Statistics and dynamics of the perturbed universe
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Document Version
Publisher's PDF, also known as Version of record

Publication date:
1995

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

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Chapter 4

Previrialization

1 Introduction

The most generally held assumption about the origins of structure in the Universe is that gravity has been the most important agent in its assembly. Whereas one can argue about the nature of the seeds that determined where structure would arise, or about the relative importance of the various hydrodynamical effects that determined its final appearance, gravity is always present; forcing the accretion of matter onto the seeds and causing densities to rise to values high enough to allow interactions of a different nature to make their appearance. However, understanding of this gravitational aspect of structure formation can still not be considered complete. This is mainly due to our lack of understanding of the so-called non-linear regime. Whilst we believe that we understand the very early stages of gravitational collapse where departures from homogeneity are small and the dynamics is well described by its linearized equations of motion, this is not so for the later phases. There, violent collapse processes do not lend themselves to quasi-equilibrium descriptions similar to those applicable in the earlier linear expansion phase. Indeed, the problem of finding a concise and consistent description of these processes remains the greatest obstacle to a complete understanding of structure formation. And even though we can now numerically simulate gravitational collapse processes for many millions of particles, our fundamental understanding of these processes has not far surpassed the simplest model devised to describe gravitational collapse, the top-hat model (e.g. Gunn & Gott, 1972).

This model describes the evolution of a uniform, spherically symmetric overdensity in an otherwise unperturbed universe. (Here and in the rest of this chapter I will talk about clusters and proto-clusters, even though the discussion is relevant to structures on all scales where gravity is the dominant force.) According to the top-hat model the proto-cluster will initially expand with the background universe, but its excess mass causes this expansion to slow down until at some moment it stops expanding altogether. Then the cluster starts to contract. Due to relaxation processes that are not further specified, this contraction stops at roughly half the maximum radius, leaving the cluster in virial equilibrium. The main quantitative predictions of this model are its maximum size, the time in which this so-called turn-around point is reached, the factor by which the cluster contracts, and the time scale for this contraction. These quantities are related to the initial density perturbation
in a straightforward manner. While its qualitative features form the archetypal model for gravitational collapse, its quantitative predictions have also been used extensively. Since its various characteristics are so simply related to its initial conditions, observations of present day structures may be used to infer typical values for the amplitude of the initial density perturbation field (e.g. Lake, 1993; Peebles, 1993). Also in the most popular analytical theory for calculating the cluster mass spectrum, that of Press & Schechter (1974; see also Bond et al., 1991), the quantitative predictions of the top-hat model are widely used. The fact that predictions of the Press-Schechter theory are quite well reproduced by cosmological N-body calculations (e.g. Efstathiou et al., 1988) supports the belief that the top-hat model is in fact a good description of the way clusters collapse even when these are far from isolated, spherically symmetric or uniform.

This belief has been called into question mainly by Peebles, who has instead proposed the so-called previrialization hypothesis. The literal meaning of the term previrialization only hints at an earlier and possibly different approach to equilibrium than predicted by the top-hat model. In the literature one finds various slightly different versions of the hypothesis, but generally, they can all be derived from the following statement, which is the form in which I will investigate the proposed effect:

(PVH) the top-hat model does not give an adequate description of gravitational collapse for clusters which contain substantial sub-structure and/or are imbedded in still larger structures.

In its most literal interpretation, one may immediately agree with this hypothesis. Structures in the Universe are not, and never were, organized in isolated, spherically symmetric and homogeneous clumps. In fact, this will turn out to be an important aspect of this work: how should one translate the above hypothesis into terms that are meaningful for the kinds of structure encountered in a universe with a realistic spectrum of density perturbations? What aspects of the model may be relevant and how should one interpret possible discrepancies? Due to problems such as these and the accompanying possibilities they offer for diverse interpretations of the hypothesis, various approaches have been made in the literature to investigate the significance of the proposed effect.

In its original form, the hypothesis was first stated by Davis & Peebles (1977):

"..the proto-cluster can be “previrialized” due to the development of non-radial motions while it is still expanding as a whole."

They proposed this hypothesis to explain certain aspects of their solutions of the BBGKY-equations. These describe the evolution of the N-point correlation functions in an infinite hierarchy. Davis & Peebles truncated this hierarchy using various assumptions, such as a scale-free spectrum of density fluctuations and a hierarchical form for the three-point correlation function. Their solutions indicate that the average separation between galaxies would initially grow, decelerate and reach a maximum value, but there was no decrease in separation afterwards. They linked this statistical result to the prediction that clusters will not recollapse after reaching maximum expansion; i.e. that the standard picture provided by the top-hat model is not applicable in these cases. Instead the cluster will be virialized.
4.1 Introduction

already due to the effects of internal and external inhomogeneities. These will have induced strong non-radial components in the velocity field of the cluster prior to turn-around, leaving the cluster near virial equilibrium at maximum expansion. This prediction was tested by Villumsen & Davis (1986) who indeed found a significant tangential component to the velocity fields in the outer parts of the clusters in their N-body simulations.

Another aspect of the PV-hypothesis challenges the relation between the amplitude of the initial density perturbation and the resulting densities, either at turn-around or after relaxation. Peebles (1990) derives the initial density needed to obtain a more or less uniform proto-cluster that is just freezing out of the universal expansion. He finds significant differences between the predictions for a cluster evolving in isolation and one that is surrounded by similar mass concentrations. In the latter case a higher initial density is required to obtain the same density at the transition epoch, and significant deviations from a purely radial velocity fields are induced.

In still another approach, Bernardeau (1994) calculates the evolution of a cluster arising from a peak in an initially Gaussian density fluctuation field. By following the evolution of the volume of the cluster in perturbation theory in the high peak limit, he finds a very good correspondence with the predictions of the top-hat model until turn-around. Although his method does not allow him to follow the relaxation phase, for lower peaks he finds tentative evidence for a dependence on the degree of small scale structure in accord with the previralization hypothesis.

Various other investigations, not always directly concerned with probing the hypothesis, have found no significant effects. Results relevant to the original work on previralization are easily obtained from cosmological N-body calculations. The global statistics that were analytically calculated by Davis & Peebles are easily obtained from these simulations and for various types of initial power spectra the predictions were not confirmed (Efstathiou et al., 1988; Davis et al., 1985). Particles do approach each other again after maximum separation. Davis et al. (1985) attribute the discrepancy to probably faulty assumptions made in the BBGKY approach.

Another negative result for the previrialization hypothesis was obtained by Evrard & Crone (1992), who investigated their interpretation of previralization, namely that

"... small-scale power leads to subclustering which significantly retards the collapse of structure on larger scales."

They performed a number of cosmological N-body simulations which only differed in a variable short wavelength cut-off in the initial power spectrum. At equal output times this did not change the abundance of high mass clusters. They explained this by the claim that the evolution of a cluster is governed directly by the power on the scale of that cluster, a claim that receives further support in the work by Little et al. (1991) and Frenk et al. (1988). They did still allow for the possibility of retarding effects on the evolution caused by structures on larger scales, but their approach was not suitable to examine this effect. In their opinion this explained the apparent discrepancy between their result and that of Peebles (1990).

Finally, Thomas & Couchman (1992) investigated the factor by which their simulated cluster(s) contracted after turn-around. Their simulations took into account hydrodynamical effects for the gas component mixed in with a dark matter component. Although the
contraction was slower than in the top-hat model which best fitted the evolution up to turn-around, the recollapse factor was about a factor of two, just as expected from the top-hat model. The mass of the dissipative material in their simulations is only 10% of the total mass, and the dissipative effects will therefore probably not have greatly influenced the large scale evolution of the clusters. The more so because this run did not include radiative cooling and, if anything, the gas was more uniformly distributed than the dark matter due to pressure effects.

In this chapter I will examine various aspects of the previrialization hypothesis. All approaches are centered on the analysis of two cosmological N-body simulations. First, I will follow the evolution of the global statistics which Davis & Peebles calculated analytically for these simulations. With these statistics I will define a global quantity that can be directly compared to the predictions of the top-hat model. My second approach is based on an investigation of individual clusters which were extracted from the simulations. I will discuss how these very non-uniform structures may nevertheless be compared to the top-hat model. Third, I will present results from two types of isolated collapse simulations for several of these clusters. The first uses a TREE code to evolve the clusters from the same initial configurations as in the original cosmological simulation, but now isolated from their surroundings. The second not only isolates the clusters, but also removes any non-radial inhomogeneities by averaging the clusters on spherical shells. Comparison of the full cosmological simulation with the first type should indicate the effects of external inhomogeneities while comparison with the second type should show the effects of internal structure on the collapse dynamics. This second simulation should approximate the relevant top-hat model most closely and should give the clearest indication about the significance of the previrialization hypothesis in its crudest form.

The outline of this chapter is as follows. In the next section I review the top-hat model and several of its generalizations in more detail. Then I will introduce the cosmological simulations, describe the method of cluster extraction and test the simulations for self-similarity using the various global statistics that I will later use in the analysis. The three following sections contain the results of the three approaches mentioned above, and I conclude with a summary and discussion of these results.

2 The top-hat model and its generalizations

The top-hat model describes the evolution of an exactly spherical, uniform overdensity in an otherwise unperturbed universe. As is well known, we may treat such a perturbation as a Friedmann universe on its own and the corresponding solutions apply. The equation of motion for the radius of the cluster is

$$\frac{d^2 R}{dt^2} = - \frac{GM(R)}{R^2}.$$  \hspace{1cm} (4.1)

The mass of the cluster is given by

$$M(R) = \frac{4\pi \bar{\rho}}{3} (1 + \frac{\delta M}{M(R)}) R^3,$$  \hspace{1cm} (4.2)
where $\delta M/M$ is constant for the top-hat model. Initially the cluster is assumed to be expanding radially with the background, $V = HR$, where $H$ is the Hubble parameter. The velocity therefore is unperturbed, but adding a velocity perturbation is actually equivalent to changing the amplitude of the initial density perturbation. The model is now completely prescribed by this amplitude, $\delta_{in} \equiv (\delta M/M)_{in}$ and the initial radius (for a description of this model including a general velocity perturbation, see Lilje & Lahav (1991)).

If the initial density is higher than the critical density, which is always the case for a positive perturbation in an Einstein-de Sitter universe, the solutions of the equations of motion are given by the familiar parametric equations:

$$R(\eta) = \frac{R_{ta}}{2}(1 - \cos(\eta))$$
$$T(\eta) = \frac{T_{ta}}{\pi}(\eta - \sin(\eta)).$$

The parameters $R_{ta}$ and $T_{ta}$ give the values of radius and time when the expansion is halted, after which the cluster contracts according to the equations. For the top-hat model these turn-around parameters are given by

$$R_{ta} = \frac{GM(R_{in})}{|E|} \approx R_{in} \left(\frac{\delta M}{M}(R_{in})\right)^{-1}$$

$$T_{ta} = \frac{\pi R_{in}^{3/2}}{\sqrt{8GM}} \approx \left(32G\bar{\rho}_{in}(\delta M/M)^{3/3\pi}\right)^{-1/2}$$

Here $E$ is the total energy, which must be negative for the structure to be bound. The fact that the turn-around time is independent of radius for the uniform top-hat model implies that at $T_{ta}$ the kinetic energy vanishes and the total energy is given by the potential energy, $E = W \propto -GM/R_{ta}$. If after recollapse the cluster reaches virial equilibrium, and if there is no significant energy loss through the escape of mass to infinity, the total energy of the final system should be $E = W/2 \propto -GM/2R_{vir}$; thus $R_{vir} = R_{ta}/2$. The timescale after turn-around in which equilibrium is reached should be on the order of $T_{ta}$.

This is essentially all there is to the evolution of the top-hat model (shown schematically in Fig. 1). Now consider some obvious generalizations to this model. Relaxing the demand of spherical symmetry means that the initial fluctuation is a uniform ellipsoid. As is well known, uniformity is conserved during its evolution, in which the shortest axis reaches turn-around first, followed by the others; in Fig. 2 I have plotted a typical case for an oblate spheroid. The corresponding sphere with the same initial density, also shown in the figure, lies between the extremes, i.e. its radius evolves on a timescale longer than that of the minor axis, but shorter than that of the major axes. In this case the final parameters can not be determined using the same argument as used above because now the timescales are
different for the various axes and the kinetic energy never vanishes. For a more detailed investigation of these models see Chapter 6.

Another generalization is the spherically symmetric overdensity with a non-trivial radial density profile. When the initial density profile is a power-law of the radius, $\delta(r) \propto r^{-7}$, I will call these \textit{cone-hat} models. For these models, the spherical symmetry actually allows one to treat the recollapse phase rigorously as a similarity solution (Fillmore & Goldreich, 1984; Bertschinger, 1985). One now considers the evolution of shells, which, until turn-around, follow the solutions of the uniform case. Due to the non-trivial density profile the collapse timescales differ for different shells, but it appears that also in this case the final radius is a fixed fraction of the turn-around radius, at least for a range of values for $\gamma$, in spite of the crossing of the shells that will necessarily occur (e.g. Fillmore & Goldreich, 1984). From this one readily calculates the final density profile from the initial density perturbation profile:

$$\rho_f \propto r^{-\alpha}, \quad \alpha = 3\gamma/(1 + \gamma).$$

(4.7)

For $\gamma = 3$, which also describes the point mass perturbation, one obtains the well known relation $\rho \propto r^{-1.25}$, (Gott, 1975).

One may extract various scaling relations between the defining parameters of this model (e.g. Faber, 1982; Padmanabhan, 1993). One of these, directly derived from equations (4.5) and (4.6), relates the turn-around time to the turn-around radius,

$$R_{ta} \propto T_{ta}^{(2+2\gamma)/3\gamma}$$

(4.8)

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Schematic drawing of the evolution of the top-hat model}
\end{figure}
For models with a power-law power spectrum of the initial density perturbation field,

$$P(k) \equiv \langle |\delta_k|^2 \rangle = Ak^n,$$  \hspace{1cm} (4.9)

a fiducial density profile can be defined from the expressions for the mass fluctuations on scale $R$. These are given by the variance of the density field smoothed on that scale by a smoothing function $W$,

$$\delta^2(R) = \sigma^2(R) \propto \int dk k^2 P(k)W(kR)dk \propto R^{-3-n},$$  \hspace{1cm} (4.10)

so $\gamma = (3 + n)/2$. In terms of the mass this may be written

$$\sigma^2(M) = \langle (\delta M/M)^2 \rangle \propto M^{-(3+n)/3}$$  \hspace{1cm} (4.11)

which gives the relation

$$R_{la} \propto T_{la}^{(5+n)/(3(3+n))}$$  \hspace{1cm} (4.12)

In Fig. 3 I show the evolution of various shells within such an overdensity. Figures like these will be used later to compare the evolution of clusters in the N-body simulations to these models.

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Figure 2: The evolution of the two axes of an oblate spheroidal overdensity in a flat universe with initial axial ratio $a_{\text{minor}} : a_{\text{major}} = 1 : 2$. The dashed line shows the corresponding sphere with the same initial overdensity.
To summarize, the predictions that characterize the top-hat and cone-hat models and that may be compared to more realistic calculations, are the following: First the qualitative behaviour: an expansion phase, turn-around, recollapse and subsequent relaxation. Then the quantitative predictions given in equations (4.5) and (4.6). To these should be added the predictions for the virialization radius and time,

\[ R_{\text{vir}} = R_{\text{ta}} / 2 \]  \hspace{1cm} (4.13)

and

\[ t_{\text{vir}} = 2t_{\text{ta}}. \]  \hspace{1cm} (4.14)

From the cone-hat model we finally have the prediction about the relation between turn-around radii and times within a collapsing cluster, Eq. 4.12.

3 Simulations: cluster extraction and self-similarity

3.1 Simulations

The two cosmic simulations that form the core of these investigations were carried out by Simon White. Both contain 10^6 particles and were evolved in an Einstein- de Sitter universe using the particle-particle particle-mesh (PPPM) algorithm (Hockney & Eastwood, 1981, see also Efstathiou et al., 1985). The initial conditions were set up in the usual manner.
(Efstathiou et al., 1985). An initial density field with a power-law spectrum of fluctuations was created by moving particles from a ‘glassy’ grid using the Zel’dovich approximation (Zel’dovich, 1970). The fourier modes used to create the perturbing field had random phases and amplitudes given by the power spectrum, \( P(k) \equiv<|\phi_k|^2> = k^n \) with \( n = -1 \) and \( n = 0 \). Only the amplitudes were changed for these two simulations, the phases were the same. At the time that the Nyquist frequency reached approximate non-linearity, the PPPM algorithm took over the evolution. The mesh used within the particle-mesh part of the algorithm contained 256 grid cells on a side, while the (comoving) smoothing length used in the particle-particle part was roughly 1/2500 of the length of the computing box. Output times were chosen such that, between them, there was a constant logarithmic increment in the mass-scale that was just going non-linear (e.g. Efstathiou et al., 1988). These scales were chosen to be approximately the same for the two simulations. The \( n = 0 \) simulation contained 9 output times, including the initial timeframe. The \( n = -1 \) simulation contained 8 output frames. Unfortunately, the output time corresponding to the second output time in the \( n = 0 \) simulation was not saved for this calculation.

Concerning the grid from which the particles were moved to create the initial conditions, the term ‘glassy’ is used to describe a configuration that is between completely regular and completely random. To avoid too much power on small scales due to discreteness (white noise) effects, one prefers anti-correlated points to complete spatial randomness (Efstathiou et al., 1985), but the standard rectangular grid leads to too regular an appearance of the later stages. A ‘glassy’ grid avoids both these complications in an elegant manner. It is generated dynamically, by evolving an initial random distribution of points using a standard N-body integrator with opposite sign for the interaction, i.e. negative gravity. This moves the particles away from each other, creating an anti-correlated point set, but without the obvious regularities of a rectangular grid. For more details and a comparison of the various types of initial particle distributions, see Baugh et al., (1994).

Fig. 4 shows a projection of the final configuration of the two simulations. Fig. 5 shows the entire evolution of a slice through these volumes, centered on the position of the largest cluster in the \( n = -1 \) simulation. A few aspects of the initial conditions are apparent already upon visual inspection of these figures. The fact the simulations started with hierarchical initial conditions is apparent from the bottom-up growth of the structures. The fact that the \( n = -1 \) spectrum is flatter than the \( n = 0 \) spectrum results in a greater range of cluster sizes in the former simulation. A greater range of mass scales reach non-linearity at roughly the same time. This is also the cause of the more homogeneous spatial distribution of the clusters in the \( n = 0 \) simulation. At the time that a certain mass scale goes non-linear and also becomes prominent visually, larger scales are less important and obvious than for \( n = -1 \). Nevertheless, as is especially notable in the slices, the positions of rich clusters are correlated between the two simulations. This is also true for their relative richness and their environment and is caused by the equal phases used in setting up the initial density field.
Figure 4: The projection of the final states of the two N-body calculations used in this work, $n=-1$ above, $n=0$ under. Only one in forty of the particles are plotted.
Figure 5: Full evolution of the simulations, shown through slices centered on the final, comoving position of the largest cluster in the $n=-1$ simulation (left column). The right column contains the corresponding slices through the $n=0$ simulation, centered on this same comoving position.
Figure 6: Self-similarity test of the simulations, n=-1, left, n=0, right. Upper pair of boxes contains the two-point correlation functions and the conditional densities, scaled as described in the text and such that correlation length is at r=1 for last output time. Middle pair contains velocity correlations, scaled in the same manner. The dots show $<v_{12}>$, the lines the radial and tangential velocity dispersions. The lower two frames show the dispersion of the counts in spherical shells.

3.2 Scaling and self-similarity

Because of the scale-free initial conditions and the Einstein-de Sitter background, the evolution should be self-similar, i.e. at different times they should appear similar modulo a
change of scales. This is apparent already from visual inspection; apart from changes of scale, consecutive output frames are similar in appearance, but quantitative measures are needed to test this rigorously. This is useful as an indication of the relative importance of undesired numerical artifacts such as discreteness effects and effects related to the finite size of the simulated volume. The scales imposed by these may well break the self-similarity. To this end I have calculated some statistics for which the scaling behaviour can be predicted: the variance of counts in cells of different volumes, $\sigma_N$, the conditional density, $\Gamma$ (see Chapter 1) and the two-point correlation function, $\xi$, the mean radial peculiar velocity between pairs of points as function of separation, $<v_{12}>$ and the corresponding variance of the velocity in radial, and tangential direction, respectively $<v_{12,||}^2>$ and $<v_{12,\perp}^2>$.

Because of the self-similarity these should obey the following scaling laws (Efstathiou et al., 1988; Davis & Peebles, 1977): the two-point correlation function, $\xi(x,t)$, with $x$ the comoving separation, should depend only on

$$s \equiv x/t^\alpha,$$  \hspace{1cm} (4.15)  

where $\alpha = 4/3(3+n)$. The same should be true for the variance of counts in cells as function of the size of the cells, $\sigma(R, t)$. Likewise the quantity $<v_{12}>/at^{\alpha-1}$ should depend only on $s$ and so also the other velocity moments. (Here $a$ is the universal scale factor.) The conditional density finally, should depend on $s$ in the combination $\Gamma t^2$. The results in Fig. 6 show that these scaling relations are satisfied very well for all observables with some slight deviations previously noted by Efstathiou et al. (1985) and (1988) and which will not be discussed further here.

When comparing the results of the two simulations and when comparing these to their counterparts in the real Universe, the question arises how to scale the calculations, both in space and in time. Since these are power-law simulations there is no natural scale to normalize them and the comparison with the Universe leaves considerable freedom. As noted, the output times were such that there was a fixed logarithmic increment in the scale of the structure just going non-linear. This implies that, for the steeper $n = 0$ spectrum, the absolute time interval between the various output frames is greater than for the corresponding frames in the $n = -1$ simulation. I will generally normalize the two simulations at the final output frame, which corresponds to assigning a higher initial redshift for the $n = 0$ simulation. Whenever a comparison with the Universe is needed I will do this by choosing a particular physical mass to correspond to the mass of a typical cluster in this last frame of the simulations. Together with the time normalization this also fixes the length scales, but I ignore the question whether or not the resulting characteristic length scales, such as the correlation length, agree with the corresponding scales in the Universe as well. This will generally not be the case.

3.3 Cluster extraction

The clusters were extracted from the simulations using the usual friends-of-friends (fof) algorithm. All points that can be connected by paths built up of links not exceeding a certain size are grouped together into clusters. This linking length is commonly taken as a fraction, $b$, of the mean interparticle distance. As a rule of thumb, this corresponds to an
overdensity $\delta \simeq 2b^{-3}$ contained within the outer contour of the cluster (Efstathiou et al., 1988).

Fig. 7 shows the multiplicity functions, i.e. the frequency distribution of cluster masses, obtained for the last six output times from both simulations; here the linking parameter $b = 0.1$. Here one sees in a quantitative fashion the effect mentioned above: the curves for $n = -1$ are much flatter than those for $n = 0$, which have more pronounced peaks, corresponding to the fact that in the former clusters at a greater range of masses go non-linear at the same time. The roughly constant separation between consecutive peaks again shows the self-similarity of the calculations. Clusters containing more than 1000 points were investigated further. To these were added all points within a radius, $R_{200}$, around the center of mass of the fof-cluster within which the overdensity was $\delta_{\text{int}} = 200$. The fof-algorithm was thus only used to determine the centers of the clusters. The value of 200 for the interior overdensity corresponds roughly to the overdensity of a top-hat cluster at the moment of virialization. I have plotted several of these rich clusters in figs. (8) and (9). They span a range in size from roughly 1500 to over 20000 points. On the left in Fig. 8 the clusters in the $n=-1$ simulation are is shown, while the right side shows the positions of the corresponding points in the other simulation. In Fig. 9, the right side shows clusters from the $n=0$ simulation, with on the left side the corresponding points in the $n=-1$ simulation. Again we note the fact that clusters in one simulation find their counterpart in the other, but more on this in a later section.
Figure 8: Three clusters from the final output frame of the n=1 simulation (left column) and the corresponding points in the n=0 simulation (right column). Plotted in comoving scales.
Figure 9: Three clusters from the final output frame of the n=0 simulation (right column) and the corresponding points in the n=-1 simulation (left column). Plotted in comoving scales.
4 Approach I. : global statistics

The first approach to the previrialization hypothesis is via the global statistics calculated in the previous section. The first observation that should be made, is the one most directly related to the original work by Davis & Peebles, (1977): the results for the pair-wise peculiar velocity correlations, $<v_{12}>$, upon which they based their previrialization hypothesis. Comparison of their Fig. 5, showing the analytical solution for $<v_{12}>$ as function of scale, with the corresponding result for the N-body simulations presented here in Fig. 6, shows that this basis is shaky. While the analytical solution never reaches values lower than minus the Hubble velocity, the sign of particles approaching each other after expansion, this clearly is the case for the simulations. A similar result was obtained by Davis et al. (1985), but

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/figure10.png}
\caption{Spatial two-point functions for all output times of the two simulations. (a) and (b) give respectively the two-point correlation function and the conditional density for the $n=-1$ simulation. (c) and (d) the corresponding quantities for $n=0$.}
\end{figure}
Figure 11: Plots of the quantity $R_M$, defined in the text, versus time for various masses $M$. The filled dots give the turn-around points, the open squares give the virialization points as determined from the top-hat model: $R_{\text{vir}} = R_{\text{ta}}/2$, $T_{\text{vir}} = 2T_{\text{ta}}$. This linear plot most clearly shows the top-hat-like behaviour of these quantities as function of time.

since their simulations contained characteristic length scales induced by the CDM power spectrum, it can, strictly speaking, not be compared with the analytical calculations which were based on scale free conditions. Nevertheless, they also find recollapse.

A more direct illustration of this general behaviour can be obtained from the spatial two-point functions, shown now in physical coordinates, in Fig. 10. For the qualitative behaviour that we are looking for, expansion, turn-around, recollapse and relaxation, we need a variable corresponding to the radius in the top-hat model. The relevant quantity can be nicely derived from the evolution of the conditional density. This gives the evolution of the local density profile around a typical point in the simulation. In this sense it can be compared directly to cluster density profiles. The conditional density has an advantage over cluster density profiles, in that it is well defined at all times, whereas we will see later that this is not so trivial for individual clusters. However, the conditional density has the disadvantage that contributions from particles in low-density regions are also included.

Using the conditional density, I have defined a family of characteristic radii, $R_M$. These are the radii of the spheres around the average point which contain a mass $M$. From the integral of the conditional density, the cumulative density $\Gamma$, these radii can be easily obtained at all output times. For several values of the mass, they are plotted as function of time in Fig. 11. They show exactly the behaviour that one would predict from a straight-
Individual clusters

Figure 12: Log-log version of Fig. 11. The lines have slope determined by the relation between the turn-around quantities, $R_{ta}$ and $T_{ta}$, Eq. 4.12 for a power-law density profile.

forward generalization of the top-hat model. The turn-around points, shown by the filled dots, are calculated from a parabolical fit of the maximum and the two points around it. In a log-log plot, Fig. 12, they lie on a straight line, the slope of which is consistent with the predicted relation (4.12). Another quantitative prediction, the overdensity at turn-around which is predicted to be $\delta_{ta} = 4.56$ from the top-hat model, is less well satisfied. Especially in the $n = -1$ simulation, the actual value of $\delta_{ta}$ for the radii shown, is low by roughly a factor 2. The open squares in Fig. 12 show the expected virialization points, obtained from the turn-around parameters by the simple relations (4.5) and (4.6). The actual times of virialization are somewhat longer and the virialization radii are greater. The recollapse factor is thus somewhat smaller than the top-hat prediction. Realizing that these statistics include contributions from non-correlated pairs also, one should not attach too much significance to this apparent quantitative failure of the top-hat predictions. It is more surprising that the prediction of the relation between the turn-around parameters for an isolated cone-hat model is so well satisfied by this global statistic.

The straightforward conclusions from this first investigation are that, at least qualitatively, gravitational collapse in a hierarchical, scale free universe does follow the naive extensions of the top-hat model to the global statistics examined here. It is difficult to judge the correspondences and differences between model and simulations, since strictly speaking the model is extended far beyond its range.
5  Approach II. : individual clusters and the cone-hat model

In this section we will consider individual clusters extracted from the simulations, and I will try to compare their evolution to the predictions of the simple top-hat and cone-hat models. Of the clusters which were extracted from the last output frame of the two simulations according to the description given above, only the richest ones, containing 1000 particles or more, were considered. The linking parameter in the fof-algorithm was $b = 0.1$. As described in a previous section, each cluster included all surrounding points which lie within the sphere enclosing an overdensity $\delta = 200$. 

In Fig. 13 four time steps out of the evolution of two rich clusters, one from each simulation are shown. I have only plotted the points that finally belong to the cluster, i.e. within the contour containing an overdensity of 200 around the fof-cluster center. Other points which would fit into the frames, but did not belong to the cluster are left out. These do have a certain influence on the dynamics, as will become apparent when considering isolated collapse simulations, but the importance of this was is hard to estimate. Indeed, this is the main complication one faces when comparing these collapses with the isolated top-hat collapse model. At no stage during their evolution, except possibly for the last, do these clusters resemble a more or less smooth cluster with a clearly defined center which corresponds to a well-defined peak in the density distribution. There is no obvious way in which to assign a single radius and/or overdensity to the clusters, but these are precisely the defining characteristics of the top-hat model. Taking the outer radius or the half-mass radius, will, at early stages, include many particles which are not part of the final cluster and which therefore contribute to the dynamics, even though they later escape from the cluster.

Since there is no obvious way to make the comparison, I have made a naive choice by defining radii in a manner comparable to the $R_M$ radii defined in the previous section. Using only the points of the final cluster, at each timestep I defined radii $R_N$ enclosing $N\%$ of the mass of that cluster for various values of $N$. They may be thought of as defining the orbits of average particles of the cluster but will not enclose a constant total mass except for the final time.

As figs. (14) shows, even for these clusters which during most of their evolution are far from centrally condensed or spherically symmetric, the $N\%$-radii, at least qualitatively, evolve just as expected for a cluster that does possess these characteristics. These figures show the $R_N$ radii for various values of $N$ for the clusters shown in Fig. 13 and a few others. The various line types correspond to different methods of choosing the origin. The obvious choice is to use the center of mass (c.o.m.) of the points in the cluster, but this is not necessarily the best because it may not correspond to the center as one would wish to define it from other considerations. For example, this center may not correspond to a peak in the density distribution and thus not give a declining density profile in the inner parts. This is especially true for the early stages of the evolution when a cluster still consists of many subclumps. A choice that then generally gives more satisfying results is given by the median of the three coordinates, instead of the mean because outliers are not weighted as heavily in that case. This is the procedure used for most of the figures here. In some
Figure 13: Four stages out of the evolution of two rich cluster, one from each of the simulations (left: \( n=-1 \); right: \( n=0 \)). Time steps are such that similar masses go non-linear in both simulations and there is a fixed logarithmic increment in this characteristic mass between the time steps.
cases there may be one sub-clump which dominates the cluster and which may be used as the protocenter. A good method to find this subcluster may be to determine the deepest potential well which corresponds often to the highest density peak. In the work on the density profiles of clusters (see Chapter 5) I will often use the c.o.m. of a number of the points with the lowest potential energy.

For the relatively crude statistic used here it it makes little difference which definition is used. The qualitative aspects of the evolution are similar and, moreover, they are exactly what is expected for the collapse of an isolated, smooth cluster with a non-negligible initial density profile: i.e. an expansion phase that from a more or less well defined turn-around point goes over into a recollapse phase that ends in equilibrium, shown by the radii levelling off to a more or less constant value.

To appreciate this and to permit firmer conclusions about quantitative aspects of the collapses process, I have averaged over a number of clusters in two different ways. To determine the recollapse factor, for each cluster the $N\%$-radii were normalized such that the turn-around radius and time were set to unity. These quantities were calculated as for the $R_M$ radii in the previous section, and Fig. 15 shows the resulting curves for $N$ ranging from 20 to 70 %. The behaviour up to and even after turn-around is very well described by the top-hat model. The fact that the maximum of the model curve is somewhat higher than the actual averaged curves is due to the parabolic fits used to determine the turn-around parameters.

![Figure 14: $R_N$ radii for the two clusters from Fig. 13 and two other clusters from each of the simulations. Three linetypes show the result for three different choices of the center of the cluster. Full line: median of three coordinate directions; dotted line: center of mass; dashed line: center of mass of the 1% of all particles that have the lowest potential energy. The clusters in the left column are extracted from the $n=1$ simulation, the clusters at the right come from the $n=0$ simulation.](image-url)
It is clear, that the inner radii recollapse by a greater factor than the outer ones. The effects of this behaviour on the resulting density profiles will be considered in more detail in Chapter 5, since clearly the constant collapse assumption leading to the predicted density profile of Eq. 4.7 is not satisfied. Another quantity that can be obtained from this figure is the relative time between turn-around and relaxation. The top-hat model does not make a precise prediction about this, but it should be of the order of a few times the turn-around time (assuming that violent relaxation processes determine the timescales). This prediction seems justified from the figures, where the inner radii seem to take somewhat longer to reach equilibrium than the outer ones.

To see whether relation (4.5) is satisfied I have performed a different averaging procedure. The curves for each individual cluster were normalized such that the turn-around radius and time for the half-mass radius, \( R_{50} \), was put to unity and the results for the different clusters were averaged again. The results are shown in Fig. 16. The straight line has slope taken from the predicted relation between turn-around radius and time from equation (4.5). It is clear that the predictions are borne out nicely, though more so for the \( n = -1 \) than for the \( n = 0 \) simulation. In the latter case especially the inner radii do not follow the trend very well. Turn-around occurs at a later time than predicted and \( R_{10} \) turns around even later than \( R_{50} \). This can be explained by the fact that the proto-clusters consist of agglomerations of smaller sub-clusters. Dynamically this has its most important effects on those scales where this inhomogeneity is most pronounced, i.e. on the smallest scales. The

\[\text{Figure 15: Averaged and normalized evolutions of } R_N \text{ versus time, for } N \text{ from 20 to 80\%. Each cluster's } R_N \text{ radius was normalized by the values of its turn-around parameters. The results were averaged over all clusters. Left: } n=-1; \text{ right: } n=0. \text{ The dashed line gives the exact top-hat model result, assumed here to recollapse completely.}\]
spherically averaged density distribution will in general not be monotonically decreasing in the inner parts when the origin is not specifically placed within a density concentration as was the case for the curves shown here. As Fig. 14 shows, taking the deepest potential well to define the center of the cluster (this is more sensitive to small scale structures) recollapse may indeed be faster for these inner radii.

The fact that the inner radii conform less well to the cone-hat model prediction is also a sign that an actual cluster does not have a perfect power-law density profile initially and the deviation is again expected to be largest in the center. If the clusters are thought to correspond to peaks in an initial density perturbation field, the profile approaches a power-law profile only at radii which correspond to the cluster-size but is flatter in the inner parts (Bardeen et al., 1986). One may use the density profiles of Bardeen et al. and compare the resulting turn-around parameters to the clusters here.

I have instead pursued a more direct approach, by simply calculating the initial density profile explicitly and deriving the expected evolution from it. As can be seen from the pictures of the initial conditions of the clusters shown earlier, one cannot just use the density profile of the cluster points themselves. Except possibly for the inner parts, the non-spherical shape of the proto-cluster would induce a severe underestimate of the density. Thus we necessarily have to introduce points that do not belong to the cluster. Furthermore, we must take velocity perturbations into account. While these were assumed absent in the top-hat models as it was presented in section 2, the initial conditions of the simulations were set up using the growing mode of the solutions to the linearized equations of motion.

Figure 16: Log-log plot of averaged $R_N$ radii. For all clusters these radii were determined and then divided by the turn-around parameters of the half-mass radius. The dots again show the turn-around points, while the line, as in Fig. 12, gives the cone-hat relation from Eq. 4.12.
In the top-hat model this effect may be included by simply increasing the initial density perturbation by a factor $5/3$ (e.g. Padmanabhan, 1993).

In Fig. 17 the cumulative density perturbation profiles are shown for the clusters used above. They were obtained using the center of mass of the cluster points, but points not in the final cluster were also included in the determination. In the same figure the integral of the two-point correlation function, $\Gamma' / \bar{p} - 1$ is also shown. The straight lines correspond to the initial density profile as calculated using Eq. 4.10 and to that calculated from the theoretical two-point correlation function, $\xi(r) \propto r^{-(n+3)}$. These two forms have both been proposed as the correct model for the initial density profile of a proto-cluster (compare Faber (1982) with Bardeen et al. (1986)). In the previous calculations I have only used the density profile from Eq. 4.10 which fits the results better, but the initial density profiles seem to be closer to the prediction from two-point correlation function. We will return to this point later.

From the initial $N$%-radii and the corresponding density perturbations we can calculate the expected turn-around time and radius from equations (4.5) and (4.6). I have plotted the ratios of these with the measured quantities for the $R_{70}$ in Fig. 18 as open squares. The stars give the result when one assumes a ‘growing-mode’ velocity perturbation. One notes that the inclusion of a velocity perturbation is surely necessary, for the radii collapse faster than the top-hat model would predict from the density perturbation alone. On the

![Figure 17: Initial density perturbation profiles and integrated two-point correlation functions (dashed line). The profiles were calculated around the center of mass of the points ending up within the final cluster only. All points were used in determining the density. The straight line with slope $-\gamma = -(3 + n)/2$ correspond to the theoretical profile as calculated from the variance of the density field on scale $R$. The line with slope $-2\gamma$ corresponds to the profile as calculated from the two-point correlation function.](image-url)
other hand, with a correction factor of $5/3$, as implied by the top-hat model, the simulated clusters lag behind the model for the inner radii. Turn-around happens later than predicted and at a larger radius. This discrepancy disappears somewhat for the larger radii, where the points cluster around the predicted value.

6 Approach III. : isolated collapse simulations

In this section we will take a more detailed look at the collapse process itself. Although we have convinced ourselves that, on average, cluster collapses show similarities to the top-hat or cone-hat models when using the appropriate variables, some doubts remain about the relevance of the absolute predictions extracted from the top-hat model. Also, the question of why the internal and external inhomogeneities do not significantly change the global behaviour of the collapse process is interesting in itself. To shed light on these issues, we will here compare the evolution of some of the clusters in the PPPM simulation to analogs that differ from these only in those aspects that are thought to be the main causes of the supposed previrialization effect. The possible effects of external structures were removed by isolating the proto-cluster from its surroundings and evolving it in isolation. The effects of non-radial internal inhomogeneities were removed by averaging the initial proto-cluster in the angular directions. Here the radius and velocity of a particle were kept constant as well as the magnitude of the tangential component of the velocity, but the angular coordinate

Figure 18: Ratios of predicted turn-around quantities to actually observed ones for 70% ($n=-1$) and 80% ($n=0$) radii. Squares denote predictions from the density profiles only, stars show predictions when a growing-mode velocity perturbation is assumed to correct the density perturbation by a factor $5/3$. 
and the direction of the tangential velocity were randomized. The resulting spherically
symmetric proto-cluster was again evolved in isolation. The first type of simulation was
performed using a version of Hernquist's TREE-code (Hernquist, 1987). The spherically
symmetric clusters were evolved with a code that calculates the gravitational potential
from an expansion of the density field on radial shells using spherical harmonics (van Albada
(1982); also Bonnecoe (1987)).

There is one complication that requires further attention. In general, these protoclusters,
removed from their surroundings and evolved in isolation, will not be bound. For while
the final configurations of the clusters as defined in this work are spherical and contain all
the points within a certain radius of the center, this is not true for the earlier stages. The
particles that during the evolution escaped from within the confines of the protocluster, will
nevertheless have affected its evolution, causing other particles to be bound to the cluster.
I have therefore redefined the protocluster to include all particles within a sphere around
the center of mass of the cluster points at the initial time. Since we intend to isolate the
cluster from its surroundings, the radius of this sphere should correspond to the scale of the
cluster. Typically the maximum separation of the cluster points to the center was chosen,
unless this proved so large that too many points would be included. Then a smaller radius
was chosen, which necessarily excluded some of the particles of the original cluster from
the redefined one (though never more than 10% and more often less than 1%).

Fig. 19 shows a set of simulations of a moderately rich cluster extracted from the \( n = -1 \)
simulation. The first column shows the evolution of the original cluster as calculated in the
cosmological simulation. The second column shows the evolution of the redefined cluster.
This one contains more particles than the original cluster, but since the radius of the sphere
was less than the maximum radius of the protocluster, 7% of its particles are left out. The
third column shows the evolution of this cluster in isolation, calculated with the TREE-
code, while the fourth column shows the evolution using the expansion (SPEXP) code from
the same initial conditions. Due to the hierarchical nature of the initial conditions this
may not be a natural or even a good choice, since this kind of code cannot follow the
collapse of sub-structure with sufficient resolution. I have included it primarily to show the
difference between the two algorithms for these kinds of systems. In fact, the expansion code
performs quite well; especially the large scale structure corresponds reasonably well with
the TREE-code result. The two last columns show the evolution of the spherically averaged
cluster calculated via the TREE and the expansion code respectively. In this case too much
resolution is not necessarily the preferred choice since the system we want to model does not
have structure on small scales. These have been smoothed away in the spherical averaging
process and the small scale structures that form in the TREE-code calculation therefore do
not correspond to any input inhomogeneities, but only to discreteness-noise effects.

The next figures, (20) and (21), show four stages from the evolution of two of these
clusters, for the PPPM, the TREE and the SPEXP calculations. Only the points belonging
to the original cluster are now plotted; i.e. the extra points included for the reasons
mentioned above are left out. These clusters are two from a total of six that were reevolved
in this way. All were taken from the \( n = -1 \) simulation originally spanning a range in mass
form 1500 to 6000 points. The redefined clusters contain up to 16000 points.

It is interesting to compare the original evolution with the corresponding isolated evo-
Figure 19: All results on cosmological and isolated simulations for one of the clusters of the $n=1$ simulation. First column only shows the points of the final cluster as defined in the text. The second column shows the full redefined cluster as evolved in the PPPM-simulation. The third column shows the TREE-simulation starting from the first frame of the second column. The fourth column shows the SPEXP-simulation of that same cluster. The Fifth column shows the TREE-simulation of the spherically averaged counterpart of the redefined cluster, while the last column shows the corresponding SPEXP-simulation.
Figure 20: Three simulations of one cluster from the $n=1$ simulation. First column shows original PPPM-evolution. Second shows TREE-simulation on same initial cluster. Third column shows SPEXP-evolution of spherically averaged cluster. Only points of original cluster are shown.
Figure 21: As Fig. 20, for different cluster.
Figure 22: $R_N$ curves for three types of cluster evolution. PPPM, TREE (dotted line) and SPEXP (dashed line). Cluster # 20 is the cluster in Fig. 20, cluster # 52 is the cluster in Fig. 21.
lution from the TREE-code simulations. All in all, these are very similar, and it is easy to find corresponding sub-structures, especially at the early stages. The only difference between the two is the absence of possible interactions with external objects. These interactions may come in a variety of forms, the most important of which are tidal and shear effects. Both of these may act to separate parts of a proto-cluster from each other, while the squeezing of the shear effect may cause other parts to approach each other. With the TREE-code simulations, remnants of these effects remain due to the velocity structure of the proto-clusters. Tidal and shear velocity fields, due to external structures, will also be present in the isolated proto-cluster; however there will no longer be an associated acceleration field, so the magnitude of these effects will probably be reduced as well. The absence of a stretching effect can be seen in the later stages of the evolution of the cluster in Fig. 20. Subclusters which in the full cosmological simulation are clearly separated from the main

![Figure 23: Density profiles for the reevolved clusters from Fig. 22. These are determined using all points from the redefined clusters (see text). The dots show the density profile, the lines show the cumulative density. Filled circles and full lines correspond to the cluster in the PPPM simulation; open squares and dotted lines to the TREE-code results; the stars and the dashed lines to the SPEXP-code. The horizontal line is at an overdensity of 5.6. The point where the cumulative density crosses this line should be compared to the turn-around point extracted from the radial velocity profiles in Fig. 24.](image-url)
body, have almost merged with it in the isolated collapse. The absence of a squeezing effect is nicely illustrated by the cluster in Fig. 21, where we see that in isolation, the cluster seems to keep its initial spherical shape somewhat longer.

These external effects are a small influence in the evolution of a cluster. They may either prolong or shorten the evolution of the main body of the cluster. In the spherically symmetric collapse these effects are altogether absent and, obviously, the appearance of the evolution is radically different. On a smooth background a core develops, sometimes accompanied, or preceded by shell-like structures. In most cases an anisotropic structure develops, which, according to Aguilar & Merrit (1985), is probably due to an instability related to the radial-orbit instability of spherical models with anisotropic orbits (e.g. Fridman & Polyachenko, 1984; Binney & Tremaine, 1987). It is once more apparent that there are no obvious characteristics which allow a comparison between the real cluster and its spherically symmetric counterpart, apart from those considered above. Fig. 22 shows these $R_N$ radii for the clusters that were re-evolved. The median values of the coordinates were used for origin and only the cluster points were used in the determination of $R_N$.

As noted from the visual appearances of the clusters, the results from the TREE-code

Figure 24: Radial velocity profiles corresponding to the density profiles in Fig. 23. Symbols have the same meaning as in that figure.
simulations are very similar to the original (PPPM) collapses. This is true both for the approach to turn-around, the values of $R_{ta}$ and $T_{ta}$, and for the behaviour after turn-around. In most cases the spherically symmetric cluster collapses faster than the other two; although this effect is significant only for the inner radii. The values of both turn-around radius and time are very similar for the 70% radius. The same is true for the relaxation phase after turn-around, where it is seen that the virialization radii are similar for all three cases. This suggests that the final density profiles should be the same as well, and this turns out to be the case as can be seen from Fig. 23.

Fig. 24 shows the corresponding final radial velocity profiles. These still show signs of collapse in the outer parts of the cluster, where relaxation has not yet been achieved. At the turn-around points, where the velocity vanishes for the first time, the cumulative density is of order 5.6, as predicted by the top-hat model. The density profiles are clearly not power-laws. This demonstrates that the one-dimensional cone-hat models (Fillmore & Goldreich 1984; Bertschinger, 1985), though describing the evolution up to turn-around reasonably well, fail to do so for the subsequent relaxation process. This will be the subject of Chapter 5.

The picture that emerges again from the approach in this section, is that the top-hat model provides a very good description of cluster collapse. Still, the reasons that are put forward in support of the previrialization hypothesis, and the results from other authors merit a further investigation why this is so. In the conclusion of this chapter I will propose a possible solution to that question.

7 Discussion and summary

In this chapter I have presented investigations on the significance of the previrialization effect. While the name implies virialisation earlier than expected, which means earlier than predicted by the top-hat model of gravitational collapse, in the literature the definition has been interpreted and extended in various manners. In this work I have extended it still further. The working hypothesis has been that the top-hat model gives an inadequate description of realistic cluster collapse. The various possible aspects of this statement allow for a wide range of approaches. My first approach is the one most directly related to the work that originally led to the hypothesis itself. In contrast to the results by Davis & Peebles (1977), I do not find evidence for any previrialization effect. In contrast to their analytical calculations, in cosmological N-body simulations the pairwise velocity does decrease below zero, indicating that particles approach each other after having reached a maximum separation. Only later does the average velocity rise to zero again, at which point one may claim that the distribution has virialized. This has previously been observed by other authors, e.g. Efstathiou et al. (1988) and Davis et al. (1985); here however not only is this qualitative feature of the previrialization hypothesis falsified, but one may also find both qualitative and quantitative corroboration for the top-hat and related models. To show this, I have defined radii which enclose a fixed mass around the average particle and follow the evolution of these. As Fig. 11 shows, these characteristic radii behave exactly as one would expect from the predictions of the top-hat model, while a comparison of figs.(12) with (3) shows that the scaling between the turn-round radius and time is just as expected
from the statistical properties of the initial perturbation field.

Similar results are obtained from the analysis of individual clusters. At no stage during their evolution do these clusters resemble the spherically symmetric, uniform top-hat model. Nevertheless, the naive choices for defining characteristic scales, namely the $R_N$ radii, enclosing a fixed fraction of the cluster mass, qualitatively, and in many aspects even quantitatively, reproduce the predictions of the top-hat and cone-hat models perfectly. All the aspects of these models, expansion, turn-around and recollapse by a factor of roughly 2, can be observed also here. Furthermore, the scaling between turn-around radius and time that one may derive on the basis of the statistical properties of the initial density perturbation field, Eq. 4.12, is very well satisfied. Deviations are seen only at the smallest scales, which can easily be understood from the fact that these clusters are simply not exactly described by one scale. The main difficulty for testing the predictions is posed by relations containing the amplitude of the initial density perturbation as in eqs.(4.5) and (4.6). These densities may be obtained from the initial output frame of the simulations using spherical averages in the standard manner. The fact that the initial cluster is in general far from spherically symmetric makes the question of the relevance of these quantities a non-trivial one. This question is further complicated by the fact that for the simulations peculiar velocities do not vanish. Correcting for this gives reasonable agreement between the predicted and the observed turn-around radius and time, but only for the outermost radii. A better test for these absolute (i.e. not just scaling) relations, comes from the final exercise that was performed on the simulations. Here first the external and later the internal inhomogeneities were removed by isolating the protocluster and later averaging them on radial shells. The comparison with these clusters, evolved in isolation, give the most direct test of the previrialization hypothesis in its original form. Furthermore, the spherically symmetrized clusters are the closest top-hat analogon to the original cluster, and comparison with this immediately provides a clue to relevance of the absolute predictions of that model. As the results again showed, on the relevant scales the various clusters evolve similarly; there is no sign of any of the effects predicted by previrialization. Time scales of turn-around and virialization agree as do the maximum radii and the subsequent collapse factor.

At the same time, these various collapse calculations most dramatically show the differences between the smooth model and the clumpy reality. How can they both result in clusters that are very similar in their final structure? Why is it that the effects that were proposed as causes for previrialization turn out to be unimportant?

To attempt to answer this question I have plotted in Fig. 25 the velocity vectors for the points of the three types of clusters at an intermediate stage of their collapse. The spherical collapse is very smooth except for a few regions that seem to be experiencing some kind of instability, possibly related to the radial-orbit instability mentioned above. The velocity fields of the clumpy collapses, both cosmological and isolated, are very chaotic; however, here seems to be order in this chaos. While non-radial motions certainly are induced by the small scale structure that is present, they are not purely random. The fact that they are induced at all, already hints at some level of organization on the smaller scales. Flow patterns are seen where particles converge on subclumps which may well start to behave as individual particles themselves, having much smaller tangential velocities and therefore being more likely to follow the general top-hat pattern of infall.
This interpretation may be tested by determining the average velocities of these sub-clumps. I have therefore applied the friends-of-friends algorithm to these proto-cluster at some earlier time steps. For the resulting sub-clumps I then calculated the average velocity, velocity dispersion, $\sigma_v$, and the second moment of the separation from the center of mass, $\sigma_r \equiv \langle |\mathbf{r} - \mathbf{r}_{\text{c.o.m.}}|^2 \rangle^{1/2}$. This quantity gives an indication of the size of the subclump. From these I calculated a virial ratio, which I approximated by $2T/W = -\sigma_v^2/(0.6GM/\sigma_r)$, where $M$ is the mass of the clump.

In Fig. 26 I show the result for a few timesteps out of the evolution of a the cluster from Fig. 21. The sub-clusters found by the fof-algorithm are drawn as circles, the sizes
Figure 26: Velocity vectors and fof-clusters for three stages in the evolution of a cluster from the $n=-1$ simulation. The clusters were determined using a linking length $l=0.02$, only clusters containing at least 40 points were drawn as circles in the right column. The full line is a logarithmic measure for the total mass of the cluster, the dashed line measures the size of the cluster as defined using $\sigma_r$ as defined in the text. The velocity vectors in the right column are scaled up by a factor five w.r.t. the vectors in the left column. The numbers north-east of the circles give the value of the virial ratio as defined in the text.
of which give an indication of their mass. The line segments give the projected center-of-
mass velocity of the respective clumps, the length is a measure of the total velocity. The
numbers attached to the squares give the virial ratio of the corresponding clump. One notes
that the virial ratios of many of the major clumps are close to unity, while deviations are
seen mainly in those clumps that are clearly disturbed, or very small, or extended. These
quantities are only rough measures of the real dynamical state of the clumps of course, but
they do indicate that the tangential velocities, though substantial, may not have such a
great influence on the large-scale dynamics.

This fact contradicts the argument in support of previrialization given by Peebles
(1993,p. 540) (see also Evrard & Crone,1992). The argument uses various moments of
the Boltzmann equation to derive the following evolution equation for the radial velocity,
(see Peebles, 1993, Eq. 22.40)

\[
\frac{\partial \langle v_r \rangle}{\partial t} + \langle v_r \rangle \frac{\partial \langle v_r \rangle}{\partial r} + \frac{1}{r^2 n} \frac{\partial}{\partial r} r^2 n \langle \delta v_r^2 \rangle - \frac{\langle v_r^2 \rangle}{r} + g = 0
\]  

(4.16)

Peebles (1993, p.540-541) argues that when the expansion stops at time \( t \) and expansion
radius \( r \), the tangential velocity will have changed by roughly a factor \( r/t \sim G \rho r \), which
just cancels the gravitational acceleration term \( g \) for a density perturbation of order unity.
Evrard & Crone (1992) investigated the ratio \( \langle v_r^2 \rangle / r/GM \) at the turn-around radius
and found that it was much smaller than one. But, as Fig. 26 shows, the gravitational
force experienced by particles, is also far from spherically symmetric. Subclumping is
strong enough to produce virialized subclusters which partake in the dynamics more or
less as individual particles with a tangential velocity that is much smaller than that of the
constituent particles.

This picture also supports of one of the main assumptions of the original version of the
Press-Schechter theory of structure formation (Press & Schechter, 1974). There the cluster
multiplicity function is calculated from the initial spectrum of density perturbation; one of
the principal assumptions is that subclusters collapse quickly and can thereafter be treated
as point particles with respect to the evolution of the larger scale structure of which they are
a part. This seems to be just what we observe in Fig. 26. Another aspect of Press-Schechter
theory is that the top-hat model is used for predicting the time a cluster forms. When,
according to linear theory, the overdensity on a certain scale has reached a critical value,
which from the top-hat model is \( \delta_c \approx 1.68 \), that scale collapses and forms a virialized cluster
that may thus be treated as a point mass for as far as the further evolution on larger scales
is concerned. As I have shown, the timescales for collapse and relaxation of the clusters in
the simulations are roughly equal to those of their spherically averaged counterparts which
have been evolved in isolation, i.e. to the closest fitting top-hat analogue, so this aspect of
Press-Schechter theory seems justified as well.

To summarize, the previrialization hypothesis as stated in the Introduction may be dis-
carded. Clusters do recollapse by about the predicted factor in about the predicted time.
This occurs after a turn-around point that occurs after the predicted period of expansion
and at roughly the predicted radius. More detailed models, predicting relations between
scales within the cluster such as relative timescales and relative spatial scales are also well
satisfied on average. The cause of this behaviour lies probably in the way in which realistic
cosmological density perturbations are organized. While sub-clustering develops and causes non-radial motions, they do so in a consistent manner. The cluster is not just heated up, but the quasi random motions actually are organized in the same way as the cluster itself is organized. The particles do not move in a smooth spherical potential, but along contorted flow lines which nevertheless follow the local force lines (see also Villumsen & Davis, 1986). As the potential itself develops into a more and more spherically concentrated shape, local clumps move along, recollapsing in a relatively much smoother potential, as if they were individual particles themselves. Finally, through complicated multiple merger processes, the cluster forms. This may partly explain the discrepant result from Peebles’ (1990) analysis, where he does find proof of the relevance of previrialization. His central assumption is that "... one can choose a transition epoch...at which a protocluster is usefully approximated as a loose concentration of particles surrounded by similar mass concentrations".

This assumption is followed by the presumption that substructure would enhance the previrialization effect. First, there is in realistic, hierarchical spectra, no such transition epoch as the pictures showing the evolution of clusters have clearly demonstrated. This was actually one of the main problems for these investigations. Secondly, as I argued, it is not necessarily true that substructure will enhance these effects. On small scales, where the comparison between the two types of clusters is worst anyway which 2 TYPES ??, substructure will certainly distort the flow. But as shown above, this does not significantly retard or even stop the recollapse on those larger scales on which the clusters are defined. It therefore seems that, though the top-hat model is a severe simplification of realistic clusters, when interpreted correctly it gives an apt description of gravitational collapse.

Some questions do remain, that seem not to be treated correctly by the standard theories. They are mainly concerned with the relevance of self-similar calculations for the relaxation phase of the evolution. First, the factor by which shells recollapse is not constant with radius as has been predicted for clusters arising out of these perturbation spectra. This furthermore resulted in density profiles that were clearly not power laws, but are similar to the results obtained from isolated collapse calculations (e.g. van Albada, 1982; see also Chapter 5). Isolated collapse calculations like these are saved from irrelevance by this work. If previrialization were significant, there would be no recollapse and probably no need for invoking processes like violent relaxation. Even though that process is yet ill understood, it seems to be active also in more realistic circumstances. Finally, the relaxation phases of the various types of clusters as shown in section 6, are very different in appearance. The clumpy collapses go through stages of hierarchical clustering with associated multiple merger processes. The spherical clusters collapse much more smoothly, while the relaxation seems to be caused by the occurrence of instabilities (Aguilar & Merrit, 1985). Both these types however result in density profiles that are very similar. This question will be the subject of the next chapter.

Acknowledgments

I want to thank Simon White for suggesting this project to me and for providing me with two cosmological N-body simulations and useful discussions. I thank the Institute of Astronomy in Cambridge for the hospitality during the month of March, 1994, when I started these investigations. I want to thank Marc Balcells and Tjeerd van Albada for the
help they gave me with the TREE-code and the spherical expansion code and for many useful discussions. I want to thank Bob Sanders for carefully reading the manuscript and many useful discussions.

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