form of the potential $V$ will be very different when expressed in terms of $\phi$ or in terms of $N$.

### 4.2 Universality classes at large $N$

In Sec. 3.3, we have seen that the window we can probe by means of CMB observations corresponds to a small portion of the whole inflationary period. This is mainly due to the suppression of the CMB power spectrum at small angular scales, as it is shown in Fig. 3.5. This sensitive region amounts to $\Delta N \approx 7$ and it is located at present around 60 e-folds before the end of inflation (we derived this number in Sec. 3.1.4 in order to account for the homogeneity and isotropy of the CMB at its largest scale). This is indeed the point when the modes relevant for the CMB power spectrum left the region of causal physics. Its position is determined by $N$ being equal to the number of e-folds between the points $N_\text{\textit{i}}$ of horizon crossing and $N_e$ where inflation ends, that is

$$N = N_\text{\textit{i}} - N_e.$$  

(4.4)

Then, the measured values of the cosmological parameters Eq. (3.55) and Eq. (3.56) constrain the form of the scalar potential just on a limited part. The practical situation is that several scenarios can give rise to the same predictions despite the details of the specific model. This situation is visually explained in Fig. 4.1.

A lower limit on $N$ can be set by the temperature of reheating [61] (see also [38,62–66]). On the other hand, there is no compelling reason to assume the number $N$, quantifying the amount of exponential expansion of the Universe, has an upper bound; in fact, it seems natural that inflation extends a long way further into the past than the portion we can observe (see [67] for a study on this topic).

The above argument seems to suggest $1/N$ as a natural small parameter to expand our cosmological variables [58,68,69] (see also [70]). This approach is also motivated by the percentage-level deviation of the Planck reported
value for the spectral index (3.55) from unity which can be interpreted as

\[ n_s = 1 - \frac{2}{N}. \quad (4.5) \]

This argument naturally leads to assume the function \( \epsilon_1 \) (or equivalently the first slow roll parameter \( \epsilon \)) scaling as

\[ \epsilon_1 = \frac{\beta}{N^p}, \quad (4.6) \]

where \( \beta \) and \( p \) are constant and we have neglected higher-order terms in \( 1/N \) as not relevant for observations. This simple assumption leads to the so-called perturbative class defined by

\[ r = \frac{16\beta}{N^p}, \quad n_s = \begin{cases} 1 - \frac{2\beta+1}{N}, & p = 1, \\ 1 - \frac{p}{N}, & p > 1, \end{cases} \quad (4.7) \]

where we have discarded the case \( p < 1 \) as generically not compatible with the current cosmological data. Eq. (4.7) identifies the families of universality classes which any specific scenario belongs to, for fixed values of \( \beta \) and \( p \).

Most of the inflationary models in literature have an equation of state parameter scaling as a power of \( 1/N \), thus falling into the perturbative class. These includes the chaotic monomial inflation scenarios, the Starobinsky model, hilltop models and many others. It is possible to consider also other
4.2 Universality classes at large $N$

functional forms for $\epsilon_1$, as it was investigated in [58] (see also [71] for a related analysis with a different approach). However, the cosmological predictions of these classes are generically more in tension with the observational data. Further, the number of well motivated models of inflation falling into these other classes is more restricted.

The analysis at large-$N$ proves to be a powerful tool in order to organize different inflationary models just in terms of their cosmological predictions. Physically different scenarios may predict the same values of $n_s$ and $r$ in the leading approximation in $1/N$. The details of any specific model, encoded in the subleading terms (higher powers of $1/N$), are washed out and not relevant for the observational predictions. Examples of these circumstances are listed in [58, 69].

Now, we want to show that the simple assumption of a scalar spectral tilt scaling as $1/N$ can exclude a consistent region of the $(n_s, r)$ plane and yield definite predictions for our cosmological variables [69, 72]. The allowed regions are shown in Fig. 4.2. In particular, given the best fit value for $n_s$ and the strict bound on $r$, we will generically expect a very low value for the tensor-to-scalar ratio, probably order $10^{-3}$.

![Figure 4.2](image)

**Figure 4.2**

Predictions of the inflationary scenarios with equation of state parameter given by Eq. (4.6) superimposed over the Planck data. Given the favored value of the spectral index Eq. (4.5), one has generically a forbidden region for value of the tensor-to-scalar ratio $r$. 
Finally, in a pure large-$N$ description, one can identify the benchmark potentials for this Ansatz. Let us recall the relation between the Hubble parameter $H$ and $\epsilon_1$ given by Eq. (3.5). Within the slow-roll approximation, employing $H^2 = V/3$, one can integrate this equation and obtain an expression for the potential in terms of $N$ which reads

$$V(N) = \begin{cases} V_0 N^{2\beta}, & p = 1, \\ V_0 \left[1 - \frac{2\beta}{(p-1)N^{p-1}}\right], & p > 1, \end{cases} \quad (4.8)$$

where $V_0$ is an integration constant related to the energy scale of inflation. By means of Eq. (4.2) and Eq. (4.6), one gets the asymptotic form of $V$ in terms of the canonical scalar field $\phi$, that is

$$V(\phi) = \begin{cases} V_0 \phi^n, & p = 1, \\ V_0 \left[1 - \exp\left(-\phi/\mu\right)\right], & p = 2, \\ V_0 \left[1 - (\phi/\mu)^n\right], & p > 1, p \neq 2, \end{cases} \quad (4.9)$$

where $\mu$ and $n$ are related to $\beta$ and $p$ as dictated by (4.2). In particular, for $p > 1$ and $p \neq 2$, the power $n$ is related to $p$ through the following equation

$$n = \frac{2(1 - p)}{2 - p}, \quad (4.10)$$

where $p < 2$ or $p > 2$ determine respectively the negative or positive sign of $n$. The inverse relation $p = p(n)$ turns out to be of the same form.

In the large-$N$ limit, any model belonging to these universality classes will have a potential asymptotically approaching well-known scenarios such as chaotic monomial inflation ($p = 1$), inverse-hilltop models ($1 < p < 2$), Starobinsky-like inflation ($p = 2$) and hilltop potentials ($p > 2$). As already explained, the reason for such simplicity is that, in this limit, we are probing just a limited part of the inflationary trajectory, close to horizon crossing. Peculiarities among different models appear when we go away from this region. In general, the situation near the end-point of inflation will be very different from one model to another, even though they belong to the same universality class.

### 4.3 The inflaton range and observations

In this Section, we intend to examine features of a variable crucial for the construction of inflationary models at high energies, namely the inflaton field range $\Delta \phi$. A crucial distinction is indeed between small- and large-field
models, defined by sub- and super-Planckian field ranges. Generic quantum corrections to a tree-level scalar potential come in higher powers of $\phi$, and hence large-field models are particularly sensitive to these. This puts the consistency of an effective field theory description [73–75] of such models into doubt. A key question in theoretical cosmology is therefore whether $\Delta \phi$ exceeds the Planck length or not.

Knowledge of the evolution of $\epsilon_1(N)$, during all e-foldings $N$ of the inflationary period, determine the field range by means of Eq. (4.2). Therefore, it is the area underneath the curve $\sqrt{2\epsilon_1(N)}$ which determines the excursion of the scalar field $\phi$ during the expansion. However, cosmological observations allow us to constrain just a small part. The situation resembles what already seen in the previous section: generally it is not possible to uniquely connect CMB data with a precise value of the inflaton excursion. This is depicted in Fig. 4.3.

However, a first estimate of $\Delta \phi$ can be obtained by the assumption that $\epsilon_1(N)$ is constant throughout inflation. This is referred as the Lyth bound\(^1\) [76] and leads to [77, 78]:

$$\Delta \phi \sim \left( \frac{r}{0.002} \right)^{1/2} \left( \frac{N_*}{60} \right),$$

where we have set the number of e-folds at horizon exit $N_*$ equal to 60 (other values allow for a similar analysis). Therefore, a sub-Planckian excursion for the inflaton field requires a very small value of $r \lesssim 2 \cdot 10^{-3}$. For monotonically increasing $\epsilon_1(N)$, this represents a lower bound. The blue rectangular area in Fig. 4.3 provides a visual representation of this bound.

In the following, we start the discussion with some specific examples where a point in the $(n_s, r)$ plane does correspond to a wide spectrum of values for $\Delta \phi$. We consider chaotic inflation models with monomial potentials as the benchmark scenarios to show such a degeneracy of the inflaton excursion. Intriguingly, we find that the field range cannot exceed an upper-bound due to the slow-roll conditions.

On the other hand, in the sub-Planckian regime and for a range of universality classes, we prove that it is possible to precisely connect observations to a unique value of $\Delta \phi$. Information on both the tensor-to-scalar ratio and the spectral tilt uniquely determines the value of $\Delta \phi$. This remarkable uni-

\(^1\)To be more precise, Lyth’s analysis concerns just the small window accessible via CMB observations. One must note that in 1997, at the time of publication of his paper, experiments could probe just around $\Delta N \approx 4$. This certainly leads to a milder bound than Eq. (4.11). On the other hand, this result is always valid within the slow-roll approximation and does not make any assumption on the form of $\epsilon_1(N)$.\]
versality of the inflaton range will lead to a stronger bound than the usual estimate given by Eq. (4.11).

4.4 Degeneracy of the inflaton range

We now discuss the field range in different classes of models. In particular, we are interested in exploring the correspondence between a specific point in the \((n_s, r)\) plane and the values of \(\Delta \phi\). We will prove that it is possible to have exactly the same cosmological predictions, in terms of the scalar tilt and the amount of gravitational waves, while the field excursion may vary over several orders of magnitude.

For simplicity we will consider the monomial inflation scenarios as benchmark models for our study. However, note that other models can be straightforwardly studied following the same reasoning.

In the following, we analyze three classes of inflationary models with a specific dependence on \(N\) for the Hubble flow parameters. Such classes, discussed at length in [58], reproduce the large-\(N\) behavior of most of the inflationary models available in the literature.

As a first case, we discuss the so-called *perturbative* class, characterized by a leading term in \(\epsilon_1\) scaling as \(1/N^p\), with \(p\) being a constant positive coefficient. Then, we analyze models where *logarithmic* terms, such as \(\ln^q(N + 1)\), appear in the leading part of \(\epsilon_1\). In a third class of models, we consider the...
4.4 Degeneracy of the inflaton range

parameter $\epsilon_1$ having a *non-perturbative* form, in the limit at large-$N$, of the type $\epsilon_1 \sim \exp(-cN)$. We will consider the possibility of letting the total number of e-folds $N$ vary over a certain interval which is related to reheating details of the specific model. Interestingly, we find an *upper bound* on $\Delta \phi$ and the total number of e-folds which sets connections among the three classes of models considered.

As final part of our analysis, we focus on the logarithmic class and we explore the possibility of playing with the power coefficient $q$, while keeping $N$ fixed. This is an alternative way to get the same predictions of quadratic inflation, while having quite different values for the inflaton range. We will consider the possibility of going beyond single-field and/or slow-roll inflation and getting a sub-Planckian $\Delta \phi$.

Throughout this section, we assume that the inflationary parameters $\epsilon_1$ and $\epsilon_2$ of each class are exact over the whole inflationary trajectory, as it happens for chaotic scenarios. In several cases, this may be a very good approximation and may capture most of the essential properties of the models falling into the specific universality classes. Anyhow, we will take advantage of a formulation purely in terms of $N$ and extract the information we are interested in, without referring to the particular form of the scalar potential $V(\phi)$. In fact, for any specific parametrization of each class, the latter may be very complicated when expressed in terms of the canonical scalar field $\phi$.

In what follows, the benchmark will be the value of $\Delta \phi$ for chaotic models corresponding to a quasi exponential expansion of $N = 60$. This sets

$$N_* = N_e + 60,$$

as corresponding to horizon exit. Moreover, all symbols with a tilde will be reserved for the classes being examined, while the benchmark models will have no tilde.

### 4.4.1 Chaotic inflation as benchmark

Chaotic scenarios are usually characterized by monomial potentials when expressed in terms of the canonical scalar field $\phi$. Further, they naturally lead to a large value of $r$ together with a super-Planckian excursion of the inflaton field.

In a large-$N$ description, the first three Hubble flow functions turn out to be

$$\epsilon_0 = h N^\beta, \quad \epsilon_1 = \frac{\beta}{N}, \quad \epsilon_2 = -\frac{1}{N},$$

where $h$ is an integration constant and $\beta$ is related to the specific universality class.
The description in terms of $N$ is exact for these models (there are no subleading corrections) and hence captures all of their fundamental features. However, even if there would be subleading corrections, e.g. at the level $1/N^2$, observables calculated at horizon exit, such as $n_s$ and $r$, will be observationally insensitive to these (as they are too much suppressed for $N \gtrsim 50$). Therefore, these are universal predictions of entire classes of models that agree in the large-$N$ limit.

The same universality holds for the inflaton range. In the case of chaotic models with parameters (4.13), the inflaton excursion $\Delta \phi$ will be basically determined just by the leading term in $N$ [79] through Eq. (4.2).

As these models receive most of their e-foldings at large-$N$, one can safely assume that restricting to the leading term of $\epsilon_1$ is a very good approximation over the relevant part of the inflationary trajectory. The expression for the inflaton field range will therefore read

$$\Delta \phi_c = 2\sqrt{2} \beta \left( N_s^{1/2} - N_e^{1/2} \right),$$

where the subscript $c$ is added in order to refer more easily to the benchmark field excursion of monomial models throughout the following part of the thesis. Further, $N_e = \beta$, when assuming that inflation ends at $\epsilon_1 = 1$, and $N_s$ is found through Eq. (4.4).

With the above relations, potentials of the type $V(\phi) = \lambda_n \phi^n$ will keep monomial form even when formulated in terms of $N$, namely $V(N) = h^2 N^{2\beta}$, and vice versa. The relation between the two power coefficients reads

$$\beta = \frac{n}{4},$$

and can be found by using Eq. (4.2). As an explicit example, a quadratic potential corresponds to $\beta = 1/2$ and an inflationary period of $N = 60$ leads to $\Delta \phi \simeq 14.14$. Of course, this is identical to the value of $\Delta \phi$ calculated through the scalar potential $V$, within the slow-roll paradigm.

4.4.2 Perturbative class

We start considering the possible degeneracies within the perturbative class of models. In this case, the relevant Hubble flow parameters for determining the observational data have the following $N$-dependence:

$$\epsilon_1 = \frac{\tilde{\beta}}{N^p}, \quad \epsilon_2 = -\frac{p}{N}.$$  

The case discussed above is easily recovered for $p = 1$ and $\tilde{\beta} = \beta$. 

We would like to reproduce the same $n_s$ and $r$ of the benchmark chaotic model through a generic perturbative model with $p \neq 1$. This translates into equating both $\epsilon_1$ and $\epsilon_2$ of (4.13) to the functions (4.16) at horizon exit, respectively at $N_*$ and $\tilde{N}_*$. As result, we have the following relations:

$$\tilde{\beta} = \beta \frac{\tilde{N}_*^p}{N_*}, \quad p = \frac{\tilde{N}_*}{N_*}, \quad (4.17)$$

This allows to express $\tilde{\beta}$ as

$$\tilde{\beta} = \beta \left\{ p N_*^{p-1} \right\}, \quad (4.18)$$

where $N_*$ is given by (4.12). Eq. (4.18) gives us an estimate of how fine-tuned the model is in order to reproduce the same predictions of the chaotic models. Curiously, for any $\beta = \mathcal{O}(1)$ (corresponding to different chaotic models), the corresponding perturbative model will start to be severely fine-tuned in the region $p > 2$, as is shown in Fig. 4.4.

![Figure 4.4](image_url)

**Figure 4.4**

Behavior of $\tilde{\beta}$ as function of $p$ in a log-plot. It generally blows up for $p \gtrsim 2$, where the perturbative model should be highly fine-tuned in order to reproduce the same $(n_s, r)$ of chaotic scenarios. The four lines correspond to the same observational predictions of models with potential of the type $V = \lambda_n \phi^n$, with $n$ respectively equal to 1, $3/2$, 2 and 3.

Demanding that inflation ends at $\epsilon_1 = 1$ turns into

$$\tilde{N}_e = \tilde{N}_* - \tilde{N} = \tilde{\beta}^{1/p}, \quad (4.19)$$

where the total number of e-foldings $\tilde{N}$ in principle could span a range of different values related to reheating properties of the model. Using (4.17),
Eq. (4.19) gives us the functional form of the total number of e-folds $\tilde{N}$ as a function of $p$, for any $\beta$, that is

$$\tilde{N} = p N_* \left[ 1 - \left( \frac{\beta}{N_*} \right)^{1/p} \right].$$

A period of inflation $\tilde{N} = 60$ necessarily corresponds to $p = 1$, which is the benchmark of our analysis. For any other value of $\tilde{N}$, there exist several possibilities with $p \neq 1$, reproducing exactly the same predictions of chaotic models, while having a viable mechanism to end inflation ($\epsilon_1 = 1$). Nevertheless, in the region $p < 2$ and for $\beta = \mathcal{O}(1)$, solutions in $p$ are highly close together and they follow a linear relation, as shown in Fig. 4.5.

The inflaton range can be easily computed by integrating Eq. (4.2) and one obtains

$$\Delta \phi = \frac{2\sqrt{2 \beta}}{2 - p} \left( \tilde{N}_*^{1-\frac{p}{2}} - \tilde{N}_e^{1-\frac{p}{2}} \right).$$

The latter formula can be written as function of $p$, for any value of $\beta$. By substituting (4.17), one gets

$$\Delta \phi = 2\sqrt{2 \beta} \frac{p}{2 - p} \left[ N_*^{1} - \beta^{\frac{2-p}{2p}} N_*^{1-\frac{p-1}{p}} \right].$$
4.4 Degeneracy of the inflaton range

The inflaton range $\Delta \phi$ as function of $p$ in two different limits for the perturbative class of models. For low values of $p$ (no fine-tuning), $\Delta \phi$ may vary over a range related to the total number of e-folds $\tilde{N}$. In the large $p$ region, the upper bounds on $\Delta \phi$ becomes evident.

Figure 4.6 shows the main results on the inflaton range, given by (4.22), for models belonging to the perturbative class. At this point, we identify the two regions and get the following conclusions:

- For small values of $p$, the inflaton excursion $\Delta \phi$ is a continuously increasing function. It has a typical dependence $p/(2 - p)$ in the region $p \lesssim 1$, where the first term of (4.22) dominates over the second one; it has a mild transition for $1 \lesssim p \lesssim 2$, while it starts to show a really different behavior in the region\(^2\) $p > 2$. The field range covers a wide spectrum of values depending on the total number of e-folds $\tilde{N}$ of this perturbative class. As a consequence, it can be quite different from the corresponding

\(^2\)The value $p = 2$ is special as the two contributions of Eq. (4.22) become the same while the factor $p/(2 - p)$ blows up.
chaotic one, which is given by $p = 1$. In particular, we can reproduce the same values $(n_s, r)$ of a quadratic potential with $N = 60$ and still have a $\Delta \phi$ running from 5 to 32, in Planck units, corresponding to $\bar{N}$ approximately between 30 and 100. Note that $p$, as well as $\Delta \phi$, cannot be arbitrarily small as we need a minimum amount of exponential expansion, quantified by $\bar{N}$.

- For large values of $p$, the inflaton range approaches a constant value, setting an upper bound on $\Delta \phi$ for each specific value of $\beta$. This can be seen explicitly by taking the limit of (4.22) for $p \to \infty$, this becomes:

$$\Delta \phi \to 2\sqrt{2}N_*^{1/2} \left[ N_*^{1/2} - \sqrt{\beta} \right]. \quad (4.23)$$

This corresponds to an upper bound also on $\bar{N}$, as can be seen again by taking the limit for $p \to \infty$ of equation (4.20), which gives:

$$\bar{N} \to N_* \ln \frac{N_*}{\beta}. \quad (4.24)$$

This limit cannot be appreciated in Fig. 4.5, given the reported limited range of $p$. Plugging the values of the parameters for quadratic inflation into (4.23) and (4.24), one gets the approximate bounds

$$\Delta \phi \to 155.56, \quad \bar{N} \to 290. \quad (4.25)$$

Curiously, the hierarchy of ranges is inverted with respect to the one present at small $p$, as it is clear by comparing the two pictures of Fig. 4.6: at higher values of the tensor-to-scalar ratio $r$, we have smaller ranges.

We will see that the bounds for $\Delta \phi$ and $\bar{N}$ found here are recovered in the next two cases we consider in Sec. 4.4.3 and 4.4.4, within the analysis of the logarithmic and non-perturbative classes of models.

### 4.4.3 Logarithmic class

As a second case, we consider models with a first subleading correction to the Hubble flow parameters. While still neglecting higher order $1/N$ terms, one can imagine including a logarithmic dependence on $N$ such as

$$\epsilon_1 = -\frac{\bar{\beta}}{N^p \ln^q (N+1)},$$

$$\epsilon_2 = -\frac{p}{N} - \frac{q}{(N+1) \ln (N+1)} . \quad (4.26)$$

Such large values of $\bar{N}$ are not necessarily realistic (see e.g. the discussion in [80] for an upper estimate); nevertheless, it is interesting to study the behaviour of the field range for such models.
4.4 Degeneracy of the inflaton range

Inflationary models having similar dependence can be found e.g. in [69].

As in the previous case, in order to mimic the observational predictions of chaotic models in terms of \((n_s, r)\), we equate \((\epsilon_1, \epsilon_2)\) of (4.13) to (4.26) at horizon exit, respectively at \(N_*\) and \(\tilde{N}_*\). As result, we obtain

\[
\tilde{\beta} = \beta \frac{\tilde{N}_*^p \ln(q(\tilde{N}_* + 1))}{N_*} \tag{4.27}
\]
\[
q = \left( \frac{1}{N_*} - \frac{p}{N_*} \right) (\tilde{N}_* + 1) \ln(\tilde{N}_* + 1) \tag{4.28}
\]

where \(\tilde{N}_* = \tilde{N}_e + \tilde{N}\) and \(\tilde{N}_e\) is determined by the condition \(\epsilon_1 = 1\):

\[
\frac{\tilde{\beta}}{\tilde{N}_e^p \ln(q(\tilde{N}_e + 1))} = 1. \tag{4.29}
\]

We follow the same approach as in the perturbative case and allow \(\tilde{N}\) to vary as function of \(p\), while fixing \(q\). The range of the inflaton \(\Delta \phi\) can be determined by integrating (4.2) as before. However, we have to rely on numerics as obtaining an analytic expression both for \(\tilde{N}\) and \(\Delta \phi\) turns out to be not as trivial as in the previous case. For this reason, we restrict our analysis just to the benchmark of a quadratic potential, namely just to \(\beta = 1/2\).

The results for the field range and the total number of e-folds are summarized in Fig. 4.7, for two different values of \(q\). As we can see, for large values of \(p\), we recover exactly the same bounds (4.25) found within the analysis of the perturbative class. This is a remarkable result, though it may be understood from the large \(p\) behaviour of \(\epsilon_1\). In this limit, \(\tilde{N}\) also increases and hence subleading terms, in \(\epsilon_n\) for \(n \geq 2\), will be increasingly irrelevant. The two lines in Fig. 4.7, corresponding to different values of \(q\), do differ for smaller values of \(p\). However, they show identical behavior when \(p\) increases, which correspond to a large-\(\tilde{N}\) limit.

### 4.4.4 Non-perturbative class

As a third class, we consider models with Hubble flow functions such as

\[
\epsilon_1 = e^{-2cN}, \quad \epsilon_2 = -2c, \tag{4.30}
\]

where \(c\) is a constant. Note that we are not including any coefficient for \(\epsilon_1\) as this can be set equal to one by a shift in \(N\).

We proceed as in the previous cases by equating \((\epsilon_1, \epsilon_2)\) of (4.13) to the functions (4.30) at horizon exit, in order to reproduce the same observational
predictions of chaotic inflation models. We get the following relations:

\[
\tilde{N}_* = \frac{1}{2c} \ln \frac{N_*}{\beta}, \quad c = \frac{1}{2N_*}.
\] (4.31)

Moreover, imposing that inflation ends at \(\epsilon_1 = 1\) translates into

\[
\tilde{N}_e = \tilde{N}_* - \tilde{N} = 0,
\] (4.32)

which can be manipulated, using (4.31), in order to get the following condition on the total number of e-foldings:

\[
\tilde{N} = N_* \ln \frac{N_*}{\beta},
\] (4.33)

expressed just in terms of parameters of the benchmark models, where \(N_*\) is given by (4.12). Eq. (4.33) fixes uniquely the total amount of exponential expansion required to give the same \((n_s, r)\) of the chaotic scenarios, with parameter \(\beta\), and to end inflation via the condition \(\epsilon_1 = 1\). Note that this coincides exactly with the large-\(p\) limit of the perturbative case, namely Eq. (4.24).

The inflaton range is given by integrating Eq. (4.2) between \(\tilde{N}_e\) and \(\tilde{N}_*\):

\[
\Delta\phi = \frac{\sqrt{2}}{c} \left(1 - e^{-c\tilde{N}_*}\right).
\] (4.34)
The latter can be written just in terms of the benchmark parameters by using (4.31) and it reads

\[ \Delta \phi = 2\sqrt{2} \left( N_* - \sqrt{\beta N_*} \right), \tag{4.35} \]

which yields the field range in terms of \( \beta \). Note that this again coincides exactly with the large-\( p \) limit of the field range in the perturbative case, that is (4.23). Fig. 4.8 shows such functional dependence; the negative slope of the curve makes explicit the inversion of hierarchy of field ranges with respect to the one which naively one would expect. In fact, lower values of \( r \) (lower values of \( \beta \)) will correspond to larger \( \Delta \phi \). This is exactly the same finding for the upper bounds in the perturbative class of models. Such behavior becomes explicit once we express Eq. (4.35) in terms of the typical inflaton range \( \Delta \phi_c \) for the chaotic models, given by Eq. (4.14). The relation turns out to be:

\[ \Delta \phi = 2\sqrt{2} N - \Delta \phi_c, \tag{4.36} \]

where \( N \) is the total number of e-folds for the benchmark chaotic models and, throughout our study, it is fixed to be equal to 60. However, it is not possible to arbitrarily decrease \( \Delta \phi \) even going to really large values of \( \beta \). In fact, by taking the limit for \( 4 \beta \rightarrow \infty \) of (4.35), we obtain

\[ \Delta \phi \rightarrow \sqrt{2} N, \tag{4.37} \]

as can be seen in the second plot of Fig. 4.8, where \( N = 60 \). This corresponds to a lower-bound on \( \tilde{N} \) which, in the same limit, approaches the benchmark number of e-folds \( N \), as it is clear by taking the limit of (4.33). The field range of \( \sqrt{2} N \) can then be understood from an \( \epsilon_1 \) parameter that is approximately equal to one during almost the entire inflationary period.

Within the non-perturbative models, it is then possible to mimic chaotic scenarios in terms of their cosmological observables \( n_s \) and \( r \). Nevertheless, both the total number of e-foldings \( \tilde{N} \) and the field excursion \( \Delta \phi \) are uniquely determined once we choose the power coefficient of the chaotic scenario, namely once we fix \( \beta \). Curiously, the resulting values perfectly correspond to the upper-limits we found in the previous sections. In the specific example of quadratic inflation, that is for \( \beta = 1/2 \), one obtains again \( \Delta \phi \approx 155.56 \) and \( \tilde{N} \approx 290 \), as expected from the discussion in sec. 4.4.2 and 4.4.3.

Note, however, that this limit appears only in the large-\( N \) limit of the non-perturbative class. Specific models of this class are discussed in [58]. An

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\(^4\)Such a limit is anyway not physical as it would correspond to an infinitely large amount of primordial gravitational waves.
The inflaton range $\Delta \phi$ as function of $\beta$ in two different limits for the non-perturbative class of models. For low values of $\beta$ (physical values for the tensor-to-scalar ratio $r$), $\Delta \phi$ is a decreasing function. The four coloured points correspond to the upper-bounds already found in sec. 4.4.2 and 4.4.3. In the large-$\beta$ limit, the inflaton range cannot arbitrarily decrease and approaches a lower-limit.

example is natural inflation [81], which has specific subleading corrections in addition to (4.30). In the limit of a large periodicity, corresponding to small $c$, this model asymptotes to quadratic inflation and therefore has the same field range as this benchmark model. The origin of this difference with (4.35) lies in the subleading corrections, that exactly become increasingly important when $c$ is small (the effective expansion parameter being $1/cN$). In other models, like hybrid inflation [82], which end by the action of a transverse symmetry breaking field, the excursion can be even smaller, and still satisfy the observational constraints. We will discuss a similar phenomenon in the next subsection.
4.4 Degeneracy of the inflaton range

4.4.5 Sub-Planckian field ranges

Now we take a different approach within the logarithmic class of models, in order to illustrate the possibility of obtaining smaller field ranges as compared to the benchmark model of quadratic inflation. The idea is to reproduce the same observational predictions in terms of \((n_s, r)\) by fixing \(\tilde{N}\) (in what follows, we assume \(\tilde{N} = 60\)) and letting \(q\) vary as a function of \(p\) through the relation (4.28).

Once again, the inflationary field range \(\Delta \phi\) can be determined by integrating (4.2) numerically. We find a striking difference between values of \(p\) that are larger or smaller than around 1.1.

We find that setting the end of inflation by \(\epsilon_1 = 1\) turns out to be possible only for \(p\) not exceeding a value around 1.1. For \(p > 1.1\) the function \(\epsilon_1(N)\), given by (4.26), never reaches the unity and, then, a viable inflationary scenario has to be ended through some other mechanism.

On the other hand, one can still set the end of slow-roll inflation via the condition \(\epsilon_2 = 1\). In this case, the field range is a decreasing function of \(p\), as showed in Fig. 4.9, and the values of \(\Delta \phi\) correspond to the distance which the canonical field \(\phi\) travels within the slow-roll approximation. For sufficiently large \(p\), such excursion becomes even sub-Planckian. However, note that these models generically would not correspond to slow-roll inflation throughout the whole period of exponential expansion and they would need to end inflation e.g. via a second field, or some other mechanism.

![Figure 4.9](image)

**Figure 4.9**  
Slow-roll field range \(\Delta \phi\) as function of \(p\). The end of slow-roll inflation is set through the condition \(\epsilon_2 = 1\). Sub-Planckian field ranges can be obtained if inflation ends through a second field or some other mechanism.
In this Section, we have investigated the implications of the CMB data for the inflationary field range. More precisely, we have tried to answer to what extent one can infer $\Delta \phi$ from a measurement of $(n_s, r)$. We have analyzed this question by comparing three different classes of models – perturbative, logarithmic and non-perturbative – to the benchmark models of chaotic inflation, with particular attention to the quadratic scenario.

Surprisingly, we have found that the field range can vary an order of magnitude; while the quadratic model implies $\Delta \phi \approx 14$ in Planck units, the non-perturbative class gives the same observables while $\Delta \phi$ is a factor 11 larger. Moreover, we have identified a continuous degeneracy in the other classes: different one-parameter families of models yield identical $(n_s, r)$ while $\Delta \phi$ spans over a quite large range. Remarkably, $\Delta \phi$ can be increased by exactly the same factor by varying this parameter in both the perturbative and the logarithmic class. Therefore, this constitutes an upper bound for these classes of models.

It might be surprising that there is an upper limit on the field range. After all, we are allowing in principle for an infinite number of e-foldings, hence one would expect it to be possible to hover just below $\epsilon_1 = 1$ for an infinitely long period in terms of $N$; such a scenario is illustrated by the upper line in Fig. 4.10. This period would contribute an infinitely large field range $\Delta \phi$ as well. This raises the question: why do we not find such infinitely large field ranges? We suspect that the answer lies in the Hubble flow equations for the slow-roll parameters. For slow-roll inflation, in the approximation where we are only keeping the lowest two slow-roll parameters, these can be written as

\[
\frac{d\epsilon}{dN} = 2\epsilon(\eta - 2\epsilon), \quad \frac{d\eta}{dN} = \epsilon(\eta - 3\epsilon).
\]  
(4.38)

Note that one cannot have both right-hand sides vanishing at the same time when $\epsilon \neq 0$; therefore it is impossible to keep $\epsilon$ constant over a large range of e-foldings. As a consequence, there is a limit on the number of e-foldings between horizon exit and the end of inflation, for a generic slow-roll model. This is a consequence of the generic lower limit on $d\epsilon/dN$, and translates into a limit on the field range during this period.

Nevertheless, the above discussion constitutes only a generic argument; in fact, specific and non-generic inflationary models could have yet larger field ranges. Examples are in fact provided by models in the perturbative and the logarithmic classes, with parameter $p < 0$. In these models the field range can be arbitrarily large. However, these models are contrary to the
large-$N$ approach that we have taken in this section, where the inflationary period approaches a De Sitter phase as $N$ becomes infinite. For $p$ negative it turns out that one has a cut-off on the number of e-folds preceding the moment of horizon exit. Therefore these do not extend infinitely into the past, approaching a De Sitter phase. In this way it turns out to be possible to evade the generic argument for the upper limit based on (4.38) above.

From the perspective of UV-sensitivity, yet more interesting is the question how small $\Delta \phi$ can be, and in particular whether it can reach sub-Planckian values. This point has been discussed in some detail recently in literature. In order to minimise the field range, one would like to have the area under the curve $\epsilon(N)$ in Fig. 4.10 as small as possible; this case is illustrated by the lower line. Starting at horizon exit, one would therefore need to suppress $\epsilon$ as fast as possible \[83–86\]. In \[87\], however, it was pointed out that this is impossible in the slow-roll approximation, exactly due to Eq. (4.38); as the right hand sides are bilinear in percent-level slow-roll parameters, these can only vary rather slowly as a function of $N$. This upper bound on the change of $\epsilon$ implies a lower bound on the field range. Amusingly, this is the exact opposite reasoning which led to the large field range discussion above.

The issue of getting a smaller $\Delta \phi$ with respect to the benchmark of the quadratic model has been investigated explicitly in the different classes. In the single-field slow-roll approximation, we have found that sub-Planckian field ranges do not seem to be possible, in agreement with the recent bound \[87\]: we could only reduce $\Delta \phi$ by a factor of three, down to $\Delta \phi \approx 5$ in Planck units. However, these classes of models allow for a much stronger reduction
of the inflationary field range, provided one allows for an alternative end of inflation (a related interesting analysis in the context of hybrid natural inflation was done in [88]). In particular, by imposing the condition $\epsilon_2 = 1$, we have found sub-Planckian inflationary trajectories that satisfy all slow-roll single-field requirements. Nevertheless, within these models, the parameter $\epsilon_1$ never reaches the unity and the inflationary expansion needs to be stopped by some other mechanism. Such models could be viable when performing a full fast-roll analysis, or when embedded e.g. in a multi-field model. Note that this type of multi-field is markedly different from those studied in Ref. [89]; in contrast to that reference, our entire inflationary trajectory is purely single-field, and we only appeal to the second field for a waterfall transition to end inflation.

4.5 Universality of the inflaton range

In the previous Section, we have shown concrete examples where a value of $n_s$ and $r$ does not correspond to a specific estimate for the inflationary field range $\Delta \phi$. This is indeed what one would expect generically from a variable depending on the entire inflationary trajectory.

Nevertheless, it is possible to identify different regions where the field range does exhibit a universal behavior. We have proved this remarkable fact in the publication [79] and we will present again the main results below.

In the following, we will restrict our analysis to the perturbative class of models, characterized by an equation of state parameter given by Eq. (4.6). We will not assume this expression to be exact but allow for subleading contributions which generically may play an important role towards the end of inflation.

4.5.1 Universality at large $N$

In order to get the expression for $\Delta \phi$, one must integrate Eq. (4.2) along the entire inflationary trajectory. By considering a large-$N$ behavior such as that in Eq. (4.6), for $p \neq 2$, we obtain

$$\Delta \phi = \frac{2\sqrt{2\beta}}{2 - p} N^{1 - \frac{p}{2}} - \phi_e,$$

(4.39)

where $\phi_e$ is a constant piece related to the value of the inflaton when inflation ends. Then, we run into two possible situations, depending on whether $p$ is smaller or larger than 2.
In the first case, for $p < 2$, the inflaton range $\Delta \phi$ is proportional to a positive power of $N$. In the large-$N$ limit, the constant part $\phi_c$ is subleading and one can argue that, within any universality class, the magnitude of the field excursion will be model-independent and therefore universal. Furthermore, given that $\Delta \phi$ keeps increasing together with $N$, one can correctly refer to such scenarios as genuine large field models.

In the second case, for $p > 2$, the $N$-dependent term of (4.39) is subleading with respect to the constant term $\phi_c$, in the large-$N$ limit. The value of $\Delta \phi$ is therefore determined by the point where inflation stops and generically not universal: for instance, $\Delta \phi$ can already obtain a super-Planckian contribution during the last e-fold [90]. This model-dependent piece is generically sub-dominant for models with $p < 2$ while it represents the main contribution when $p > 2$.

Finally, the remaining possibility is $p = 2$ where the functional form of the field range reads

$$\Delta \phi = \sqrt{2 \beta} \ln N - \phi_c.$$  \hfill (4.40)

The log-dependence leads to a situation where $\Delta \phi$ mildly increases together with $N$. The special role of this point, corresponding to Starobinkylke scenarios, has been recently highlighted in the context of the inflationary attractors [91–95] as well as non-compact symmetry breaking [96]. Moreover, a change of behavior around the point $p = 2$ was noticed also in the analysis on the degeneracy of the inflaton range done in [60] and presented in the previous Section. Here we stress its peculiarity also as marking the separation between a region of authentic large field models ($p < 2$), whose $\Delta \phi$ exhibits universality features, and a region ($p > 2$) where models can have the same $r$ and $n_s$ at leading order (and, thus, belonging to the same universality class) but still very different field ranges.

### 4.5.2 Universality at small $\mu$

The results presented above are obtained in a pure large-$N$ expansion, that is, in the limit $N \to \infty$. However, physical values usually amount to an exponential expansion of around 50 to 60 e-foldings preceding the end of inflation. Although the latter is a big number, the universal regime can be easily affected by tuning specific parameters of the models.

For large enough values of $N$, any model, characterized by an equation of state parameter such as Eq. (4.6), will be represented by a potential, which is parameterized as a small deviation from the benchmarks potentials (4.9). Specifically, for $p > 1$ and $p \neq 2$, the generic form of $V$ will include higher
order corrections and read
\[
V(\phi) = V_0 \left[ 1 - \left( \frac{\phi}{\mu} \right)^n + \sum_{q=n\pm1}^{\pm\infty} c_q \left( \frac{\phi}{\mu} \right)^q \right],
\]
where \( n \) is related to \( p \) through Eq. (4.10) and the plus or minus sign depends respectively on \( p > 2 \) or \( p < 2 \). Then, the coefficients \( c_q \) parameterize the deviation from hilltop or inverse-hilltop models respectively.

Now we show that, at small \( \mu \) and for finite values of \( N \), we recover universality: in addition to the cosmological observables \( n_s \) and \( r \), the inflaton excursion will be model-independent. Interestingly, this is exactly the regime we will consider to derive the field range bound in the next Section.

The spectral index \( n_s \) and tensor-to-scalar ratio \( r \) will be generically insensitive to higher order terms in the expansion (4.41) as they are calculated at horizon exit. In fact, the inflationary regime is restricted to the region \( \phi < \mu \), for hilltop models \((p > 2)\), and \( \phi > \mu \), for inverse hilltop potentials \((1 < p < 2)\); therefore, the farther one is located from the end-point of inflation the more one can ignore higher order corrections in the scalar potential. Then, the large-\( N \) regime provides an accurate estimate of such observables which, at small \( \mu \), read
\[
\begin{align*}
n_s &= 1 - \frac{p}{N}, \\
r &= 2^{p-2} (p-2)^{2p-2} \frac{\mu^{2p-2}}{(p-1)^{p-2} N^p}. \tag{4.42}
\end{align*}
\]
The coefficients \( c_q \) will appear only in subleading terms in \( N \). The family of models represented by Eq. (4.41) will have identical behavior in the small-\( \mu \) limit and for large enough values of \( N \). Conversely, this is generically not the case for large values of \( \mu \); in such a limit, the end-point of inflation is pushed towards the region where the coefficients \( c_q \) play an important role and dissimilarities become important; consequently, going 50-60 e-foldings back, even the point at horizon crossing will start to be sensitive to \( c_q \) corrections. For large values of \( \mu \), the large-\( N \) expansion is not well defined and scenarios belonging to the same universality class at small \( \mu \), may give quite different predictions in terms of \( n_s \) and \( r \).

In the limit of large \( N \) and small \( \mu \), the field range turns out to be
\[
\Delta \phi = \left[ \frac{2 - p}{\sqrt{2(1 - p)}} \right]^{1-\frac{2}{p}} \mu^{2-\frac{2}{p}} - \frac{(\frac{p}{2} - 1)^{p-2} \mu^{p-1} N^{1-p}}{(p-1)^{\frac{p}{2}-1}}, \tag{4.43}
\]
where the first term is clearly related to the end-point of inflation while the second one is the \( N \)-dependent term. For the reasons given above, \( c_q \) corrections will not enter the \( N \)-dependent part which gives the main contribution
to the field range for $1 < p < 2$ while it is subleading for $p > 2$. Things are different when calculating the end-point $\phi_e$; this piece is sensitive to higher-order corrections in $\mu$. As soon as $\mu$ increases, this point is pushed away towards a region where differences among the models begin to appear. If, for simplicity, we focus on the case $n = 3$ (examples belonging to this universality class are hilltop inflation and the models referred to as RIPI and MSSMI in [59]) and consider terms up to fifth order in the expansion (4.41), the end-point reads

$$\phi_e = \sqrt{\frac{2}{3}} \mu^{3/2} + \frac{2\sqrt{2}}{9} c_4 \mu^2 + \frac{5(4c_4^2 + 3c_5)}{27 \cdot 2^{1/4} \sqrt{3}} \mu^{5/2}. \quad (4.44)$$

Crucially, the coefficients $c_q$ appear just with higher powers of $\mu$; this holds even for other values of $n$ (both positive and negative) as well as the special point $p = 2$. This implies that one obtains universal predictions in the small-$\mu$ limit, not just in terms of $n_s$ and $r$, but also in terms of $\Delta \phi$, whose form approaches Eq. (4.43).

## 4.6 The Lyth bound with a tilt

In Sec. 4.3, we have seen that the Lyth bound provides an optimal estimate of the field range, given a measurement of $r$ which is simply related to $\epsilon_1$ through Eq. (4.1). However, starting from the same value of $\epsilon_1$ at horizon crossing, one can imagine different behaviors $\epsilon_1(N)$ that give rise to either smaller [83, 87, 99] or larger areas [60]. This situation is shown in Fig. 4.3.

We would like to show that this estimate becomes stronger when one takes the additional information of the spectral index into account. In particular, given the redshifted value (3.55) and assuming $r$ to be small, the dependence $r \propto 16\epsilon_1(N)$ is tilted upwards at horizon crossing\(^6\). The natural history therefore leads to a larger area than that of the corresponding rectangle. As a consequence, the requirement $\Delta \phi = 1$ implies a lower value of $r$, as illustrated by the blue line in Fig. 4.11. This is our main message: by including constraints on $n_s$ one can strengthen considerably the Lyth bound.\(^5\)

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\(^5\) The Lyth bound can also be evaded using multiple scalar [89] or vector fields [97]. An extension to fast roll can be found in [98].

\(^6\) Note that our approach differs from [84, 86], which also include the spectral tilt in their expressions: while these references derive a minimal value for $\Delta \phi$, we aim to provide a generic estimate by making use of its universal properties.
to the results on the universality of $\Delta \phi$ in the sub-Planckian regime\textsuperscript{7}, we will show that the reported value (3.55) leads to $r \lesssim 2 \cdot 10^{-5}$ for sub-Planckian field ranges. This constitutes a bound which is two orders of magnitude stronger than the usual estimate as given by Eq. (4.11).

\textbf{Figure 4.11}

\textit{Two curves indicating $\sqrt{2 \epsilon_1}$ with identical areas $\Delta \phi = 1$. The flat curve depicts the Lyth bound, while the tilted curve indicates the improvement when taking the spectral index into account.}

\subsection{4.6.1 Strengthening the Lyth bound}

We now use the results derived in the previous Section in order to revisit the discussion on small- and large-field excursions and derive a stronger field range bound than the usual estimate Eq. (4.11).

The findings on the universality of the field range translate into the possibility of inferring an accurate estimate of $\Delta \phi$ given a point in the $(n_s, r)$ plane. This is certainly true in the small-$\mu$ limit where $\Delta \phi$ is given by Eq. (4.43). One can properly argue that sub-Planckian field ranges will be model-independent and uniquely determined by a measurement of the cosmological observables. The situation changes when $\mu$ increases; already for $\mu \gtrsim \mathcal{O}(1)$, in the region $p > 2$ (corresponding to $n_s \lesssim 0.96$), universality breaks down (as can be seen from Eq. (4.44) where each contribution is order one); differently, for $p < 2$, universality can hold even for some orders of magnitude larger than the reduced Planck mass $M_P = 1$, thanks to the dominant $N$-dependent term as set by Eq. (4.39).

\textsuperscript{7}Strictly speaking, this is true for values $\Delta \phi \lesssim 10^{-2}$, which define more accurately small field inflation. In this region $\mu < 1$ and thus sub-leading corrections are suppressed, strengthening the results on universality.
4.6 The Lyth bound with a tilt

Field ranges corresponding to $\Delta \phi = (0.1, 1, 10)$ in the plane $(n_s, \log_{10}(r))$. The green straight dashed lines represent the asymptotic behaviour for large $\rho$. The yellow area corresponds to sub-Planckian values of the field excursion and, then, to the universality region.

Then, if we plot lines of constant $\Delta \phi$ in a $(n_s, r)$ plane, the one corresponding to unity $\Delta \phi = 1$ will be a good estimate of the border above which universality breaks down, regardless the value of $n_s$. This will be taken as the new, stronger bound. As can be seen from Fig. 4.12, the line is tilted as it is a function also of the spectral index $n_s$. Interestingly, for $n_s = 1$ it approaches the value of the original Lyth bound, which is a constant value not depending on the tilt. On the other hand, in the Planck-range, an excellent fit is provided by the following expressions, corresponding to the (green) dashed straight lines in Fig. 4.12,

$$
\log_{10} r = -1.0 + 25.5 (n_s - 1), \quad \Delta \phi = 10,
$$

$$
\log_{10} r = -2.0 + 68.0 (n_s - 1), \quad \Delta \phi = 1.0, \quad (4.45)
$$

$$
\log_{10} r = -2.35 + 123 (n_s - 1), \quad \Delta \phi = 0.1.
$$

The range of values of $(n_s, r)$ consistent within those of Planck reduces the values of $\Delta \phi$ during inflation by at least an order of magnitude. For the central value $n_s \simeq 0.96$, imposing that $\Delta \phi \leq 1$ leads to the bound $r \lesssim 2 \cdot 10^{-5}$, which is two orders of magnitude below the usual Lyth bound.

On the other hand, if we impose that the ratio $r$ be bigger than a certain value, then we find a lower bound on $\Delta \phi$. Fig. 4.13 shows the field range
as a function of the scalar spectral index for different values of the ratio $r$. Again, in the range consistent with Planck, the field range is always super-Planckian, for all values of the ratio $r \gtrsim 2 \cdot 10^{-5}$. This conclusion can only be avoided by going to unrealistically large spectral indices $n_s$ close to 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4.13}
\caption{The range of field values corresponding to $r = 0.2, 0.1, 0.04, 0.01, 0.001, 0.00001$ in the plane $(n_s, \Delta \phi)$.}
\end{figure}

Similarly to the original Lyth bound, the relations (4.45) provide generic estimates of the field range, which could be avoided only by a very specific (non-generic) behavior of $\epsilon_1(N)$. However the existence of such counterexamples is of limited importance: one would like to understand when large field inflation is expected given a measurement of $r$ even if there might be fine-tuned models which give smaller field ranges for this value of $r$.

Given the central value for $n_s$ from Planck, our results imply that super-Planckian field ranges require a tensor-to-scalar ratio that exceeds $2 \cdot 10^{-5}$. Planned future CMB experiments, such as CORe \cite{100,101} and PRISM \cite{102–104}, might bring the sensitivity down to $10^{-4}$. In contrast to what one would conclude from the original Lyth bound, our results imply that a small detectable $r$ still corresponds to super-Planckian field ranges.