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On the Riccati Equations of the $\mathcal{H}_\infty$ Control Problem for Discrete Time-Varying Systems

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Abstract

In this paper we investigate the relationship between the different Riccati equations that appear in the $\mathcal{H}_\infty$ control problem for linear discrete time-varying systems. Once we obtain this relation we can reformulate the conditions under which the $\mathcal{H}_\infty$ output feedback problem is solvable. In contrary to the conditions in terms of two coupled Riccati equations found in [15], the new conditions are stated in terms of two uncoupled Riccati equations and a coupling condition.

1 Introduction

In the past decade a burst of research activity has taken place in the field of $\mathcal{H}_\infty$ control. In the wake of the pioneering paper [4] on the "Riccati state space approach" for linear continuous time-invariant systems, a number of extensions have been published treating linear discrete time-invariant systems, and Linear Time-Varying (LTV) systems.

In [10] a solution to the discrete time-invariant $\mathcal{H}_\infty$ control problem with measurement feedback is given. The conditions are stated in terms of two coupled Riccati equations, i.e., one of these equations is not stated in terms of the original system matrices. In [16] the relation with another Riccati equation in terms of the original system matrices is given. With this relation the conditions to solve the $\mathcal{H}_\infty$ control problem with measurement feedback can be given via two uncoupled Riccati equations an additional coupling condition.

In [15] sufficient conditions in terms of two Riccati equations have been given to solve the infinite horizon $\mathcal{H}_\infty$ output feedback problem for linear discrete time-varying systems. Similar to the time-invariant case, the drawback of these Riccati equations is that they are coupled. Therefore, we develop a relationship with another discrete time-varying Riccati equation, which is uncoupled.

The paper is organized as follows. In Section 2 we give a brief overview of the notation and the representation of a state space model of LTV systems used throughout the paper. The set up is based on [3], and [11]. In Section 3 we give an extension of a result of [8] to LTV systems. A summary of the result of [15] is given in Section 4. The conditions to solve the $\mathcal{H}_\infty$ output feedback problem for LTV systems are stated. Then, we use Section 3 to develop the relationship with another Riccati equation in Section 5, and reformulate the result of Section 4 in terms of two uncoupled Riccati equations with an additional coupling condition. Finally, in Section 6 we give some conclusions.

2 Preliminaries

In this section, we introduce the notation used in representing Linear Time-Varying (LTV) systems.

A state space realization of the LTV system P to be controlled, is denoted on a local time scale as:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k + D_k u_k \end{aligned} \qquad (1)$$

where x_k, u_k and y_k are (finite dimensional) column vectors in respectively $\mathbb{C}^{N_k}, \mathbb{C}^{M_k}$ and $\mathbb{C}^{L_k}$ and the matrices $\{A_k, B_k, C_k, D_k\}$ are bounded matrices of appropriate dimensions.

To denote the state space representation more compactly, we introduce as done [3] and [11], the dimension space sequences $\mathcal{B}$,

$$\mathcal{B} = \cdots \times \boxed{\mathcal{B}_0} \times \mathcal{B}_1 \times \cdots$$

where $\mathcal{B}_k = \mathbb{C}^{N_k}$ and the square box identifies the space of the 0−th entry. In a similar way, we introduce the dimension space sequence $\mathcal{M}$ and $\mathcal{N}$ from the integer sequences $\{M_k\}$ and $\{L_k\}$. It is allowed that some integers in these sequences are zero. The space of sequences in $\mathcal{B}$ with finite 2-norm will be denoted by $\ell_2^{\mathcal{B}}$. Next we stack the sequence of state vectors x_k, input vectors u_k and output vectors y_k into ∞-dimensional column vectors x, u and y; denoted explicitly for the state vector sequence as,

$$x = \begin{bmatrix} \vdots \\ x_{-1} \\ \boxed{x_0} \\ x_1 \\ \vdots \end{bmatrix}$$

where the square identifies the position of the 0-th entry. Let $\mathcal{B}^{(-1)}$ denote the shifted dimension space sequence of $\mathcal{B}$, i.e.

$$\mathcal{B}^{(-1)} = \cdots \times \boxed{\mathcal{B}_1} \times \mathcal{B}_2 \times \cdots$$

and let $\mathcal{D}(\mathcal{M},\mathcal{N})$ denote the Hilbert space of bounded diagonal operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$, then we can stack the system operators A_k, B_k, C_k and D_k into the diagonal operators A, B, C and D, as (denoted only explicitly for A):

$$A = \text{diag}\left[\begin{matrix} \cdots & A_{-1} & \boxed{A_0} & A_1 & \cdots \end{matrix}\right] \in \mathcal{D}(\mathcal{B},\mathcal{B}^{(-1)}),$$

$$C \in \mathcal{D}(\mathcal{B},\mathcal{N}), \;\; B \in \mathcal{D}(\mathcal{M},\mathcal{B}^{(-1)}), \;\; D \in \mathcal{D}(\mathcal{M},\mathcal{N}).$$

Let the causal bilateral shift operator on sequences be denoted by Z, such that,

$$Z \begin{bmatrix} \vdots \\ x_{-1} \\ \boxed{x_0} \\ x_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x_{-2} \\ \boxed{x_{-1}} \\ x_0 \\ \vdots \end{bmatrix}$$

then $Zx \in \mathcal{B}^{(1)}$. Furthermore, the k-th diagonal shift of an operator X is $X^{(k)} = Z^k A Z^{*k}$. Then a compact notation on a global time scale of the state space representation (1) is:

$$\begin{aligned} Z^{-1}x &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad (2)$$

also denoted as

$$\mathbf{P} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

With this notation it is possible to represent a LTV system as an operator. Let the transition operator $\Phi(j,k)$ of the system with state space representation (2) be defined as,

$$\Phi(j,k) = \begin{cases} A_{k-1}\cdots A_{j+1}A_j & j < k \\ I & j = k \\ \text{undefined} & j > k \end{cases}$$

and let $\lim_{k\to\infty} \Phi(j,k) = 0 \;\forall j < \infty$, then the inverse of the operator $(I - ZA)$ exists. Then the operator representation of the (*asymptotically stable*) LTV system P becomes:

$$P = D + C(I - ZA)^{-1}ZB \qquad (3)$$

Let ℓ_A denote the spectral radius of the operator ZA, i.e., $\ell_A = \lim_{n\to\infty} \|(ZA)^n\|^{1/n}$. If $\ell_A < 1$ then the realization is asymptotically stable (which is, in the case of LTV systems, equivalent to being exponentially stable). The transfer operator P is *lower* triangular. In general the Hilbert space of bounded lower operators acting from $\ell_2^{\mathcal{M}}$ to $\ell_2^{\mathcal{N}}$ is denoted by $\mathcal{L}(\mathcal{M},\mathcal{N})$ or denoted in short by $\mathcal{L}$. When the dimension N_k of the state vector is finite for all k then the operator represented as in Eq. (3) is *locally finite*. In the same way as $\mathcal{L}$, we denote the space of bounded operators by $\mathcal{X}(\mathcal{M},\mathcal{N})$ and the space of bounded *upper* triangular operators by $\mathcal{U}(\mathcal{M},\mathcal{N})$.

Finally, operators representing input-output maps are sometimes indexed. In this way, the input-output map T_{wz} relates the input sequence w to the output sequence z.

3 An equivalent representation of the Riccati Equation

We will generalize the results of [8] for the discrete time-invariant Riccati equation to a certain extend to the discrete time-varying Riccati equation. It is not possible to generalize all of the results, since they are based on an eigenvalue decomposition. The representation we get here is useful for getting a relation between different Riccati equations that appear in the solution of the LTV $\mathcal{H}_\infty$ problem, as will be seen in the next section. We are concerned with the discrete time-varying (forward) algebraic Riccati equation of the form (we adopt the notation of [8])

$$\begin{aligned} X = \;& -F^*X^{(-1)}G_1(G_2 + G_1^*X^{(-1)}G_1)^{-1}G_1^*X^{(-1)}F \\ & +F^*X^{(-1)}F + H. \end{aligned} \qquad (4)$$

Here $H,\ X \in \mathcal{D}(\mathcal{B},\mathcal{B})$, $F \in \mathcal{D}(\mathcal{B},\mathcal{B}^{(-1)})$, $G_1 \in \mathcal{D}(\mathcal{M},\mathcal{B}^{(-1)})$, $G_2 \in \mathcal{D}(\mathcal{M},\mathcal{M})$, and $G_2 = G_2^*$, $H = H^*$. We assume that (F, G_1) is a stabilizable pair, and that (C, F) is a detectable pair, where $C^*C = H$. Finally, we define $G := G_1G_2^{-1}G_1^*$.

From the discrete maximum principle (see e.g. Whittle [17]) we obtain (similar to the time-invariant case) the Hamiltonian difference equations

$$\begin{pmatrix} I & G_k \\ 0 & F_k^* \end{pmatrix} \begin{pmatrix} x_{k+1} \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} F_k & 0 \\ -H_k & I \end{pmatrix} \begin{pmatrix} x_k \\ z_k \end{pmatrix} \qquad (5)$$

for every time-step, where x_k denotes the state at time t_k and z_k denotes the corresponding adjoint vector. We can rewrite this as

$$\begin{pmatrix} I & G \\ 0 & F^* \end{pmatrix} \begin{pmatrix} Z^{-1}x \\ Z^{-1}z \end{pmatrix} = \begin{pmatrix} F & 0 \\ -H & I \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}. \qquad (6)$$

Now we can state the following theorem

Theorem 1 *Assume that there exists a stabilizing solution X of the algebraic Riccati equation (4), i.e., such that*

$$F^{\times} = F - G_1(G_2 + G_1^* X^{(-1)} G_1)^{-1} G_1^* X^{(-1)} F \quad (7)$$

fulfills $\ell_{F^\times} < 1$. Then X can be written as $X = PQ^{-1}$ for any Q non-singular, and Q and P that fulfill

$$\begin{pmatrix} I & G \\ 0 & F^* \end{pmatrix} \begin{pmatrix} Q^{(-1)} \\ P^{(-1)} \end{pmatrix} S = \begin{pmatrix} F & 0 \\ -H & I \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} \quad (8)$$

where $S = Q^{-(-1)} F^{\times} Q$.

PROOF We can rewrite (4) as

$$X - H = F^* X^{(-1)} F^{\times} \quad (9)$$

Take an arbitrary non-singular $Q \in \mathcal{D}(\mathcal{B},\mathcal{B})$, define $S := Q^{-(-1)} F^{\times} Q$, and set $P := XQ$ (thus $X = PQ^{-1}$). Substitute this in (9), then

$$P - HQ = F^* P^{(-1)} Q^{-(-1)} F^{\times} Q = F^* P^{(-1)} S. \quad (10)$$

Furthermore, (7) yields

$$G_1^* X^{(-1)} Q^{(-1)} S Q^{-1} = G_2 (G_2 + G_1^* X^{(-1)} G_1)^{-1} G_1^* X^{(-1)} F$$

Multiplication from the right by $G_1 G_2^{-1}$, using (7), and using $X = PQ^{-1}$ yields (see also [8])

$$G P^{(-1)} S = FQ - Q^{(-1)} S \quad (11)$$

Now we obtain (8) from (10) and (11). □

The reverse implication of the previous theorem also holds, but is not stated here. For the backward algebraic Riccati equation we can easily obtain a similar result. We state this result without proof.

Theorem 2 *Assume there exists a solution to the algebraic Riccati equation*

$$\begin{aligned} X^{(-1)} = &\ -FXG_1^*(G_2 + G_1 X G_1^*)^{-1} G_1 X F^* \\ &+ FXF^* + H \end{aligned} \quad (12)$$

where $H, X \in \mathcal{D}(\mathcal{B},\mathcal{B})$, $F \in \mathcal{D}(\mathcal{B},\mathcal{B}^{(-1)})$, $G_1 \in \mathcal{D}(\mathcal{B},\mathcal{N})$, $G_2 \in \mathcal{D}(\mathcal{N},\mathcal{N})$ and $G_2 = G_2^*$, $H = H^*$, and $G := G_1^* G_2^{-1} G_1$, such that

$$F^{\times} := F - FXG_1^*(G_2 + G_1 X G_1^*)^{-1} G_1$$

is asymptotically stable (i.e., $\ell_{F^\times} < 1$). Then such solution X to equation (12) can be written as $X = PQ^{-1}$ for any Q non-singular, and Q and P that fulfill

$$\begin{pmatrix} I & G \\ 0 & F \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} S = \begin{pmatrix} F^* & 0 \\ -H & I \end{pmatrix} \begin{pmatrix} Q^{(-1)} \\ P^{(-1)} \end{pmatrix} \quad (13)$$

where $S = Q^{-1} (F^{\times})^* Q^{(-1)}$.

4 $\mathcal{H}_\infty$ Output Feedback

In this section we summarize the results of [15].

Let the time-varying system T be given with state space realization:

$$\begin{aligned} Z^{-1}x &= Ax + B_1 w + B_2 u \\ z &= C_1 x + \quad\quad + D_{21} u \\ y &= C_2 x + D_{12} w + D_{22} u \end{aligned} \quad (14)$$

where $A \in \mathcal{D}(\mathcal{B},\mathcal{B}^{(-1)})$, $B_1 \in \mathcal{D}(\mathcal{M}_1,\mathcal{B}^{(-1)})$, $B_2 \in \mathcal{D}(\mathcal{M}_2,\mathcal{B}^{(-1)})$, $C_1 \in \mathcal{D}(\mathcal{B},\mathcal{N}_1)$, $C_2 \in \mathcal{D}(\mathcal{B},\mathcal{N}_2)$, $D_{12} \in \mathcal{D}(\mathcal{M}_1,\mathcal{N}_2)$, $D_{21} \in \mathcal{D}(\mathcal{M}_2,\mathcal{N}_1)$, $D_{22} \in \mathcal{D}(\mathcal{M}_2,\mathcal{N}_2)$, and x is the state sequence, w the exogenous input sequence (disturbances), u the control input sequence, y the measured output sequence, and z the to-be-controlled output sequence. Note that we do not assume that the A-operator of (14) has $\ell_A < 1$, or in other words, that we allow the system to be unstable (i.e., the state sequence x may be unbounded). In that case the operator $(I - AZ)^{-1}$ is not bounded, which implies that T does not exists, i.e., $T \notin \mathcal{U}(\mathcal{M}_1 + \mathcal{M}_2, \mathcal{N}_1 + \mathcal{B})$.

Consider the time-varying controller K with state space realization:

$$\begin{aligned} Z^{-1}\xi &= \Phi\xi + \Psi_1 y \\ u &= \Psi_2 \xi + \Psi_3 y \end{aligned} \quad (15)$$

where Φ, Ψ_1, Ψ_2 and Ψ_3 are bounded diagonal operators and where the state dimensions still has to be determined. Both systems are connected as displayed in Figure 1.

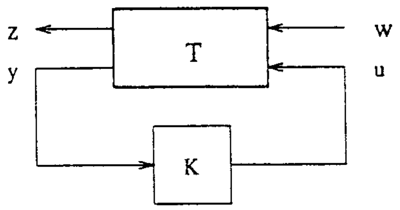

Figure 1: Block-schematic representation of the $\mathcal{H}_\infty$ output feedback problem.

Now, let $\Gamma_c \in \mathcal{D}(\mathcal{M}_1,\mathcal{M}_1)$ be a prescribed level of disturbance attenuation, such that $\Gamma_c \gg 0$, and assume that there exists a solution $M_c \in \mathcal{D}(\mathcal{B},\mathcal{B})$ of,

$$M_c = A^* M_c^{(-1)} A + E_c W_c^{-1} E_c^* + C_1^* C_1 \quad (16)$$

with $E_c = C_1^* D_c + A^* M_c^{(-1)} B_c$, $D_c = \begin{bmatrix} 0 & D_{21} \end{bmatrix}$, $B_c = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, and

$$W_c = \begin{bmatrix} \Gamma_c I & 0 \\ 0 & 0 \end{bmatrix} - D_c^* D_c - B_c^* M_c^{(-1)} B_c$$

such that

$$\Gamma_c I - B_1^* M_c^{(-1)} B_1 \gg 0$$

$M_c \geq 0$ and the operator $A_c^\times$, defined as:

$$A_c^\times = A + B_c W_c^{-1} E_c^*$$

is asymptotically stable. Now define

$$\begin{array}{lcl} U_1 & := & \Gamma_c I - B_1^* M_c^{(-1)} B_1 \\ \tilde{B}_1 & := & A^* M_c^{(-1)} B_1 \\ \tilde{B}_2 & := & C_1^* D_{21} + A^* M_c^{(-1)} B_2 \\ \tilde{B}_3 & := & +B_2^* M_c^{(-1)} B_1 \\ U_2 & := & B_2^* M_c^{(-1)} B_2 + \tilde{B}_3 U_1^{-1} \tilde{B}_3^* + D_{21}^* D_{21} \\ U_3 & := & \tilde{B}_2 + \tilde{B}_1 U_1^{-1} \tilde{B}_3^* \end{array} \tag{17}$$

Then under the following assumptions:

Assumptions 3

1. *The pair (A , B_2) is uniformly stabilizable, the operator $D_{21}^* D_{21} \gg 0$ and $\Gamma_c = \gamma^2 I_{\mathcal{M}_1}$ is chosen such that a solution to (16) as above exists.*
2. *The pair $(\overline{C}_2 \ , \ \overline{A}) := (C_2 + D_{12} U_1^{-1} \tilde{B}_1^* \ , \ A + B_1 U_1^{-1} \tilde{B}_1^*)$, with the quantities $\tilde{B}_1$ and U_1 defined in (17), is uniformly detectable, and the operator $D_{12} D_{12}^* \gg 0$,*

we can state the $\mathcal{H}_\infty$ output feedback problem as follows (Figure 1): For a given level of disturbance attenuation $\Gamma_c = \gamma^2 I_{\mathcal{M}_1}$ with $\gamma > 0$, find a state space realization $\{\Phi, \Psi_1, \Psi_2, \Psi_3\}$ of the controller K in Eq. (15), such that:

1. The A-operator of the closed-loop system in Figure 1, which has the following form:

$$A_{\text{cl}} = \begin{bmatrix} A + B_2(I - \Psi_3 D_{22})^{-1} \Psi_3 C_2 & B_2 (I - \Psi_3 D_{22})^{-1} \Psi_2 \\ \Psi_1 C_2 + \Psi_1 D_{22} (I - \Psi_3 D_{22})^{-1} \Psi_3 C_2 & \Phi + \Psi_1 D_{22} (I - \Psi_3 D_{22})^{-1} \Psi_2 \end{bmatrix}$$

   is asymptotically stable. When this is the case, the closed-loop system depicted in Figure 1 is internally stable.

2. The operator T_{wz} between w and z in Figure 1 satisfies $\Gamma_c I - T_{wz}^* T_{wz} \gg 0$.

Now define

$$\begin{array}{lcl} \overline{A} & := & A + B_1 U_1^{-1} \tilde{B}_1^* \\ \overline{B}_1 & := & B_1 U_1^{-\frac{1}{2}} \Gamma_c^{\frac{1}{2}} \\ \overline{B}_2 & := & B_2 + B_1 U_1^{-1} \tilde{B}_3^* \\ \overline{C}_1 & := & U_2^{-\frac{1}{2}} U_3^* \\ \overline{C}_2 & := & C_2 + D_{12} U_1^{-1} \\ \overline{D}_{21} & := & U_2^{\frac{1}{2}} \\ \overline{D}_{12} & := & D_{12} U_1^{\frac{1}{2}} \Gamma_c^{\frac{1}{2}} \\ \overline{D}_{22} & := & D_{22} + D_{12} U_1^{-1} \tilde{B}_3^* \end{array} \tag{18}$$

Let $\overline{\Gamma}_o \in \mathcal{D}(\mathcal{M}_2, \mathcal{M}_2)$, be a prescribed disturbance attenuation level such that $\overline{\Gamma}_o \gg 0$. Now assume there exists a solution $\overline{M}_o \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ of

$$\overline{M}_o^{(-1)} = \overline{A}\, \overline{M}_o \overline{A}^* + \overline{E}_o^* \overline{W}_o^{-1} \overline{E}_o + \overline{B}_1 \overline{B}_1^* \tag{19}$$

with $\overline{E}_o = \overline{D}_o \overline{B}_1^* + \overline{C}_o \overline{M}_o \overline{A}^*$, $\overline{D}_o = \begin{bmatrix} 0 \\ \overline{D}_{12} \end{bmatrix}$, $\overline{C}_o = \begin{bmatrix} \overline{C}_1 \\ \overline{C}_2 \end{bmatrix}$, and

$$\overline{W}_o = \begin{bmatrix} \overline{\Gamma}_o I & 0 \\ 0 & 0 \end{bmatrix} - \overline{D}_o \overline{D}_o^* - \overline{C}_o \overline{M}_o \overline{C}_o^*$$

such that

$$\overline{\Gamma}_o I - \overline{C}_1 \overline{M}_o \overline{C}_1^* \gg 0,$$

$\overline{M}_o \geq 0$ and the operator $\overline{A}_o^\times$, defined as:

$$\overline{A}_o^\times = \overline{A} + \overline{E}_o^* \overline{W}_o^{-1} \overline{C}_o$$

is asymptotically stable. Now we state a theorem by [15].

Theorem 4 *Let T be a locally finite operator with state space realization in Eqs. (14) and satisfying the Assumptions 3. Furthermore, let $\Gamma_c = \gamma^2 I_{\mathcal{M}_1}$ be a prescribed disturbance attenuation level with $\gamma > 0$. For this Γ_c, let M_c be a solution to the Riccati equation (16) satisfying the corresponding conditions. Let this M_c define the state space representation of the LTV system $\overline{T}$ with system matrices as in Eq. (18). Let $\overline{\Gamma}_o = \gamma^2 I_{\mathcal{M}_2}$ and let $\overline{M}_o$ be a solution to the Riccati equation (19) satisfying the corresponding conditions, then there exists an controller that solves the $\mathcal{H}_\infty$ output feedback problem.*

In [15] an explicit expression for the controller that solves the output feedback problem is given.

5 The relation with other conditions

It is well known that for continuous time-invariant systems the Riccati equations that occur in the solution of the $\mathcal{H}_\infty$ output feedback problem are given as two uncoupled equations, together with a coupling condition. For time-invariant discrete time systems this has been investigated in [16], and a relationship between three Riccati equations has been found. Clearly, in the discrete time-varying case the Riccati equations of Theorem 4 are coupled. Therefore, in this section we generalize the result of [16] to LTV systems.

Consider the stabilizing solution M_c to the algebraic Riccati equation (16), the stabilizing solution $\overline{M}_o$ to the algebraic Riccati equation (19), and additionally, assume that for a prescribed level of disturbance attenuation $\Gamma_o \in$

$\mathcal{D}(\mathcal{N}_1, \mathcal{N}_1)$, $\Gamma_o \gg 0$, there exists a solution $M_o \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ of

$$M_o^{(-1)} = AM_oA^* + E_o^*W_o^{-1}E_o + B_1B_1^* \quad (20)$$

with $E_o = D_oB_1^* + C_oM_oA^*$, $D_o = \begin{bmatrix} 0 \\ D_{12} \end{bmatrix}$, $C_o = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, and

$$W_o = \begin{bmatrix} \Gamma_o I & 0 \\ 0 & 0 \end{bmatrix} - D_oD_o^* - C_oM_oC_o^*$$

such that

$$\Gamma_o I - C_1M_oC_1^* \gg 0,$$

$M_o \geq 0$ and the operator $A_o^\times$, defined as:

$$A_o^\times = A + E_o^*W_o^{-1}C_o$$

is asymptotically stable. Then we can give the following relation

Theorem 5 *If the prescribed disturbance levels are $\Gamma_c = \gamma^2 I_{\mathcal{M}_1}$, $\Gamma_o = \gamma^2 I_{\mathcal{N}_1}$, and $\overline{\Gamma}_o = \gamma^2 I_{\mathcal{M}_2}$, then*

$$\overline{M}_o = M_o(I - \gamma^{-2}M_cM_o)^{-1} \quad (21)$$

PROOF First we define

$$\begin{array}{rcl} R_o & := & D_oD_o^* - \begin{bmatrix} \Gamma_o I & 0 \\ 0 & 0 \end{bmatrix} \\ F_o & := & A - B_1D_o^*R_o^{-1}C_o \\ G_o & := & C_o^*R_o^{-1}C_o \\ H_o & := & B_1B_1^* - B_1D_o^*R_o^{-1}D_oB_1^* \\ \overline{R}_o & := & \overline{D}_o\overline{D}_o^* - \begin{bmatrix} \overline{\Gamma}_o I & 0 \\ 0 & 0 \end{bmatrix} \\ \overline{F}_o & := & \overline{A} - \overline{B}_1\overline{D}_o^*\overline{R}_o^{-1}\overline{C}_o \\ \overline{G}_o & := & \overline{C}_o^*\overline{R}_o^{-1}\overline{C}_o \\ \overline{H}_o & := & \overline{B}_1\overline{B}_1^* - \overline{B}_1\overline{D}_o^*\overline{R}_o^{-1}\overline{D}_o\overline{B}_1^* \end{array} \quad (22)$$

By Theorem 2 we know that M_o, and $\overline{M}_o$ can be written as $M_o = PQ^{-1}$ and $\overline{M}_o = \overline{P}\,\overline{Q}^{-1}$, respectively, where Q, and $\overline{Q}$ are non-singular, and Q, P and $\overline{Q}$, $\overline{P}$, respectively fulfill

$$\begin{array}{l} L\begin{pmatrix} Q \\ P \end{pmatrix} S = N\begin{pmatrix} Q^{(-1)} \\ P^{(-1)} \end{pmatrix}, \\ L := \begin{pmatrix} I & G_o \\ 0 & F_o \end{pmatrix} \text{ and } N :=; \begin{pmatrix} F_o^* & 0 \\ -H_o & I \end{pmatrix} \end{array} \quad (23)$$

where $S = Q^{-1}(F_o^\times)^*Q^{(-1)}$, and

$$\begin{array}{l} \overline{L}\begin{pmatrix} \overline{Q} \\ \overline{P} \end{pmatrix} \overline{S} = \overline{N}\begin{pmatrix} \overline{Q}^{(-1)} \\ \overline{P}^{(-1)} \end{pmatrix}, \\ \overline{L} := \begin{pmatrix} I & \overline{G}_o \\ 0 & \overline{F}_o \end{pmatrix} \text{ and } \overline{N} := \begin{pmatrix} \overline{F}_o^* & 0 \\ -\overline{H}_o & I \end{pmatrix} \end{array} \quad (24)$$

where $\overline{S} = \overline{Q}^{-1}(\overline{F}_o^\times)^*\overline{Q}^{(-1)}$. Now we follow the proof of Theorem 3 of [16]. First assume that there exists a similarity transformation $W \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ such that

$$\begin{array}{l} \overline{L}\begin{pmatrix} I & -\gamma^{-2}M_c \\ 0 & I \end{pmatrix} = WL \text{ and} \\ \overline{N}\begin{pmatrix} I & -\gamma^{-2}M_c^{(-1)} \\ 0 & I \end{pmatrix} = WN \end{array} \quad (25)$$

Then

$$\begin{array}{l} L\begin{pmatrix} Q \\ P \end{pmatrix} S = N\begin{pmatrix} Q^{(-1)} \\ P^{(-1)} \end{pmatrix} \Rightarrow \\ WL\begin{pmatrix} Q \\ P \end{pmatrix} S = WN\begin{pmatrix} Q^{(-1)} \\ P^{(-1)} \end{pmatrix} \Rightarrow \\ \overline{L}\begin{pmatrix} I & -\gamma^{-2}M_c \\ 0 & I \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix} S = \\ \overline{N}\begin{pmatrix} I & -\gamma^{-2}M_c^{(-1)} \\ 0 & I \end{pmatrix}\begin{pmatrix} Q^{(-1)} \\ P^{(-1)} \end{pmatrix} \end{array}$$

Together with (24) this implies that there exists a $R \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ such that

$$\begin{pmatrix} I & -\gamma^{-2}M_c \\ 0 & I \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \overline{Q} \\ \overline{P} \end{pmatrix} R^{-1}$$

Then

$$\begin{array}{l} R\overline{Q}^{-1} = (Q - \gamma^{-2}M_cP)^{-1} \text{ and} \\ \overline{P}R^{-1} = P \Rightarrow \overline{M}_o = \overline{P}\,\overline{Q}^{-1} = P(Q - \gamma^{-2}M_cP)^{-1} \\ \qquad = M_o(I - \gamma^{-2}M_cM_o)^{-1} \end{array}$$

Hence, the only part that is left to prove is the existence of W. Write W as

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \quad (26)$$

then it follows from (25) that

$$\begin{pmatrix} I & \overline{G}_o - \gamma^{-2}M_c \\ 0 & \overline{F}_o \end{pmatrix} = \begin{pmatrix} W_{11} & W_{11}G_o + W_{12}F_o \\ W_{21} & W_{21}G_o + W_{22}F_o \end{pmatrix}$$

and

$$\begin{pmatrix} \overline{F}_o^* & -\gamma^{-2}\overline{F}_o^*M_c^{(-1)} \\ -\overline{H}_o & \gamma^{-2}\overline{H}_oM_c^{(-1)} + I \end{pmatrix} = \begin{pmatrix} W_{11}F_o^* - W_{12}H_o & W_{12} \\ W_{21}F_o^* - W_{22}H_o & W_{22} \end{pmatrix}$$

Therefore, W exists and is of the form (26) with $W_{11} = I$, $W_{21} = 0$, $W_{12} = -\gamma^{-2}\overline{F}_o^*M_c^{(-1)}$, and $W_{22} = \gamma^{-2}\overline{H}_oM_c^{(-1)} + I$ if

$$\overline{F}_o = (I + \gamma^{-2}\overline{H}_oM_c^{(-1)})F_o \quad (27)$$

$$\overline{F}_o^* = F_o^* + \gamma^{-2}\overline{F}_o^*M_c^{(-1)})H_o \quad (28)$$

$$\overline{H}_o = (I + \gamma^{-2}\overline{H}_oM_c^{(-1)})H_o \quad (29)$$

$$\overline{G}_o = G_o - \gamma^{-2}\overline{F}_o^*M_c^{(-1)}F_o + \gamma^{-2}M_c \quad (30)$$

We can prove (27), (28), (29), and (30) by straightforward, but huge and technical calculations. We refer to [16], and [13] for the details. □

The relation of this theorem means that we can reformulate Theorem 4 as follows

Corollary 6 *Let T be a locally finite operator with state space realization in Eqs. (14). Let (A, B_2) be uniformly stabilizable, (C_2, A) uniformly detectable, $D_{21}^* D_{21} \gg 0$, $D_{12} D_{12}^* \gg 0$, and $\Gamma_c = \gamma^2 I_{\mathcal{M}_1}$ be a prescribed disturbance attenuation level with $\gamma > 0$. For this Γ_c, let M_c be a solution to the Riccati equation (16) satisfying the corresponding conditions. Let $\Gamma_o = \gamma^2 I_{\mathcal{N}_1}$ and let M_o be a solution to the Riccati equation (20) satisfying the corresponding conditions. Assume that the coupling condition $\gamma^2 I - M_c M_o \gg 0$ is fulfilled. Then there exists an controller that solves the $\mathcal{H}_\infty$ output feedback problem.*

6 Conclusion

We developed a relationship between three Riccati equations that appear in the $\mathcal{H}_\infty$ control problem for linear discrete time-varying systems. Now the conditions for solvability of the $\mathcal{H}_\infty$ control problem are stated in terms of two uncoupled Riccati equations and a coupling condition.

On the solvability of the Riccati equations, more research has to be done. In this view, the generalization of the time-invariant concept of zeros on the unit circle has to be investigated. Probably the possibility of dividing the system in a causal and anti-causal part plays an important role in this case.

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