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Modeling and Control of Heat Networks with Storage: the Single-Producer Multiple-Consumer Case

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***Abstract*—In heat networks, energy storage in the form of hot water in a tank is a viable approach to balancing supply and demand. In order to store a desired amount of energy, both the volume and temperature of the water in the tank need to converge to desired setpoints. To this end we provide a provably correct internal model controller that guarantees these tracking goals and is robust against parameter uncertainties. In order to design this controller and analyse the closed loop system we derive a nonlinear model from first principles. This model describes temperature and volume dynamics of a setup consisting of a single producer with a storage tank and multiple consumers. We show that the control goal can be achieved by only measuring the aggregated flow rate of the consumers, the volume of the storage device and the corresponding temperature.**

***Index Terms*—District heating systems, Heat exchanger networks, Thermal storage, Output regulation.**

I. Introduction

Energy demand is rising and a considerable fraction of the energy is consumed in the form of heat. While high quality resources such as natural gas are used for heating, waste heat is left unutilized in most cases. Since this increase in energy consumption has a negative impact on the environment, there is a need for more energy efficient systems. One of the proposed solutions is the integration of heat networks into the existent infrastructure. Such heat networks are commonly referred to as district heating systems (DHS) when found in an urban area or heat exchanger networks (HEN) in industrial environments.

Although DHS's have been around since the early 1900s, recent years have witnessed a renewed interest in heating systems for several reasons. One of these reasons is the recent advances in geothermal energy production methods. Another reason is the change in governmental policies with the intention to reduce carbon dioxide emissions. Since adding a DHS to existing infrastructure often increases the energy efficiency of the overall system, the total carbon dioxide emissions drop. Two examples of higher energy efficiencies due to a DHS are utilizing waste heat and incorporating environmentally friendly sources such as biomass incinerators. A third example is the integration of combined heat power plants (CHP) into a DHS. Since a CHP can produce both heat and electricity it has a better energy efficiency compared to conventional power plants that produce only electricity.

One of the reasons why waste heat is not yet shared between different companies on a large scale is the mismatch in supply and demand. Heat may be produced when it is not needed, and conversely, may not be available when needed. Thus it is imperative to have an efficient system to store heat and provide it when needed. For this reason storage elements are included in DHS's. Furthermore if prices in such a network are time varying, a storage device can be used to store energy until prices rise, thus increasing revenues for all parties.

Literature review. General modeling principals of chemical and thermodynamic systems can be found in [1]. Models for heat exchangers are widely available, see *e.g.* [2] where controllability and observability are also investigated. There is a wide variety of possibilities for thermal storage. A survey of different techniques can be found in [3]. The most common way is to use water tanks with a fixed volume that can be heated and cooled. In such tanks there are three layers, one with hot water, one with cold water and a separation layer called a thermocline, which has a steep thermal gradient. This type of storage is referred to as stratification and is studied in detail in both [4] and [5] using a numerical evaluation. An alternative is to have an empty tank which can be filled/drained with hot water but has the disadvantages of lower efficiency and higher dissipation rates. Recent interest has shifted towards heat storage in phase-changing materials, which have some interesting properties such as low dissipation rates.

In the 1980s the synthesis and control of DHS's and HENs were mostly focused on the steady state optimal design of heat exchangers, which resulted in simple engineering techniques such as the pinch method [6]. With the introduction of linear optimization techniques such as model predictive control, the focus later shifted to the design of optimal controllers where optimal steady states were considered. An example of this approach can be found in [7]. The same approach applied to cooling systems can be found in [8] and [9]. Although cooling systems serve a different purpose, the principles and dynamics that describe heating systems also apply. Thus the control methodologies used in this paper can easily be applied to cooling systems. A linear programming model for optimal resource management of a combined heat power plant, in combination with a distribution network is presented in [10]. The modeling and control of a similar system is provided in

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[11]. In case the demand is not known in advance, prediction methods, as studied in [12] and [13], can be applied. Recently [14] solved an optimization problem to maximize the heat transfer in HEN's with stream splits. Note that most of these methods give optimal values for steady state behavior, but the dynamics of these systems are often neglected.

In previous works by one of the authors [15], [16] a pressure regulation problem was solved for nonlinear hydraulic networks, however temperature dynamics were neglected. Due to consumers that require fixed temperatures, the stability of the temperatures in DHS's is important. A common way to provide this stability is to control the flow and heat injection by PID controllers [17]. Examples of stability analysis in connection with PID control and energy systems are given in [2], where a heat exchanger is considered, and [18], where a linearised model of a thermal network is studied.

Main contribution. In this paper we propose a provably correct internal model (IM) controller for a DHS with a storage device. To this end we derive a dynamic model of a DHS with a single producer, a storage device and multiple consumers. This model is obtained by interconnecting models of components derived from first principles. The model is non-linear due to the contribution of flowrates, volumes and temperatures in the storage tank. Consumers can each extract a desired amount of heat from a heat exchanger and set their own flowrate. To guarantee that the demand of these consumers is met, two controllers are introduced. The first controller is a proportional controller that regulates the flowrate of the producer, depending on the aggregated flow of the consumers. The second controller regulates the heat injection in a heat exchanger located at the producer. The two controllers in closed-loop to the nonlinear model are shown to guarantee the desired convergence property of the overall system. The controller is robust to parametric uncertainties possibly present in the model. An additional advantage of the proposed approach is that the control goal is achieved without any measurement of the consumers' demand.

Outline. The paper is organized as follows: In Section II we introduce the framework of interest. This is followed by the problem formulation in Section III and the controller design in Section IV. The main result is stated in Section V followed by a case study with corresponding simulations in Section VI. The conclusions and future work are given in Section VII. A number of proofs, some model derivations and the controllers which are designed for the case study can be found in the appendix.

Notation. We denote by $\mathbb{R}$ and $\mathbb{R}_{>0}$ the set of real numbers and strictly positive real numbers respectively. Given a vector $x \in \mathbb{R}^n$, x^T is considered its transpose. If the entries of x are functions of time then the time derivative of x is denoted as $\dot{x} := \frac{dx}{dt}$ unless stated otherwise. For the sake of notational simplicity, for time-dependent variables we drop the explicit dependency on t as long as no confusion arises. We define the operator $\Lambda(x) := \text{diag}\begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}$ as the diagonal matrix of elements x_i, and $\text{block.diag}\begin{pmatrix} A_1 & A_2 & \dots & A_n \end{pmatrix}$ as the block-diagonal matrix for which the block diagonal matrices are A_i for $i = 1, ..., n$. The identity matrix of dimension n is given by I_n and a $n \times m$ matrix containing only zeros is denoted by $\mathbf{0}_{n\times m}$. Finally we denote the vector of all ones by $\mathbf{1}_n := \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^T$.

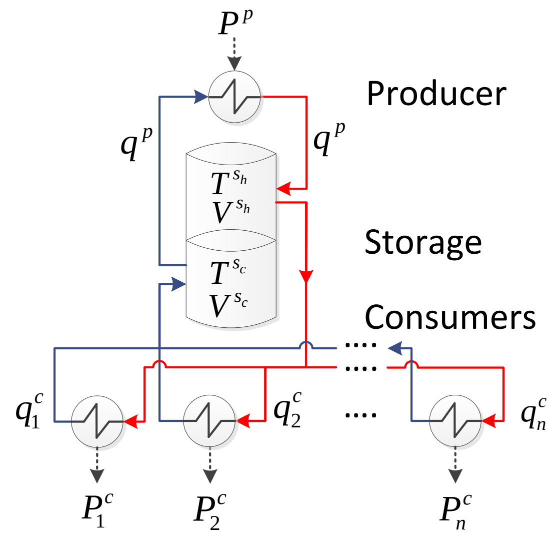

Fig. 1. Topology of a district heating network.

II. SYSTEM MODEL

Motivated by industrial waste-to-energy plants (see *e.g.* [19]) and CHP, we consider a setup with a single producer of heat. Such a producer provides steam that is converted to electricity by a turbine and can extract steam or hot water at several locations. This heated water or steam is fed through a heat exchanger that separates two flows with a heat conductor. The heat exchanged between these two streams is considered the injection of heat to the network. The heat is delivered to n consumers that come in the form of industrial or residential buildings, each having its own demand which is modeled as a combination of periodic and constant signals. Due to possible variable heat prices and demand patterns, there might be situations in which, in addition to balancing demand and supply, it is desired to store energy. To this end we introduce a storage tank containing heated water.

Remark 1 Note that the heat exchangers separate the fluid in the network from both the fluids of the consumers and producer. This separation ensures that no contamination can enter the network and storage device. In certain cases it can however be beneficial to bypass the heat exchangers (and storage) by adding a direct connections between a producer and consumer. For example, when the producers generates steam, such a connection can be used to further heat up the hot water delivered to the consumer.

A. Topology of a district heating system

The topology of the district heating system considered in this paper is depicted in Figure 1. A producer supplies heated water or steam[1] to a heat exchanger which is connected to a

[1]In case of steam delivery to the heat exchanger a phase-changer heat exchanger can be used and fits in this approach.

network. The heat transfer in the heat exchanger is considered to be the power injection P^p of the producer. Water in the network is heated in the heat exchanger and transported to a storage tank that has two layers of water with variable volume. The top layer is fed by the water that comes from the heat exchanger of the producer. Each consumer also has a heat exchanger from which it extracts the heat. This heat exchanger connects the outlet of the hot layer and the inlet of the cold layer of the storage tank. In order to derive a model of this topology we first introduce each component.

B. Storage tank

The storage tank that we consider uses a stratification principle where hot water on the top is separated from cold water at the bottom by a thermocline. This device has four valves, two at the top and two at the bottom. These valves are used as in- and out-lets of the hot and cold part of the storage device. The temperatures of the top and bottom layer are approximately constant with respect to the height of the tank. On the contrary, the thermocline is a thin layer with a steep temperature gradient [5], [8]. We neglect the thermocline which allows us to model the storage device as two separate storage tanks that are placed on top of each other[2]. This is motivated by a low heat exchange rate between the hot and the cold layer of the storage tank. Additionally it is possible to add an insulation layer that decreases the heat exchange even further. Such a layer can also prevent mixing during long discharging or charging periods which causes a more severe heat exchange rate. Furthermore we assume that each layer is perfectly stirred. We denote the total capacity of the storage tank by $V^{\max}$ and consider the storage device to be always completely filled with water. To this end we assume the initial condition to satisfy

$$V^{s_h}(0) + V^{s_c}(0) = V^{\max}, \tag{1}$$

where V^{s_h} and V^{s_c} are volumes of the hot and cold layer respectively. Furthermore the following constraint is fulfilled [3]:

$$V^{s_h}(t) + V^{s_c}(t) = V^{\max} \quad \text{for all } t > 0. \tag{2}$$

Additionally let

$$\mathcal{V} := \left[V^{\min}, V^{\max} - V^{\min}\right], \tag{3}$$

be the set admissible volumes of both layers in the storage with $0 < 2V^{\min} \leq V^{\max} < \infty$. We furthermore assume both $V^{s_h}(0), V^{s_c}(0) \in \mathcal{V}$. For reasons of simplicity, the heat exchanges between both the two layers and the environment are neglected for the time being. We define the temperatures in the hot and cold layer of the storage tank as T^{s_h} and T^{s_c} respectively. The temperature in the heat exchanger of the producer is denoted by T^p while its volume is defined as V^p. The temperature and volume in the heat exchanger of consumer j is given by T_j^c and V_j^c respectively. These variables are collected in the vectors

$$T^c := \begin{pmatrix} T_1^c & T_2^c & \ldots & T_n^c \end{pmatrix}^T$$
$$V^c := \begin{pmatrix} V_1^c & V_2^c & \ldots & V_n^c \end{pmatrix}^T,$$

where n is the number of consumers.

C. Producers and consumers

The power demand of the consumers is given by

$$P_j^c \in \mathbb{R}_{>0} \qquad j = 1, 2, \ldots, n.$$

Assuming that the used fluid is incompressible and since there is no storage nor loss of mass in a heat exchanger, the inflow rate equals the outflow rate. Hence, it is enough to associate to each heat exchanger a single flow variable. The flow through the heat exchanger of consumer j is denoted as q_j^c. By aggregating these entries we have

$$q^c := \begin{pmatrix} q_1^c & q_2^c & \ldots & q_n^c \end{pmatrix}^T. \tag{4}$$

Similarly we define the flow passing through the heat exchanger of the producer by

$$q^p \in \mathbb{R}_{>0}. \tag{5}$$

D. Sensors

We restrict ourself to a setup which allows only for measurements that are located at the producers and storage devices. This implies that there is no need for communication between the controllers and the consumers. We assume that both the volume and temperature of the hot layer in the storage tank are measured and these measurements are communicated to the producer. Furthermore we assume that the aggregated return flow of the consumers is measured.

E. Nomenclature

The notation of all physical variables and parameters that will be used can be found in to following table:

Variable	Description	Unit
V	Volume[34]	m^3
q	Flow rate[3]	m^3/s
W	Work	J
H	Enthalpy	J
Φ	Heat exchange	J
P	Heat exchange rate[3]	J/s
E	Total energy[45]	J
$\hat{E}$	Relative energy[4]	J
T	Temperature[34]	$^\circ C$
T^{ref}	Reference temperature	$^\circ C$
$\hat{T}$	Relative temperature[34]	$^\circ C$
C_p	Specific heat	$J/(Kg^\circ C)$
ρ	Density	kg/m^3
m	Mass	kg
U	Heat transfer coefficient	$W/(m^{2\circ}C)$
A_h	Area	m^2

[2]Under this assumption, a topology with separate hot and cold water storage tanks results in exactly the same model. The results presented in this work can therefore also be applied to such a network.

[3]Details of how this constraint is fulfilled can be found in the proof of Lemma 1.

F. Model of the district heating system

The open loop dynamics are obtained by interconnecting the models of the individual components (heat exchangers, storage) such that the network of Figure 1 is obtained. The derivations of the model of the individual components as well the model of the overall system can be found in Appendix A. The resulting model is given as

$$\begin{aligned} V^p\dot{T}^p &= \left(T^{s_c} - T^p\right) q^p + P^p \\ V^{s_h}\dot{T}^{s_h} &= \left(T^p - T^{s_h}\right) q^p \\ V^{s_c}\dot{T}^{s_c} &= \sum_{i=1}^{n} \left(T_i^c - T^{s_c}\right) q_i^c \\ V_j^c\dot{T}_j^c &= \left(T^{s_h} - T_j^c\right) q_j^c - P_j^c,\ j = 1, 2, \ldots, n \\ \dot{V}^{s_h} &= q^p - \sum_{i=1}^{n} q_i^c \\ \dot{V}^{s_c} &= \sum_{i=1}^{n} q_i^c - q^p. \end{aligned} \tag{6}$$

Having this model allows us to formally formulate the control problem we would like to solve.

III. Problem formulation

Optimization techniques aiming at maximizing profit, *e.g.* by shifting loads in time, can be commonly found in the power systems literature [20]. Motivated by these techniques which provide optimal storage levels, we define a setpoint tracking problem with the objective to store a desired amount of energy. To this end we consider a setpoint tracking problem with setpoints for both the temperature and volume of the hot layer in the storage device. The motivation for this approach is that a combination of a temperature and a volume defines the amount of energy that is stored, as shown in Appendix A-B. Hence the regulation problem is defined as follows:

Problem 1 For system (6), design a controller that, given the unmeasured power demand P_j^c, $j = 1, 2, \ldots, n$, regulates the heat injection P^p and the flow rates q in such a way that any solution of the closed-loop system satisfies

$$\lim_{t\to\infty} \left(V^{s_h} - V^{s_h*}\right) = 0 \tag{7}$$

$$\lim_{t\to\infty} \left(T^{s_h} - T^{s_h*}\right) = 0, \tag{8}$$

where T^{s_h*} and V^{s_h*} are desired setpoints for the temperature and volume, respectively. □

Since the setpoint V^{s_h*} cannot exceed the capacity of the storage tank we introduce the following standing assumption:

Assumption 1 We assume $V^{s_h*} \in \mathcal{V}$ with V^{s_h*} being a setpoint and $\mathcal{V}$ is defined as in (3).

[3] Variables denoted with superscript p refer to producers while c refers to a consumer. In the case of a superscript c, the subscript i is used to distinguish between consumers.

[4] The hot and cold part of the storage tank is denoted by superscript s_h and s_c respectively.

[5] For any control volume the superscripts in and out mean the inflow and outflow respectively.

A. Model power demand

Each consumer j extracts an unknown demand $P_j^c(t)$ which can be regarded as a disturbance. For Problem 1 to be solvable, this disturbance needs to be rejected. In the scenario considered in this paper, the sensors are only placed at the storage device and therefore $P_j^c(t)$ is not available to the controller. We assume that the disturbance consists of a linear combination of constants and sinusoidal signals with unknown amplitudes and phases. A large class of disturbance signals can be modelled by resorting to a sufficiently large number of frequencies ([21]). In order to design a controller, the frequencies of the sinusoidal signals must be known. Such frequencies can be obtained from historical data or heat demand predictions based on weather forecasts. If these historical data are not available, adaptive and robust methods to deal with the case of unknown frequencies are available ([21]). The investigation of these methods in the present context, however, goes beyond the scope of the paper.

Before introducing the model representing the demand, we collect the power demand P_j^c of each consumer j and the setpoint T^{s_h*} in one vector $d \in \mathbb{R}^{n+1}$ such that

$$d := \begin{pmatrix} -P_1^c & \ldots & -P_n^c & T^{s_h*} \end{pmatrix}^T.$$

We assume that the demand P_j^c for each each consumer j is not measured and also T^{s_h*} is not communicated to the controller. Therefore d is not available to our controller, but we do impose some structure on it. To this end we assume each d_i is produced by an exosystem which is described by

$$\begin{aligned} \dot{w}_i &= S_i w_i \\ d_i &= \Gamma_i w_i \end{aligned} \qquad i = 1, ..., n+1, \tag{9}$$

where $\Gamma_i \in \mathbb{R}^{1\times s_i}$, $S_i \in \mathbb{R}^{s_i\times s_i}$ and s_i is the dimension of w_i. Furthermore we set $S_{n+1} = 0$, $\Gamma_{n+1} = 1$ and initialize $w_n(0) = T^{s_h*}$ such that $d_{n+1} = T^{s_h*}$. We define $S := \text{block.diag}\begin{pmatrix} S_1 & \ldots & S_{n+1} \end{pmatrix}$ and $\Gamma := \text{block.diag}\begin{pmatrix} \Gamma_1 & \ldots & \Gamma_{n+1} \end{pmatrix}$ and collect

$$w := \begin{pmatrix} w_1^T & \ldots & w_{n+1}^T \end{pmatrix}^T,$$

so that

$$\dot{w} = Sw, \tag{10}$$

with $S \in \mathbb{R}^{s\times s}$ and $s = \sum_{i=1}^{n+1} s_i$. We impose that all eigenvalues of S have zero real part and multiplicity one in the minimal polynomial. This assumption implies that all trajectories of (10) are bounded and none of them decay to zero as $t \to \infty$. Note that this structure imposed on d allows for constant and sinusoidal signals as well as any linear combination of them. Finally we assume that matrix S is available for the design of the controller.

Remark 2 Note that taking $S_{n+1} = 0$ implies that T^{s_h*} is a constant setpoint. Although it is possible in our approach to allow time varying trajectories for T^{s_h*}, this possibility is not considered due to the lack of practical relevance.

B. Compact form

We now write system (6) in a more compact form. Before doing this, we need to define the inputs and outputs of the system. Because we can control both the heat injection and the flow rates in the pipes, these are taken as inputs and are given as

$$u = P^p$$
$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} q^p \\ q^c \end{pmatrix},$$

respectively. Note here that the heat injection u is a single input while the flows in the pipes v are aggregated in a vector. In line with Problem 1 we define the tracking error of the system as

$$e = T^{s_h} - T^{s_h *},$$

which is available to the controller. Furthermore we collect temperatures and volumes in the vectors

$$x := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} T^p \\ T^{s_h} \\ T^{s_c} \\ T^c \end{pmatrix}$$
$$z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} V^{s_h} \\ V^{s_c} \end{pmatrix},$$

so that (6) becomes

$$M(z)\dot{x} = A(v)x + B_1 u + Pw \tag{11}$$
$$\dot{z} = B_2 v \tag{12}$$
$$e = Cx + Qw, \tag{13}$$

with the matrices given by

$$M(z) = \text{block.diag}\begin{pmatrix} V^p & \Lambda(z) & \Lambda(V^c) \end{pmatrix} \tag{14}$$

$$A(v) = \begin{pmatrix} -v_1 & 0 & v_1 & \mathbf{0}_{1\times n} \\ v_1 & -v_1 & 0 & \mathbf{0}_{1\times n} \\ 0 & 0 & -\mathbf{1}_n^T v_2 & v_2^T \\ \mathbf{0}_{n\times 1} & v_2 & \mathbf{0}_{n\times 1} & -\Lambda(v_2) \end{pmatrix} \tag{15}$$

$$B_1 = \begin{pmatrix} 1 \\ \mathbf{0}_{(n+2)\times 1} \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 & -\mathbf{1}_n^T \\ -1 & \mathbf{1}_n^T \end{pmatrix} \tag{16}$$

$$P = \begin{pmatrix} \mathbf{0}_{3\times(s-1)} & \mathbf{0}_{3\times 1} \\ -\Gamma & \mathbf{0}_{n\times 1} \end{pmatrix} \quad Q = \begin{pmatrix} \mathbf{0}_{1\times n} & 1 \end{pmatrix} \tag{17}$$

$$C = \begin{pmatrix} 0 & 1 & 0 & \mathbf{0}_{1\times n} \end{pmatrix}. \tag{18}$$

Observe now that $A(v)$ is a time varying matrix due to the dynamics of $v(t)$. Also note that (8) and (7) are equivalent to

$$\lim_{t\to\infty} e = 0 \tag{19}$$
$$\lim_{t\to\infty} (z_1 - z_1^*) = 0, \tag{20}$$

where $z_1^* := V^{s_h *}$.

IV. Controller design

The design of the controller consists of two parts. The first part derives the controller that regulates the flows in the pipes such that (7), *i.e.* (20), is satisfied. We assume that the consumers set their flowrate to a constant depending on their demand. This flowrate can be set by the network operator, a local controller or the consumer itself but is not communicated to the producer or the storage device. However, the aggregated flow that returns to the cold layer in the storage device is measured and available for the controllers. This measurement along with a measurement of the volume in the hot storage layer is communicated to the producer. The second part derives the controller of u which achieves (8), *i.e.* (19). We show that the temperature dynamics (11) in closed loop with the flow controller asymptotically converges to a linear time invariant system with unknown parameters. We design an IM controller for such linear time invariant system. Finally we combine the two controllers and analyse the stability of the closed loop system.

A. Control of flow rates

We now design a control input for the flowrate v such that $\lim_{t\to\infty}(z_1 - z_1^*) = 0$. The proposed controller for the flowrates is given as

$$\begin{aligned} v_1 &= \mathbf{1}_n^T v_2^* + \alpha(z_1^* - z_1) \\ v_2 &= v_2^*, \end{aligned} \tag{21}$$

where $\alpha \in \mathbb{R}_{>0}$ is the gain of the controller and $v_2^* \in \mathbb{R}_{>0}^n$ is the vector of flowrates at the consumers.

Lemma 1 System (12) with controller (21) has solution $z(t)$ such that

$$z_1(t) = e^{-\alpha t}(z_1(0) - z_1^*) + z_1^*, \tag{22}$$

and

$$z(t) \in \mathcal{V} \times \mathcal{V}, \tag{23}$$

for all $t \geq 0$ and constraint (2) is satisfied.

Proof: Evaluating (12) with input (21) provides

$$\dot{z}_1 = \alpha(z_1^* - z_1), \tag{24}$$

which solution yields (22). Taken together with Assumption 1, it is easily verified that (23) is satisfied. Observe that by (12) we have

$$\dot{z}_2(t) + \dot{z}_1(t) = 0, \tag{25}$$

and also $z_2(0) + z_1(0) = V^{\max}$ which implies that (2) is satisfied. □

Observe furthermore that controller (21) is completely independent from any temperature dynamics. Using Lemma 1 we conclude that (20) is satisfied.

Remark 3 Note that controller (21) sets $v_2 = v_2^*$ which is a constant flowrate. Since this flowrate is in practice set by a separate controller located at the consumers it is restrictive to assume this flowrate to be constant. However if the timesteps are taken small enough these flowrates can be approximated by a slowly varying signal. The effect of this slow variation is not taken into account in the analysis.

B. Control of heat injection

Next we provide a controller that regulates the heat injection u, such that (8), *i.e.* (19) is satisfied. To do this, in this section we restrict ourselves to the case where flows and volumes are constant, *i.e.* $\dot{z} = 0$. Note that this restriction is only used for the design of the controller and will be relaxed in the analysis of the closed loop system. In order to satisfy $\dot{z} = 0$ we set $v_1^* = \mathbf{1}_n^T v_2^*$, where we recall that v_2^* are the flowrates that are set by the consumers. Since these flowrates are strictly positive this implies that also $v_1^* > 0$. We now define

$$\begin{aligned}\bar{A} &= M(z^*)^{-1}A(v^*)\\ \bar{B} &= M(z^*)^{-1}B_1\\ \bar{P} &= M(z^*)^{-1}P,\end{aligned} \tag{26}$$

so that (11)-(13) take the form of the linear time invariant system

$$\begin{aligned}\dot{x} &= \bar{A}x + \bar{B}u + \bar{P}w\\ e &= Cx + Qw.\end{aligned} \tag{27}$$

For this system we consider a controller of the form

$$\begin{aligned}\dot{\xi} &= F\xi + Ge\\ u &= H\xi + Ke,\end{aligned} \tag{28}$$

with $\xi \in \mathbb{R}^l$. Furthermore let F, G, H and K be matrices such that there exist matrices Π, Σ and R that solve

$$\begin{aligned}\Sigma S &= F\Sigma\\ R &= H\Sigma,\end{aligned} \tag{29}$$

and

$$\begin{aligned}\Pi S &= \bar{A}\Pi + \bar{B}R + \bar{P}\\ 0 &= C\Pi + Q.\end{aligned} \tag{30}$$

These are the celebrated regulator equations. We now prove that a controller of the form (28) exists and is able to solve the output regulation problem.

Theorem 1 There exist F, G, H and K such that the solutions of the closed-loop system (27)-(28) with exosystem (10) are bounded and satisfy (19), *i.e.*

$$\lim_{t\to\infty} e(t) = 0.$$

Proof: The proof can be found in Appendix B-A. □

Remark 4 Knowledge of the matrix $\bar{A}$ in (26) requires knowledge of the vector v^* related to the steady-state flowrates. In practice, v^* need not be perfectly known. As shown in [21], the controller in (28) turns out to be robust against model parametric uncertainties, and, hence, against uncertainty in the estimate of v^*.

Remark 5 Note that $\xi \in \mathbb{R}^l$ with $l = s+r$ (for details see *e.g.* [21]) with s the dimension of S and r the relative degree[6] of system (27). Therefore we see that the state of the controller scales linearly with s. The intuition behind this is that the controller needs to keep track of all the, possibly different, demand patterns of the consumers.

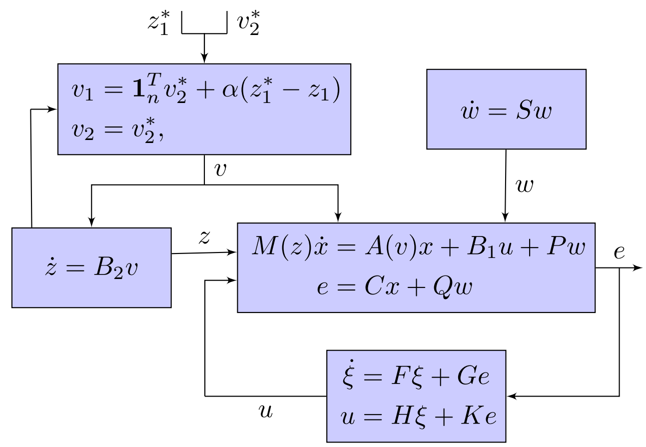

Fig. 2. Block diagram of the closed loop system

V. Main result

Having designed u and v in (21) and (28), respectively, we combine the two controllers and show that both volume and temperature converge to the desired setpoints (see Problem 1). To do this we write system (11)-(13) in closed loop with the two controllers which leads to a non-linear system whose block diagram is given in Figure 2. This is followed by an analysis of the stability and convergence of the entire system. The first step is to write (11) in closed loop with controller (28) to obtain

$$\begin{aligned}M(z)\dot{x} &= (A(v)+B_1KC)x + B_1H\xi + (B_1KQ+P)w\\ \dot{\xi} &= GCx + F\xi + GQw.\end{aligned} \tag{31}$$

Note that applying controller (21) ensures that $M(z)$ is invertible due to Lemma 1. Therefore we can define

$$\begin{aligned}A_e &= M(z)^{-1}A(v) - \bar{A}\\ B_e &= M(z)^{-1}B_1 - \bar{B}\\ P_e &= M(z)^{-1}P - \bar{P},\end{aligned} \tag{32}$$

with $\bar{A}$, $\bar{B}$ and $\bar{P}$ being as in (26). Now we perform the coordinate change

$$\zeta := \begin{pmatrix} x - \Pi w \\ \xi - \Sigma w \end{pmatrix}. \tag{33}$$

Bearing in mind (29) and (30), (33) allows us to write closed loop system (31) as

$$\dot{\zeta} = (A' + F'(t))\zeta + B'(t)w, \tag{34}$$

with

$$A' = \begin{pmatrix} \bar{A} + \bar{B}K\bar{C} & \bar{B}H \\ GC & F \end{pmatrix} \tag{35}$$

$$F'(t) = \begin{pmatrix} A_e(t) + B_e(t)KC & B_e(t)H \\ 0 & 0 \end{pmatrix} \tag{36}$$

$$B'(t) = \begin{pmatrix} P_e(t) + B_e(t)R + A_e(t)\Pi \\ 0 \end{pmatrix}. \tag{37}$$

[6]For large scale systems it holds that $s >> r$.

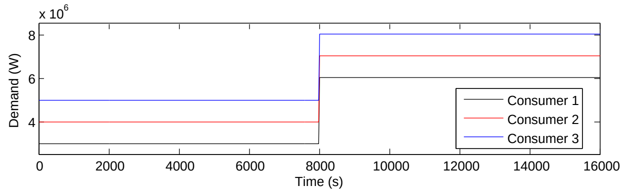

Fig. 3. Piece-wise constant demand pattern.

Note that in contrast to what happens in linear time invariant systems, (37) only vanishes in case $z = z^*$ and $v = v^*$. Consider now system (34) with (33) and observe that $\zeta = 0$ implies that $x = \Pi w$. Due to (30) we know that $C\Pi + Q = 0$ and therefore also that

$$e = Cx + Qw = 0.$$

Consider any solution to (34) with exosystem (10). Proving that these solutions are bounded and $\lim_{t\to\infty} \zeta = 0$ implies that (19) is satisfied. Since in Section IV-A it was already proven that (20) holds, this would also imply that Problem 1 is solved.

Theorem 2 Consider system (34) where w is as in (10). Then, for any initial condition, ζ is bounded and satisfies

$$\lim_{t\to\infty} \zeta(t) = 0.$$

Proof: The proof can be found in Appendix B-B. □

VI. Case studies

In order to illustrate the model and the proposed controller we consider three case studies. In the first case study we consider three different consumers, each having a constant heat demand. Two time intervals are considered with different setpoints for the volume of the storage, while the temperature in the storage device is kept to a fixed level. The volume setpoints of the hot layer are set such that water is stored in the first time interval while it is drained during the second time interval. As a result, energy is stored in the first time interval while it is drained in the second interval. An incentive to do this can *e.g.* be a higher cost for production in the second time intervals. Another reason could be to secure supply in case of unforeseen interruptions of the heat production. The second case study is identical to the first one with the exception that a time varying demand is considered. Finally a real demand pattern is considered.

A. Constant demand

In the first case study we consider two intervals with piece-wise constant demand and different setpoints as was motivated in Section III. We use a thermal storage which is designed to operate at $85°C$. The temperature should be kept around this value as much as possible, while the change in volume modifies the stored amount of energy. For this reason we take the setpoint and inital condition of the temperature in the hot

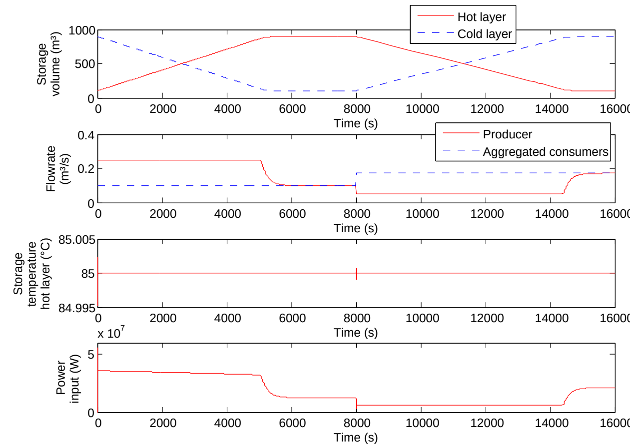

Fig. 4. Three consumers with constant demand.

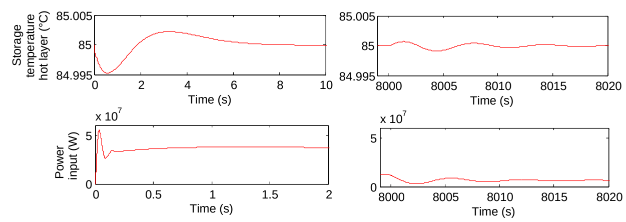

Fig. 5. Magnification of Figure 4. The storage temperature of the hot layer and corresponding power injection.

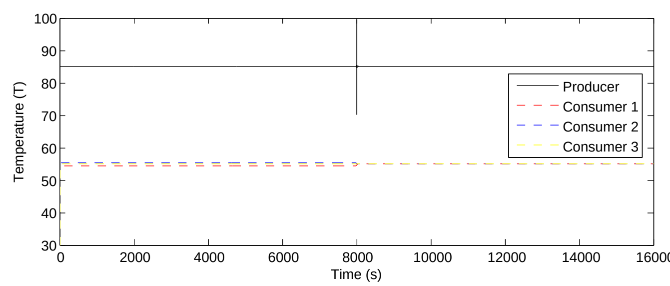

Fig. 6. Temperatures of the heat exchangers.

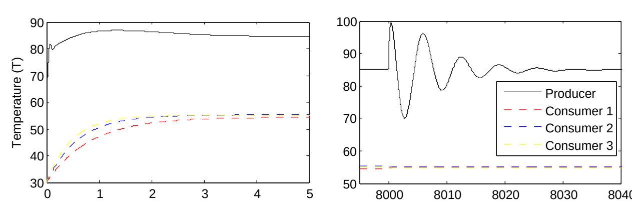

Fig. 7. Magnification of Figure 6. Temperature of all the heat exchangers at $t = 0$ and during the transition between two time intervals.

layer of the storage tank equal to $T^{s_h*} = T^{s_h}(0) = 85°C$. The density and specific heat are given by $\rho = 975kg/m^3$ and $C_p = 4190J/Kg°C$, respectively. The total volume of the storage tank is $V^{\max} = 1000m^3$ and the volume of the hot layer is initialized at $V^{s_h}(0) = 100m^3$. Each heat exchanger has a volume of $V_i^c = V^p = \frac{2}{100}m^3$. The temperatures of the heat exchangers are initialized to $T^p(0) = 70°C$ for the producer and $T_i^c(0) = 30°C$ for each consumer i. Controller (28) is given in Appendix C and the gain α in (21) regulating the flow is set to $\alpha = 0.005$. To avoid non-positive and too high flow rates we impose a saturation on the corresponding control input. This saturation is given by $0.05m^3/s \leq q^p \leq 0.25m^3/s$. The extension of the present analysis to the case of input saturations is left for further research.

The two time intervals we consider are given by $0s \leq t < 8000s$ and $8000s \leq t < 16000s$. The volume setpoints are $V^{s_h*} = 900m^3$ and $V^{s_h*} = 100m^3$ for the first and second time interval respectively. Since the setpoint for temperature in the hot storage tank is fixed this implies that the energy is stored in the first time interval while it is drained in the latter time interval. The demand of the three consumers are given in Figure 3. The flowrates of the consumers in the first time interval are given by $q_1^c = 0.024m^3/s$, $q_2^c = 0.033m^3/s$ and $q_3^c = 0.041m^3/s$ and in the second time interval they are set to $q_1^c = 0.049m^3/s$, $q_2^c = 0.057m^3/s$ and $q_3^c = 0.065m^3/s$.

The results are shown in Figures 4, 5, 6 and 7 where the latter two figures display the temperatures in the heat exchangers. Figure 5 illustrates the evolution of the temperature of the hot layer, which converges to the desired $T^{s_h*} = 85°C$, and the corresponding power input of the producer. As a consequence of the initialization and the choice of the setpoints of the volume of the hot layer we see from Figure 4 that the storage is filled in the first time interval and drained in the second time interval. The changes in volume are a consequence of a difference in the aggregated flowrate of the consumers and the flowrate of the producer. At the end of both time intervals these flowrates are equal to each other causing a constant volume of the hot and cold layers in the storage tank. Furthermore it can be seen that in both time intervals the setpoints of the volume are achieved.

It can be seen that the heat injection is approximately proportional to the flowrate of the producer. This can be understood from the dynamics of heat exchanger located at the producer as in (6). From this equation in follows that at steady state (e.g. $\dot{T}^p = 0$) the power injection is proportional to $q^p(T^{S_c} - T^p)$. Since the consumer has both a constant flowrate and a constant heat demand, T^{S_c} converges also to a constant value. Furthermore, since $T^p = 85°C$ it follows that the heat injection is proportional to the flowrate of the producer. Draining the storage tank results in a lower production despite the increased demand.

In Figure 6 it can be seen that the temperatures in the heat exchangers stay constant during both time intervals. However Figure 7 shows some transient behaviour at $t = 0s$, and some oscillations at $t = 8000s$. These oscillations are due to the transient behaviour as a consequence of new setpoints, flowrates and heat extraction rates.

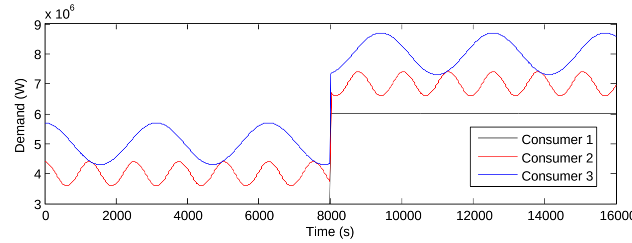

Fig. 8. Time varying demand pattern.

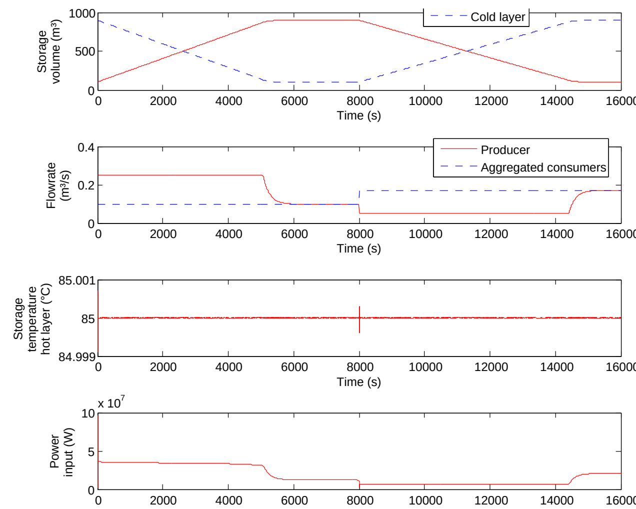

Fig. 9. Three consumers with time varying demand.

B. Time varying demand

The second case study we consider is equivalent to the case study in Section VI-A with the exception that a time varying demand is considered, as illustrated in Figure 8. For this reason we take all the initializations, setpoints and parameters identical to the previous case study. However, the controller (28) is different due to the time varying demand, and is given in Appendix C. In Figures 9, 10, 11 and 12 we see the outcome of the simulation. Figure 10 shows that approximately $T^{s_h*} = 85°C$ within ten seconds despite the time varying demand. Furthermore Figure 9 shows that also the volume setpoints are achieved at the end of each time interval. Since the time varying demand does not influence the flowrates we see the same behaviour of the volumes as in the case study in

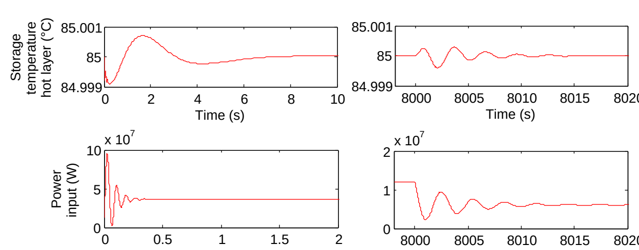

Fig. 10. Magnification of Figure 9. Storage temperature and the power injection from $t = 0$ and $t = 8000$.

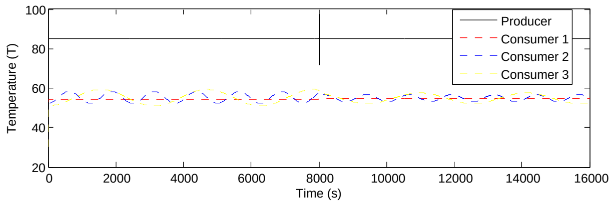

Fig. 11. Temperatures of the heat exchangers under time varying demand.

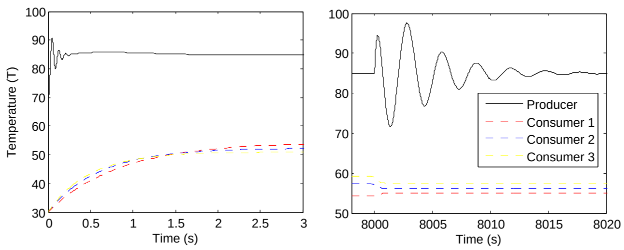

Fig. 12. Magnification of Figure 11. Temperatures of the heat exchangers from $t = 0$ and $t = 8000$.

Section VI-A. Similar as in the previous case study we see a spike in the heat injection at $t = 8000s$ due to new setpoints, flowrates and extraction rates. In Figure 11 the temperatures of the heat exchangers of the consumers clearly show the impact of the time-varying demand. Finally in Figure 12 we see a similar transient behaviour at $t = 0s$ and $t = 8000s$ as in the previous case study due to new setpoints, flowrates and demand patterns.

Remark 6 Additional simulations have shown that the performance of the controller is not significantly deteriorated in the presence of small-range frequency uncertainties (around 20%) in the consumer demand pattern. This is partly due to the the large integral gain in the controller (cf. Appendix C) and partly due to the slow dynamics of the system. As pointed out in Section III-A, adaptive and robust methods [21] do exist that can be used to compensate for larger uncertainties.

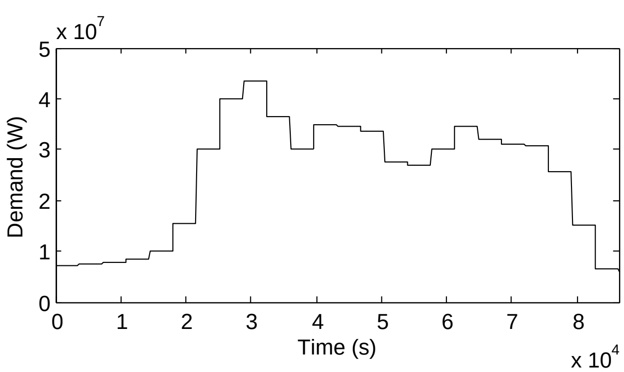

Fig. 13. Heat demand pattern of a whole day.

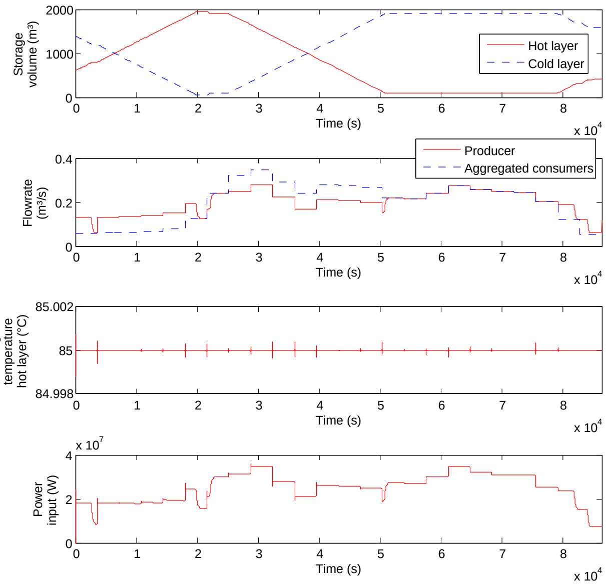

Fig. 14. Simulation results under a real demand pattern.

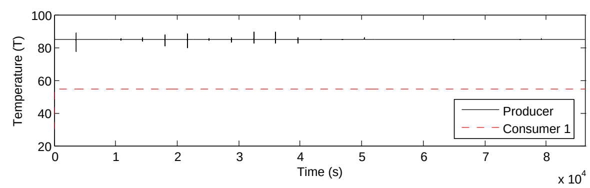

Fig. 15. Temperatures in the heat exchangers under real demand.

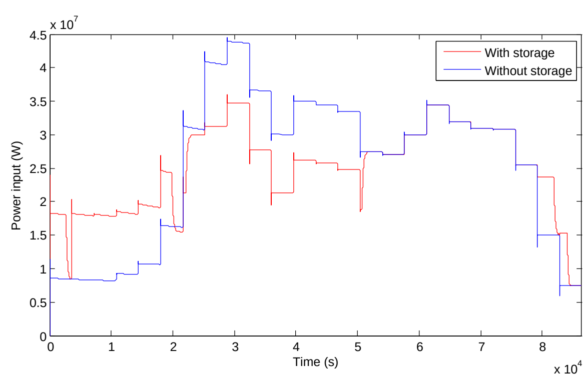

Fig. 16. Comparison of the power injection between two setups, one with and one without storage.

Time	1	2	3	4	5	6	7	
Setpoint	7.25	7.5	7.75	8.5	10	15.5	30	
8	9	10	11	12	13	14	15	16
40	43.5	36.5	30	35	34	33.5	27.5	27
17	18	19	20	21	22	23	24	25
30	34.5	32	31	30.75	25.5	15	6.5	6

TABLE I
VOLUME SETPOINTS WITH THE TIME IN HOURS.

C. Real demand

In the final case study we investigate the performance of the controller in the presence of a real demand pattern. We simulate a complete day with a storage device of $V^{\max} = 2000m^3$ where heat is stored during low demand and drained during high demand. To this end we let $0s \leq t \leq 86400s$ and use a demand pattern that is provided in [5], to which we applied a discretization with a resolution of one hour. This demand pattern can be found in Figure 13. The aggregated flowrate of the consumers is set proportional to the demand at all time.

Again we set $\rho = 975kg/m^3$ and $C_p = 4190J/Kg°C$ and the volumes of the heat exchangers are given by $V^p = V^c = 0.02m^3$. The initialization of the temperature of the heat exchangers is given by $T^p(0) = 50°C$ and $T^c(0) = 50°C$. The temperature setpoint throughout the day is $T^{S_h*} = 85°C$ and therefore also the initial temperature is set to $T^{S_h}(0) = 85°C$. The initial temperature for the cold layer is $T^{S_c}(0) = 30°C$ and the volume of the hot layer is initialized at $V^{S_h} = 600m^3$. The gain for the controller of the flowrate is set to $a = 0.005$. Again we assume a non-negative flow rate and set a maximum flowrate to avoid flowrates that are too large which is given by $0.05m^3/s \leq q^p \leq 0.5m^3/s$. The controller (28) that is used remains identical throughout the whole simulations and given in Appendix C.

The setpoints for the volume of the hot layer are given in Table I. In contrast to the previous case studies, we chose these setpoints such that most time intervals are too short to achieve steady state. As a consequence the storage will continuously fill or drain over multiple time-intervals. The setpoints are also chosen such that peaks are shaved while the storage is filled during low demand. The result of the simulation is given in Figure 14 and the heat exchanger temperatures are given in Figure 15. In Figure 14 it can be seen that the temperature in the hot storage layer converges to the desired $T^{S_h*} = 85°C$. The controller is able to maintain this temperature throughout the whole day despite the changing demand. Also in the time-intervals where the volume is not able to achieve steady state the temperature of the hot storage layer is kept close to the desired $T^{S_h*} = 85°C$. It can be seen that the storage is almost continuously filled until $20000s$ after which it is continuously drained until $53000s$. After $53000s$ the level of the storage is kept constant since it is almost fully drained and after $79200s$ the storage is filled again to the same level as the initial condition to prepare for the next day. As a consequence loads are shifted resulting in lower peaks compared to a setup without a storage tank which can be seen in Figure 16.

Note that the demand is a piecewise constant signal due to the discretization. Since the flowrate of the consumers is set proportional to this demand the flowrate has the same behaviour. This implies that the controller receives a step function as input at the beginning of each hour. As a consequence we observe in Figure 15 some small spikes of the temperature in the heat exchanger located at the producer which is the transient response to these step-inputs.

VII. CONCLUSION AND FUTURE RESEARCH

We presented a model of a district heating system with a storage device, single producer and multiple consumers. The proposed controller is able to regulate the energy level in the storage device to a desired setpoint in spite of unmeasured, possibly time varying, heat demand.

We plan to investigate the possibility of extending our findings to a topology with multiple storage tanks. In that case each storage tank is connected to multiple consumers. Also we would like to investigate the case where one has different setpoints for both volume and temperature in each of those storage tanks. There is a recent interest from the industry into temperature cascading, in which the leftover heat coming from a consumer is re-used by other consumers. This motivates us to interconnect multiple network clusters with each other, where each cluster corresponds to a network as in this paper. In this situation the output of the consumers in a cluster is connected to another cluster where it is used as an input. Lastly, motivated by practical reasons, it is of interest to study the controller under the presence of heat dissipation and time delays.

APPENDIX A
MODEL COMPONENTS

In this section we derive the adopted models for the heat exchanger and storage device to make the paper as self-contained as possible. Furthermore we derive a relation between temperature, volume and energy in order to motivate Problem 1. In order to provide this relation we first introduce the notion of enthalpy.

A. Enthalpy

Consider a control volume of an open system where the change in energy ΔE is given by

$$\Delta E = E^{in} - E^{out} + W + \Phi, \tag{38}$$

with E^{in} and E^{out} the total in and outflow of energy by mass streams. The total work supplied by the surroundings is given by W, while Φ is the heat exchanged with the surroundings. The total amount of energy in a control volume can be expressed in terms of *enthalpy* which is defined as

$$H = E + pV \tag{39}$$

where E is the energy, p the pressure and V the volume. Since we deal with temperatures between 0°C and 100°C at atmospheric pressure, water only exists in liquid form and no phase changes or chemical reactions occur. Under the additional assumption of a constant heat capacity c_p, enthalpy is only a function of the temperature (see *e.g.* [1]). This implies that

$$H - H|_{T=T^{\text{ref}}} = mc_p(T - T^{\text{ref}}), \tag{40}$$

where m is the mass of the control volume, T its temperature, T^{ref} an arbitrary fixed reference temperature and $H|_{T=T^{\text{ref}}}$ is the enthalpy associated with this reference temperature. The mass satisfies $m = \rho V$ with ρ being the density.

B. Relative energy

In this section we provide a relation between a volume, temperature and stored energy. This relations provides the main motivation for the formulation of Problem 1. In order to provide such a relation we consider a volume of heated water. Since such a body of water can only be used for heating if its temperature is lower that the outside temperature we compare the heat of the water to the outside temperature. To this end we introduce a relative temperature as

$$\hat{T} = T - T^{ref}, \tag{41}$$

where T^{ref} is a constant reference temperature which can be taken equal to the average outside temperature. Additionally we define a relative energy as

$$\hat{E} := E - E|_{T=T^{ref}}, \tag{42}$$

where $E|_{T=T^{ref}}$ is the energy of a control volume V^{s_h} at temperature T^{ref}. Due to definition (42), together with (39) and (40) we can write

$$\begin{aligned}\hat{E} =& H - pV - (H - pV)|_{T=T^{ex}} \\ =& mc_p(T - T^{ref}) + H|_{T=T^{ref}} \\ & - pV - (H|_{T=T^{ref}} - pV) \\ =& c_p\rho V\hat{T}.\end{aligned} \tag{43}$$

As a consequence, $\hat{E}$ is expressed as a product of a volume and a relative temperature. Using the identity

$$\hat{T}^{s_h} - \hat{T}^{s_h*} = T^{s_h} - T^{s_h*}, \tag{44}$$

and bearing in mind relation (43), we observe that solving Problem 1 guarantees

$$\lim_{t\to\infty} |\hat{E}^{s_h} - \hat{E}^{s_h*}| = 0, \tag{45}$$

where $\hat{E}^{s_h*} := c_p\rho V^{s_h*}\hat{T}^{s_h*}$. For this reason we are able to split an energy setpoint tracking problem into a volume and temperature setpoint tracking problem. In practice a fixed $\hat{T}^{s_h*}$ can be chosen such that a minimal operating temperatures will be guaranteed. By varying the volume V^{s_h*} different energy levels can be achieved.

C. Temperature dynamics

Since we assume perfect mixing within the control volume, the inside temperature T and outflow temperature are equal. The inflow temperature on the other hand is denoted by T^{in}. After straightforward but lengthy derivation (see *e.g.* [1]), the temperature dynamics of this control volume are obtained by (38) and (40) and read as

$$V\dot{T} = q^{in}(T^{in} - T) + P, \tag{46}$$

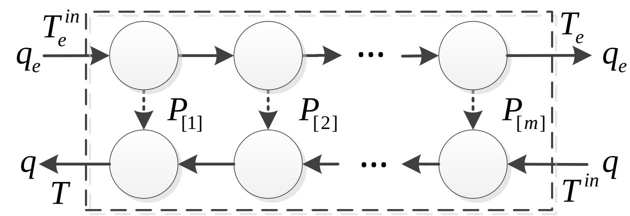

Fig. 17. Model of a heat exchanger adopted from [2] with m cells with power transfer in cell k given by $P_{[k]}$. Since the inflow equals the outflow, flow rates q_e and q are defined only once. T_e^{in} and T^{in} are the inflow and T_e and T the outflow temperatures.

where q^{in} is the inflow rate and $P := \frac{1}{\rho c_p}\dot{\Phi}$ is the heat injection or extraction rate. Note that $\dot{\Phi}$ is considered to be a rate variable instead of the time derivative. Now (46) can be used to model a heat exchanger and storage device.

D. Heat exchanger model

Heat exchangers are used for energy exchange between at least two fluid phase streams (gas or liquid), a hot and a cold stream. Since heat exchangers are placed between every consumer and the network as well as between the producer and the network, the injection and extraction of power only goes via a heat exchanger. In Figure 17 the model for a heat exchanger is presented, which consists of m cells connected in series. Each each cell has two balance volumes separated by a heat conducting element. We restrict ourselves to a model where $m = 1$, and assume that both volumes in the cell are perfectly stirred. To simplify the model, we assume the following quantities to be constant in each cell: volume, mass, specific heat c_P, density ρ, heat transfer coefficient U and contact area of the heat conducting element A_h. Using (46), the dynamics for the temperatures of the two volumes are then given by

$$\dot{T}_e = \frac{q_e}{V_e^h}\left(T_e^{in} - T_e\right) - \frac{1}{V_e^h}P \tag{47}$$

$$\dot{T} = \frac{q}{V^h}\left(T^{in} - T\right) + \frac{1}{V^h}P, \tag{48}$$

where $V_e^h > 0$ and $V^h > 0$ are the volumes in the upper and lower cell, respectively. The heat transfer rate P can be made explicit by

$$P = \frac{UA_h}{c_p\rho}\left(T_e - T\right). \tag{49}$$

At the producer side we make the assumption that, instead of adjusting temperature T_e^{in} and flow rate q_e, we are able to control the heat injection P directly. This results in a heat exchanger being modeled solely by (48).

Remark 7 From (49) we observe that the extraction rate depends linearly on $T_e - T$. Since the consumer can only control q_e and not T_e^{in} we see from (47) that the heat exchange becomes $P = 0$ when $T_e = T = T_e^{in} = T^{in}$. Since this is independent of q_e, this situation should be avoided by providing sufficient heated water T^{in} compared to T_e^{in} but also by setting the flowrate q sufficiently high. A similar remark is made in [2] where it is stated that a heat exchanger is controllable with the exception of the singular point $T_e = T_e^{in}$, $T = T^{in}$.

E. Storage model

To model the heat storage tank we again refer to (46) with $P = 0$ due to the lack of heat generation or absorption (loss). This implies

$$V^{S_h}\dot{T}^{S_h} = q^{in}(T^{in} - T^{S_h}), \tag{50}$$

where V^{S_h} and T^{S_h} are the storage volume and temperature of the hot layer. The volume dynamics of the hot layer is

$$\dot{V}^{S_h} = q^{in} - q^{out}, \tag{51}$$

where q^{in} is the inflow and q^{out} the outflow. The dynamics for T^{S_c} and V^{S_c} are obtained by replacing S_c with S_h in (50) and (51) respectively.

F. Model derivation

When an inlet of a component is fed by multiple pipes each having different flow rates and temperatures the resulting temperature and flow rate are obtained by mass and energy conservation laws. These are given by

$$q^{in} = \sum_{i=1}^{n} q_i^{in}, \tag{52}$$

$$q^{in}T^{in} = \sum_{i=1}^{n} q_i^{in}T_i^{in}, \tag{53}$$

where q_i^{in} is the flow rate and T_i^{in} the temperature of the fluid passing through pipe i, while n is the total number of pipes. By combining (48)-(53) and considering the topology of Figure 1 we obtain model (6).

Appendix B
Proofs

A. Proof of Theorem 1

For system (27) it is well known (see *e.g.* [21]) that a controller of the form (28) solves the robust output regulation problem if there exists Π, Σ and R such that (29) and (30) are satiesfied. Lemma 1.4.2 in [21] states that the regulator equations have a solution for all $\bar{A}$, $\bar{B}$, $\bar{P}$, C and Q if and only if

$$\det\begin{pmatrix} \bar{A} - \lambda I & \bar{B} \\ C & 0 \end{pmatrix} \neq 0, \tag{54}$$

for all λ which are eigenvalues of S. The matrix on the left hand side of (54) is given by

$$\begin{pmatrix} -v_1 - \lambda & 0 & v_1 & \mathbf{0}_{1\times n} & 1 \\ v_1 & -v_1 - \lambda & 0 & \mathbf{0}_{1\times n} & 0 \\ 0 & 0 & -\mathbf{1}_n^T v_2 - \lambda & v_2^T & 0 \\ \mathbf{0}_{n\times 1} & v_2 & \mathbf{0}_{n\times 1} & -\Lambda(v_2) - \lambda I_n & \mathbf{0}_{n\times 1} \\ 0 & 1 & 0 & \mathbf{0}_{1\times n} & 0 \end{pmatrix}. \tag{55}$$

Due to the sparsity of this matrix we observe that the last row and column have only one non-zero entry. Therefore condition (54) holds, if and only if

$$\det\begin{pmatrix} v_1 & 0 & \mathbf{0}_{1\times n} \\ 0 & -\mathbf{1}_n^T v_2 - \lambda & v_2^T \\ \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times 1} & -\Lambda(v_2) - \lambda I_n \end{pmatrix} \neq 0. \tag{56}$$

By assumption we have that all eigenvalues of S have zero real part which implies that $\lambda = bi$, where $b \in \mathbb{R}$. Also by the upper triangular form of (56), we clearly see this is equivalent to

$$v_1(\mathbf{1}_n^T v_2 + bi)\prod_{j=1}^{n}((v_2)_j + bi) \neq 0, \tag{57}$$

where $(v_2)_j$ is the j-th element of vector v_2. Due to $v_1 > 0$ and $(v_2)_j > 0$, it follows that (57) clearly holds, which implies there exists Π, Σ and R such that (29) and (30) are satisfied and therefore proves Theorem 1.

B. Proof of Theorem 2

In order to prove Theorem 2 we consider volumes and flow rates that are not equal to their reference values. We regard the discrepancy between the actual quantities and the reference values as a disturbance in the dynamics. We investigate next the stability properties of these dynamics. First we prove exponential stability of the origin of the system without the disturbance and then prove that convergence to the origin is preserved for the perturbed system. We now introduce some helpful lemmas that substantiate the proof.

Lemma 2 Suppose $\dot{x}(t) = A(t)x(t)$ with $x(t_0) = x_0$ is uniformly exponentially stable[10]. Also suppose there exists a finite constant β such that for all τ we have that

$$\int_{\tau}^{\infty} ||F(\sigma)||d\sigma \leq \beta, \tag{58}$$

holds for all $\tau > t_0$. This implies that

$$\dot{z}(t) = [A(t) + F(t)]\, z(t), \tag{59}$$

is also uniformly exponentially stable.

Proof: See [22, Theorem 8.5]. □

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Lemma 3 Consider system (12) with v_1 as in (21), then there exist $\beta_1, \beta_2, \beta_3 \in \mathbb{R}_{>0}$ such that

$$\int_{\tau}^{\infty} \left|\frac{1}{z_1(\sigma)} - \frac{1}{z_1^*}\right| d\sigma \leq \beta_1, \tag{61}$$

$$\int_{\tau}^{\infty} \left|\frac{1}{z_1(\sigma)}\left(v_1(\sigma) - v_1^*\right)\right| d\sigma \leq \beta_2, \tag{62}$$

and

$$\int_{\tau}^{\infty} |z_1(\sigma) - z_1^*|\, d\sigma \leq \beta_3. \tag{63}$$

[10] A linear time varying system is *uniformly exponentially stable* if there exist finite positive constants γ and λ such that its solution satisfies

$$\|x(t)\| \leq \gamma e^{-\lambda(t-t_0)}\|x_0\| \qquad t \geq t_0, \tag{60}$$

for any initial condition x_0 and any $t_0 > 0$.

Proof: Observe that (63) is equivalent to finding β_3 such that

$$|(z_1(0)-z_1^*)|\int_\tau^\infty e^{-\alpha t}d\sigma \le \beta_3. \tag{64}$$

Since

$$|(z_1(0)-z_1^*)| \le V^{\max}-2V^{\min}, \tag{65}$$

β_3 can be taken as

$$\beta_3 = \frac{V^{\max}-2V^{\min}}{\alpha}. \tag{66}$$

Consider now (61) for which it holds that

$$\int_\tau^\infty \left|\frac{1}{z_1(\sigma)}-\frac{1}{z_1^*}\right| d\sigma \le \int_0^\infty \left|\frac{z_1^*-z_1(\sigma)}{z_1^* z_1(\sigma)}\right| d\sigma. \tag{67}$$

By Lemma 1 we know we can bound

$$\frac{1}{z_1^* z_1(\sigma)} \le \frac{1}{(V^{\min})^2} \quad \text{for all } \sigma \ge 0, \tag{68}$$

which, in light of (65), (67) and (22), gives

$$\begin{aligned}&\int_\tau^\infty \left|\frac{1}{z_1(\sigma)}-\frac{1}{z_1^*}\right| d\sigma \\ \le& \frac{1}{(V^{\min})^2}\int_0^\infty e^{-\alpha\sigma}\left|z_1^*-z_1(0)\right| d\sigma.\end{aligned}$$

Therefore taking

$$\beta_1 = \frac{\eta}{\alpha} \tag{69}$$

with $\eta = \frac{V^{\max}-2V^{\min}}{(V^{\min})^2}$ implies that (61) is satisfied. Now finally consider (62) and observe that by definition (21) we have that

$$\begin{aligned}&\frac{1}{\alpha z_1^*}\int_\tau^\infty \left|\frac{1}{z_1(\sigma)}\left(v_1(\sigma)-v_1^*\right)\right| d\sigma \\ =&\int_\tau^\infty \left|\frac{1}{z_1(\sigma)}-\frac{1}{z_1^*}\right| d\sigma.\end{aligned}$$

For this reason along with (69), taking $\beta_2 := \frac{\eta}{\alpha^2 V^{\min}}$ implies that (61) is satisfied. This concludes the proof. □

Lemma 4 The origin of

$$\dot\zeta = (A' + F'(t))\zeta, \tag{70}$$

is uniformly exponentially stable with A' and F' given in (35) and (36).

Proof: Let $A(t)$ and $F(t)$ from Lemma 2 be equal to A' and $F'(t)$ as given in (35) and (36). Therefore it is trivially uniformly exponentially stable[10]. Clearly A' is a Hurwitz matrix due to the designed IM controller. By defining

$$G = \begin{pmatrix} KC & H \\ 0 & 0\end{pmatrix},$$

and using $\|AB\| \le \|A\|\|B\|$, condition (58) is satisfied if

$$\begin{aligned}&\int_\tau^\infty \|F'(\sigma)\| d\sigma \\ \le& \int_\tau^\infty \|A_e(\sigma)\|d\sigma + \|G\|\int_\tau^\infty \|B_e(\sigma)\|d\sigma \le \beta.\end{aligned} \tag{71}$$

Since $v_2 = v_2^*$, we can write $A(v) = \Lambda(v^e)A(v^*)$ where

$$v^e = \begin{pmatrix} \frac{v_1}{v_1^*} & \frac{v_1}{v_1^*} & 1 & \mathbf{1}_n^T \end{pmatrix}^T \tag{72}$$

which implies

$$\|A_e\| \le \|M(z)^{-1}\Lambda(v^e) - M(z^*)^{-1}\|\|A(v^*)\|, \tag{73}$$

and similarly

$$\|B_e\| \le \|M(z)^{-1} - M(z^*)^{-1}\|\|B\|. \tag{74}$$

Since all time varying entries on the right hand side of both (73) and (74) enter in diagonal form, (71) is satisfied if there exists $\beta_{1,i}$ and $\beta_{2,i}$ such that

$$\begin{aligned}\int_\tau^\infty \left|\frac{v_i^e(\sigma)}{M(z(\sigma))_{ii}} - \frac{1}{M(z^*)_{ii}}\right| d\sigma \le \beta_{1,i} \\ \int_\tau^\infty \left|\frac{1}{M(z(\sigma))_{ii}} - \frac{1}{M(z^*)_{ii}}\right| d\sigma \le \beta_{2,i},\end{aligned} \tag{75}$$

is satisfied for all i, where $M(z(\sigma))_{ii}$ is the i-th diagonal component of $M(z(\sigma))$. Notice that

$$\begin{aligned}&\frac{v_i^e(\sigma)}{M(z(\sigma))_{ii}} - \frac{1}{M(z^*)_{ii}} \\ =&\begin{cases} \frac{1}{v_1^* V^P}\left(v_1 - v_1^*\right) & \text{for } i=1 \\ \frac{1}{v_1^*}\left(\frac{v_1}{z_1} - \frac{v_1^*}{z_1^*}\right) & \text{for } i=2 \\ \frac{1}{z_2} - \frac{1}{z_2^*} & \text{for } i=3 \\ 0 & \text{for } i \ge 4,\end{cases}\end{aligned} \tag{76}$$

and

$$\frac{1}{M(z(\sigma))_{ii}} - \frac{1}{M(z^*)_{ii}} = \begin{cases} 0 & \text{for } i=1 \\ \frac{1}{z_1} - \frac{1}{z_1^*} & \text{for } i=2 \\ \frac{1}{z_2} - \frac{1}{z_2^*} & \text{for } i=3 \\ 0 & \text{for } i \ge 4,\end{cases} \tag{77}$$

Recall that $v_1^* = \mathbf{1}_n^T v_2^*$, by definition of (21) we know

$$\frac{1}{v_1^* V^P}\left(v_1 - v_1^*\right) = \frac{\alpha}{v_1^* V^P}\left(z_1^* - z_1\right). \tag{78}$$

and

$$\frac{1}{v_1^*}\left(\frac{v_1}{z_1} - \frac{v_1^*}{z_1^*}\right) = \frac{\alpha}{\mathbf{1}_n^T v_2^*}\frac{z_1^* - z_1)}{z_1} + \frac{1}{z_1} - \frac{1}{z_1^*}. \tag{79}$$

Evaluating (75) in light of (76)-(79), we see it is sufficient to prove there exists a $\tilde\beta_1$, $\tilde\beta_2$, $\tilde\beta_3$ and $\tilde\beta_4$ such that

$$\int_\tau^\infty \left|\frac{z_1^* - z_1}{z_1}\right| d\sigma \le \tilde\beta_1, \tag{80}$$

$$\int_\tau^\infty |z_1 - z_1^*| d\sigma \le \tilde\beta_2, \tag{81}$$

$$\int_\tau^\infty \left|\frac{1}{z_1} - \frac{1}{z_1^*}\right| d\sigma \le \tilde\beta_3, \tag{82}$$

$$\int_\tau^\infty \left|\frac{1}{z_2} - \frac{1}{z_2^*}\right| d\sigma \le \tilde\beta_4. \tag{83}$$

By Lemma 3 we have that (80), (81) and (82) are satisfied. Due to the similar dynamics of z_1 and z_2 given by (12), it is easily checked that also (83) is satisfied. Therefore we can conclude that there exist β that satisfies condition (58) which concludes the proof. $\square$

Now we are ready to prove Theorem 1. To this end we define

$$u' := B'w, \tag{84}$$

such that system (34) becomes

$$\dot{\zeta} = (A' + F'(t))\zeta + u'. \tag{85}$$

We know by Lemma 4 that the origin of (85) is uniformly exponentially stable when $u' = 0$. For this reason (see for instance [22]), there exists a state transition matrix $\Phi(t,s)$ and parameters $\mu \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{>0}$ such that

$$||\Phi(t,t_0)|| \leq \mu e^{-\lambda(t-t_0)}. \tag{86}$$

We also know that the solution $\zeta(t)$ is given by

$$\zeta(t) = \Phi(t,t_0)\zeta(t_0) + \int_{t_0}^{t} \Phi(t,\tau)u'(\tau)d\tau.$$

Since we can bound $||B'(t)w(t)|| < \gamma < \infty$ for any $t > 0$ we can also bound $||\zeta||$ by (see *e.g.* [23])

$$\begin{aligned} ||\zeta|| &\leq ||\Phi(t,t_0)\zeta(t_0) + \int_{t_0}^{t} \Phi(t,\tau)u'(\tau)d\tau|| \\ &\leq \mu e^{-\lambda(t-t_0)}||\zeta(t_0)|| + \int_{t_0}^{t} \mu e^{-\lambda(t-\tau)}||u'(\tau)||d\tau \\ &\leq \mu e^{-\lambda(t-t_0)}||\zeta(t_0)|| + \frac{\mu\gamma}{\lambda} \sup_{\tau\in[t_0,t)} ||u'(\tau)||, \end{aligned}$$

implying that $||\zeta||$ is bounded. Now consider the defined input (84). Since $\lim_{t\to\infty} z = z^*$ and $\lim_{t\to\infty} v = v^*$ we know that $\lim_{t\to\infty} A_e(t) = 0$, $\lim_{t\to\infty} B_e(t) = 0$ and $\lim_{t\to\infty} P_e(t) = 0$. This implies $\lim_{t\to\infty} B'(t) = 0$ from which we conclude

$$\lim_{t\to\infty} ||u'|| = \lim_{t\to\infty} ||B'w|| = 0. \tag{87}$$

By the boundedness of $||\zeta||$ and (87), we conclude that $\lim_{t\to\infty} ||\zeta|| = 0$, which proves Theorem 2.

Appendix C
Controller design

In this section the numerical values of controller (28) are given for the case studies presented in Section VI. These numerical values have been obtained by following the design procedure detailed in [21, Section 1.5]. This controller is given by

$$\begin{aligned} F &= \begin{pmatrix} 2 & -3 & 1 & -40 & -400 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -200 & 1 \\ 0 & 0 & 0 & -10000 & 0 \end{pmatrix} \\ G &= \begin{pmatrix} 0 & 0 & 0 & 600 & 90000 \end{pmatrix}^T \\ H &= \begin{pmatrix} 3 & -3 & 1 & -30 & -300 \end{pmatrix}, \end{aligned} \tag{88}$$

and $K = 0$. The second case study with time varying demand is presented in Section VI-B. In contrast to (88), which is designed for $S = 0$, we have that $S \neq 0$. As a consequence a different controller is obtained, and is given by

$$\begin{aligned} F &= \begin{pmatrix} 4 & -10 & 10 & -5 & 1 & -90 & -900 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -600 & 1 \\ 0 & 0 & 0 & 0 & 0 & -90000 & 0 \end{pmatrix} \\ G &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 600 & 90000 \end{pmatrix}^T \\ H &= \begin{pmatrix} 5 & -10 & 10 & -5 & 1 & -90 & -900 \end{pmatrix}, \end{aligned}$$

with again $K = 0$. Finally in Section VI-C the last case study with a real demand patern is presented. The controller that is used is given by

$$\begin{aligned} F &= \begin{pmatrix} 2 & -3 & 1 & -80 & -800 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1800 & 1 \\ 0 & 0 & 0 & -810000 & 0 \end{pmatrix} \\ G &= \begin{pmatrix} 0 & 0 & 0 & 1800 & 810000 \end{pmatrix}^T \\ H &= \begin{pmatrix} 3 & -3 & 1 & -80 & -800 \end{pmatrix}, \end{aligned} \tag{89}$$

with $K = 0$. The controller in (88) has the same structure as the one in (89) since they are both computed relatively to $S = 0$. However, the gain parameters in (89) have been chosen higher as to account for all the possible volumes setpoints which are reported in Table I.

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