3. Mathematical Structure of the equations

Before one attempts to solve the set of governing partial differential equations describing a viscoelastic fluid one has to know how the variables or combinations of the variables are connected and how they react to perturbations. In the first section the notion of the type of a partial differential equation will be explained. In the following two sections the type of a Maxwell type model and a Jeffrey's type model will be investigated. The type of a set of partial differential equations gives an indication how the equations react to very small waves; either the very small waves are convected through the computational domain with a finite velocity, indicating that the set of partial differential equations is of hyperbolic type, or the waves are damped by the equations indicating that the governing equations are of parabolic type. If the convective velocity is infinite and a variable in the interior of the computational domain is directly influenced by its value at the boundary of the computational domain the set of equations is said to be of elliptic type. The classification according to type is established by the symbol of the set of partial differential equations. It turns out that the equations describing viscoelastic fluids are of mixed elliptic/hyperbolic type for Maxwell type models and elliptic/hyperbolic/parabolic for the Jeffrey type models. The elliptic part of the equations is associated with infinite speed of compression waves, whereas the hyperbolic part can be divided in a part which is convected along the streamlines and a part which constitutes a wave equation. This wave equation is connected to the elastic shear waves in the fluid and may be further divided in space in a part where the shear wave velocity exceeds the fluid velocity, the subcritical region, and a part where the velocity of the shear waves is smaller than the fluid velocity, the supercritical region. In the case of stationary equations, in which the local time derivative is set to zero, the transition from subcritical to supercritical changes the type of the set of equations from elliptic to hyperbolic, but in the time dependent case the equations remain hyperbolic and only the domain of dependence changes.

The distinction between the various types is necessary, since variables belonging to a different type require a different treatment numerically. So the results in this chapter will be used in chapter 5 to obtain a sound numerical scheme for the system of partial differential equations governing viscoelastic fluid flow.
3.1 Classification of PDEs

In order to abbreviate all the partial derivatives the notion of multi-indices for PDEs is adopted. A multi-index is given by
\[ \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n), \]
where all \( \alpha_i \) are non-negative integers. The order of \( \alpha \) is computed as
\[ |\alpha| = \sum_{i=0}^{n} \alpha_i. \]

Next the partial differential operator \( D^\alpha \) is introduced by
\[ D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_0^{\alpha_0} \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}. \]

A linear \( m \)-th order PDE may now be expressed as
\[ P(\vec{x}, D)u = \sum_{|\alpha| \leq m} a_\alpha(\vec{x})D^\alpha u, \]
where \( u : \mathbb{R}^{n+1} \to \mathbb{R}, \vec{x} = (x_0, x_1, \ldots, x_n) \), in which \( x_0 \) is the time coordinate and \( x_1, \ldots, x_n \) are the spatial coordinates. The symbol or Fourier symbol of the PDE operator \( P(\vec{x}, D) \) is defined as
\[ P(\vec{x}, i\vec{\xi}) := \sum_{|\alpha| \leq m} a_\alpha(\vec{x})(i\vec{\xi})^\alpha, \]
where
\[ \vec{\xi}^\alpha = \xi_0^{\alpha_0} \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}. \]

The principal part of the symbol is defined by
\[ P_p(\vec{x}, i\vec{\xi}) := \sum_{|\alpha| = m} a_\alpha(\vec{x})(i\vec{\xi})^\alpha. \]

For example the symbol of the heat equation \( u_t - ku_{xx} \) is given by \( i\xi_0 + k\xi_1^2 \) and its principal part simply by \( k\xi_1^2 \).

The symbol of a PDE is derived from the representation of the solution in terms of Fourier modes. Suppose the solution \( u(\vec{x}) \) possesses a Fourier representation, i.e.
\[ u(\vec{x}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n+1}} \hat{u}(\vec{\xi}) e^{-i\vec{\xi} \cdot \vec{x}} d\vec{\xi}, \]
then
\[ P(\vec{x}, D)u(\vec{x}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{u}(\vec{\xi}) P(\vec{x}, D)e^{-i\vec{\xi} \cdot \vec{x}} d\vec{\xi} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{u}(\vec{\xi}) P(\vec{x}, i\vec{\xi}) e^{i\vec{\xi} \cdot \vec{x}} d\vec{\xi}. \]

So the Fourier transform of \( P(\vec{x}, D)u(\vec{x}) \) is given by \( P(\vec{x}, i\vec{\xi})\hat{u}(\vec{\xi}) \), which means that if for all \( \vec{x} \) and \( \vec{\xi} \) \( P(\vec{x}, i\vec{\xi})\hat{u}(\vec{\xi}) = 0 \), then \( P(\vec{x}, D)u(\vec{x}) = 0 \).
Attention will be confined to a set of non-linear partial differential equations of first order which are quasi-linear. Such a set of equations may be written as

$$\sum_{k=0}^{n} A_k \frac{\partial \vec{u}}{\partial x_k} = \vec{f},$$

(3.1)

where \( \vec{x} = (t, x_1, x_2, \ldots, x_n) \), \( \vec{u} = (u_1, u_2, \ldots, u_m) \) and the \( m \times m \)-matrices \( A_k = A_k(\vec{x}, \vec{u}) \) only depend on \( \vec{x} \) and \( \vec{u} \), but not on the derivatives of \( \vec{u} \). If \( A_k \) is independent of \( \vec{u} \) and \( \vec{f} = C \vec{u} \), then the system is said to be linear. If, in addition, this system may be written in the form

$$\sum_{k=0}^{n} \frac{\partial F_k(\vec{u})}{\partial x_k} = \vec{f},$$

then the system is said to be in conservation form.

The system (3.1) is called elliptic at \( \vec{x} \) if

$$\det \left( \sum_{k=0}^{n} A_k(\vec{x}, \vec{u}) \xi_k \right) \neq 0, \quad \forall \xi \in \mathbb{R}^{n+1} \setminus \{0\}.$$

If, on the other hand, there exist \( n \) linear independent vectors \( \vec{\xi} \) with \( \vec{\xi}_0 = 1 \) which satisfy

$$\det \left( \sum_{k=0}^{n} A_k(\vec{x}, \vec{u}) \xi_k \right) = 0, \quad \vec{\xi} \in \mathbb{R}^{n+1},$$

then the system is called hyperbolic in time. If this is the case one can find for every vector \( \vec{\kappa} \in \mathbb{R}^n \) \( n \) real numbers \( \alpha_i \) such that

$$\det \left( \alpha_i A_0 - \sum_{k=1}^{n} A_k \kappa_k \right) = 0.$$

This means that for every \( \alpha_i \) there exists a non-trivial row vector \( \vec{v}_i \) such that

$$\vec{v}^{(i)} \sum_{k=1}^{n} A_k \kappa_k = \alpha_i v^{(i)} A_0,$$

in which in the right hand side no summation is intended. The real values \( \alpha_i \) are called the characteristic velocities, whereas \( \sum_{k=1}^{m} v^{(i)}_{k} u_k \) is the \( i \)-th characteristic variable belonging to the characteristic velocity \( \alpha_i \).

In order to illustrate the above definitions it is instructive to look at the following examples

**Example 1.** This example will be used in chapter 5 where the discretization of the shear stress and the shear rate may be written in this form. The one dimensional wave equation, \( n = 1 \), is given by

$$\frac{\partial^2 u}{\partial t^2} - c_s^2 \frac{\partial^2 u}{\partial x^2} = 0.$$

Introducing \( v = u_t \) and \( w = u_x \) this equation may be written as

$$\frac{\partial v}{\partial t} - c_s^2 \frac{\partial w}{\partial x} = 0.$$
This leads to the differential operator $P$

$$P(x, D) = P(D) = \left( \begin{array}{cc} \frac{\partial}{\partial t} & -c^2 \frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x} & \frac{\partial}{\partial t} \end{array} \right),$$

and so its symbol

$$P(i\xi) = i \left( \begin{array}{cc} \xi_0 & -c^2 \xi_1 \\ -\xi_1 & \xi_0 \end{array} \right).$$

For $\xi_0 = \pm c\xi_1$ this operator admits a non-trivial solution $\tilde{u}(\xi)$, so this system is hyperbolic. This may also be written as an eigenvalue problem with $A_0 = I$ and $A_1$,

$$A_1 = \left( \begin{array}{cc} 0 & c^2 \\ 1 & 0 \end{array} \right).$$

The eigenvalues $\alpha_i$ follow from

$$\det (\alpha_i I - A_1) = \alpha_i^2 - c^2 = 0.$$

So $\alpha_1 = c$ and $\alpha_2 = -c$ and if $c^2 \geq 0$ then this system is hyperbolic. The characteristic variables may be found by computing the accompanying left eigenvector. These left eigenvectors are $\tilde{v}_1 = (1, c)$ and $\tilde{v}_2 = (1, -c)$, leading to the characteristic variables $R_1 = v + cw$ and $R_2 = v - cw$. Writing the system in terms of $R_1$ and $R_2$ instead of $v$ and $w$ gives

$$\frac{\partial R_1}{\partial t} = \frac{\partial v}{\partial t} + c \frac{\partial w}{\partial t} = c^2 \frac{\partial w}{\partial x} + c \frac{\partial v}{\partial x} = c \frac{\partial R_1}{\partial x},$$

and similarly

$$\frac{\partial R_2}{\partial t} = -c \frac{\partial R_2}{\partial x}.$$

So the equations for $R_1$ and $R_2$ are totally decoupled. The quantity $R_1$ moves with a velocity $c$ to the right, whereas $R_2$ moves with a velocity $c$ to the left, which is consistent with d’Alembert’s solution of the wave equation.

**Example 2.** This example will be used in the rest of this chapter and in chapter 5, since the pressure in the system of PDEs satisfies the Poisson equation, which is the inhomogeneous form of the equation to be discussed here. The Laplace equation in two dimensions, $n = 2$ is given by

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Again after introducing the auxiliary variables $v = \phi_x$ and $w = \phi_y$ this leads to a first order system in which $A_0 = 0$, $A_1 = I$ and $A_2$:

$$A_2 = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right).$$
This system is known as the Cauchy-Riemann equations. There are no real eigenvalues \( \alpha \) so this system is elliptic. Equivalently the first order system may be written as \( P \bar{u} = 0 \), in which the differential operator may be represented as

\[
P(\bar{\xi}, D) = P(D) = \left( \begin{array}{cc}
\frac{\partial}{\partial \bar{x}} & -\frac{\partial}{\partial \bar{y}} \\
\frac{\partial}{\partial \bar{y}} & \frac{\partial}{\partial \bar{x}}
\end{array} \right) \quad \Rightarrow \quad P(i\bar{\xi}) = i \left( \begin{array}{cc}
\xi_1 & -\xi_2 \\
\xi_2 & \xi_1
\end{array} \right).
\]

The determinant of the symbol is equal to zero if \( \xi_1^2 + \xi_2^2 = 0 \), so \( \det P(\bar{\xi}) \neq 0 \) for all \( \xi_1, \xi_2 \in \mathbb{R} \setminus 0 \), which shows again that the differential equation is elliptic.

Although the above two examples suggest that every system is either of elliptic or hyperbolic type this is not true. First order systems may be of mixed type, i.e. both elliptic and hyperbolic. Furthermore the matrix \( A_0 \) may be singular even though the system is time dependent. An example of this type is given by the inviscid, incompressible Euler equations in two dimensions, \( n = 2 \).

**Example 3.** Since for incompressible materials the system of governing PDEs is usually of mixed type it is illustrative to investigate the inviscid, incompressible Euler equations in 2D

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
\rho \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = 0, \\
\rho \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = 0
\]

where \( \rho \) is the density, \( u \) the component of the velocity in the \( x \)-direction, \( v \) the component of the velocity in the \( y \)-direction and \( p \) the pressure. In fact this model fits in the framework outlined in the previous chapter where the extra stress tensor \( \tau \equiv 0 \). This set of first order quasi-linear equations may be put in the form (3.1) by setting

\[
A_0 = \begin{pmatrix} 
0 & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & \rho 
\end{pmatrix}, \quad A_1 = \begin{pmatrix} 
0 & 1 & 0 \\
1 & \rho u & 0 \\
0 & 0 & \rho u 
\end{pmatrix}, \quad A_2 = \begin{pmatrix} 
0 & 0 & 1 \\
0 & \rho v & 0 \\
1 & 0 & \rho v 
\end{pmatrix}.
\]

The eigenvalues \( \alpha \) depend on the linear combination of \( A_1 \) and \( A_2 \) determined by the vector \( \vec{\kappa} \).

\[
\det (\alpha A_0 + \kappa_1 A_1 + \kappa_2 A_2) = (\alpha - \rho \vec{\kappa} \cdot \bar{u}) = 0,
\]

since \( \|\vec{\kappa}\| = 1 \). So there is only one real eigenvalue \( \alpha \) and thus one characteristic variable. The remaining part of this set of PDEs is elliptic. The elliptic part represents the infinitely fast compression waves, due to the incompressibility constraint \( \text{div} \bar{u} = 0 \), and the system becomes totally hyperbolic again if a small amount of compressibility is allowed. Since the formation of the characteristic equation is independent of the pressure and the incompressibility constraint, they must be contained in the elliptic part. This can be shown by taking the divergence of the momentum equation which gives

\[
\Delta p = -\rho \vec{\nabla} \cdot [(\vec{u} \cdot \vec{\nabla}) \vec{u}],
\]
which is a Poisson equation for the pressure $p$ which was shown to be elliptic. Determination of the type in terms of the simple wave solution only gives the hyperbolic part (in time) of the equations whereas determination in terms of the symbol also yields the elliptic part. The differential operator for the incompressible Euler equation is given by

$$
P(x, \phi, D) = \begin{pmatrix}
0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial x} & \rho \frac{\partial^2}{\partial x^2} & 0 \\
\frac{\partial}{\partial y} & 0 & \rho \frac{\partial^2}{\partial y^2}
\end{pmatrix},
$$

and its symbol by

$$
P(x, \phi, i\xi) = i \begin{pmatrix}
0 & \xi_1 & \xi_2 \\
\xi_1 & \rho(\xi_0 + u\xi_1 + v\xi_2) & 0 \\
\xi_2 & 0 & \rho(\xi_0 + u\xi_1 + v\xi_2)
\end{pmatrix},
$$
in which $\phi = (p, u, v)^T$. This operator yields a non trivial solution if $\det P(x, i\xi) = 0$, leading to

$$
\left(\xi_1^2 + \xi_2^2\right) (\xi_0 + u\xi_1 + v\xi_2) = 0
$$
The first factor is equal to the term resulting from the elliptic Laplace equation, signifying that the set of PDEs contains an elliptic part, whereas the second factor admits real solutions in which $\xi_0$ may be chosen equal to 1, indicating that this part is hyperbolic in time.

A different approach towards systems of partial differential equations of mixed elliptic/hyperbolic type, is the following. Suppose, in the previous example, the set of velocity vectors in which the solution has to be found, satisfies $\text{div} \vec{u} = 0$. Then the first equation in the incompressible Euler equations is superfluous and the pressure gradient, which acts as a Lagrange multiplier to satisfy the incompressibility constraint may be deleted from the momentum equation. The equations in this case reduce to

$$
\frac{D\vec{u}}{Dt} = 0.
$$
This system is purely hyperbolic with two characteristic velocities, $\alpha = \vec{\kappa} \cdot \vec{u} (2x)$, instead of one. In the numerical treatment of the incompressible Euler equations this idea is employed. One assumes that $\text{div}_h \vec{u}_h = 0$, calculates $\vec{u}_{h+1}^n$ in those two characteristic directions, which will in general not satisfy the incompressibility constraint and then use the pressure gradient to ensure that the new discrete velocity field satisfies $\text{div}_h \vec{u}_{h+1}^n = 0$. The SIMPLE-type methods are based on this approach (see Pantakar [43]).

If lower order terms appear in the differential equation, they usually do not affect the type of the differential equation, so only the principal part of the symbol determines the type. However, it is not always true that the principal part of the symbol is equal to the symbol of the differential equation in which only the highest order derivatives are retained as will be shown in the next example.

**Example 4.** This example will serve as an illustration of the determination of the type of Jeffrey type models, which contain an additional Newtonian viscosity term. In example 1
the wave equation was rewritten as a first order system but the wave equation is obviously equivalent to the following system

\[
\frac{\partial v}{\partial t} - w = 0, \\
\frac{\partial w}{\partial t} - c^2 \frac{\partial^2 v}{\partial x^2} = 0.
\]

This corresponds to the differential operator

\[
P(D) = \begin{pmatrix}
\frac{\partial}{\partial x} & -1 \\
-c^2 \frac{\partial^2}{\partial x^2} & \frac{\partial}{\partial t}
\end{pmatrix}.
\]

Simply taking the highest order derivative as the principal part of the differential operator would give

\[
P(D) = \begin{pmatrix}
0 & 0 \\
-c^2 \frac{\partial^2}{\partial x^2} & 0
\end{pmatrix},
\]

with symbol

\[
P_p(i\xi) = \begin{pmatrix}
0 & 0 \\
c^2 \xi^2 & 0
\end{pmatrix}.
\]

Although this shows that the system is hyperbolic for all \(\xi\), the relation between \(\xi_0\) and \(\xi_1\) is lost. If however the symbol is taken of the original differential operator which still contains the lower order derivatives and the determinant is computed then this gives

\[
det P(i\xi) = \begin{vmatrix}
i\xi_0 & -1 \\
c^2 \xi_1 & i\xi_0
\end{vmatrix} = -c^2 \xi_0^2 + c^2 \xi_1^2 = 0.
\]

This result is the same as the system of first order partial differential equation which is necessary since the type of an equation does not depend in which format the equation is written. So the procedure is to determine the symbol of the differential operator, to compute its determinant, which is a polynomial equation in \(\xi_0, \xi_1, \ldots, \xi_n\) and to calculate the zeros of the principal part of this polynomial equation.

### 3.2 The type of the UCM-model

The UCM model may be written in the following form

\[
A_0 \frac{\partial \tilde{\phi}}{\partial t} + A_1 \frac{\partial \tilde{\phi}}{\partial x} + A_2 \frac{\partial \tilde{\phi}}{\partial y} + S \tilde{\phi} = \bar{f},
\]

where \(\tilde{\phi} = (p, u, v, \tau_{11}, \tau_{22}, \tau_{12})^T\) and \(\bar{f} = \tilde{\phi}\). The matrices appearing in this PDE are

\[
A_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho & 0 & 0 & 0 & 0 \\
0 & 0 & \rho & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]
Since one is interested in the behaviour for very small waves, i.e., for \( \eta \ll 1 \), a simple wave solution of the form

\[
\phi = e^{i(\chi t + \vec{k} \cdot \vec{x})}
\]

divides this equation by \( \vec{k} \). Whereas the last two eigenvalues constitute a certain quantity along the streamlines, whereas the last two eigenvalues constitute a principal part of the set of PDEs is retained. A non-trivial solution of this form exists if for every direction \( \vec{k} \) a \( \chi \) exists such that

\[
\det (\chi A_0 + \kappa_1 A_1 + \kappa_2 A_2) = 0.
\]

This requirement leads to

\[
\begin{vmatrix}
0 & \kappa_1 & \kappa_2 & 0 & 0 & 0 \\
\kappa_1 & \rho \nu & 0 & -\kappa_1 & 0 & -\kappa_2 \\
\kappa_2 & 0 & \rho \nu & 0 & -\kappa_2 & -\kappa_1 \\
0 & -2\kappa_1(\tau_{11} + \eta/\lambda) - 2\kappa_2\tau_{12} & 0 & 0 & -\kappa_1 & 0 \\
0 & 0 & -2\kappa_2(\tau_{22} + \eta/\lambda) - 2\kappa_1\tau_{12} & 0 & -\kappa_1 & 0 \\
0 & \kappa_2(\tau_{22} + \eta/\lambda) & -\kappa_1(\tau_{11} + \eta/\lambda) & 0 & 0 & \nu
\end{vmatrix} = 0,
\]

in which \( \nu = \vec{k} \cdot \vec{u} - \chi \). Evaluating this determinant gives

\[
\nu^2 \left[ \vec{k}^T \mathbf{\tau_A} \vec{k} - \rho \nu^2 \right] = 0,
\] (3.2)

so if \( \mathbf{\tau_A} \) is positive definite, then there are 4 real eigenvalues \( \chi \) for every direction \( \vec{k} \). Either \( \chi = \vec{k} \cdot \vec{u} \) (2x) or \( \chi = \vec{k} \cdot \vec{u} \pm \sqrt{\vec{k}^T \mathbf{\tau_A} \vec{k} / \rho} \). The first two eigenvalues represent convection of a certain quantity along the streamlines, whereas the last two eigenvalues constitute a
wave equation. If at least one of the eigenvalues of $\mathbf{A}$ becomes negative, then the wave solution $\hat{\phi}(\vec{x}, t)$ grows exponentially in time proportional to the wave number $||\vec{k}||$, thus violating well-posedness, as will be shown in a later section.

Another way to analyze the UCM-model is to assume that there exists a non-trivial solution $\hat{\phi}$ for all $\vec{k}$, which do not have to be necessarily large. In this case the lower order terms may not be neglected. The requirement for a non-trivial solution is given by

$$\det (\chi A_0 + \kappa_1 A_1 + \kappa_2 A_2 - iS) = 0,$$

where the influence of $f$ may be neglected or treated as an intermediate initial value by means of Duhamel’s principle. Evaluating this determinant yields

$$\left(\nu - \frac{i}{\lambda}\right)^2 \left[\vec{k}^T \mathbf{A} \vec{k} - \rho \nu \left(\nu - \frac{i}{\lambda}\right)\right] = 0.$$

The solutions for $\chi$ are

- $\chi = \vec{k} \cdot \vec{u} + \frac{1}{\lambda} \left(2\kappa\right)$,
- $\chi = \vec{k} \cdot \vec{u} + \frac{1}{2\lambda} \pm \sqrt{\left[\vec{k}^T \mathbf{A} \vec{k}\right] / \rho - 1/(4\lambda^2)}$.

If now the limit $||\vec{k}|| \to \infty$ is taken the same characteristic velocities are retrieved as in the case where first the limit was taken and then the eigenvalues were computed. The above characteristic velocities for finite wave numbers show that if $\mathbf{A}$ is positive definite $\text{Im}(\chi) \geq 0$ for all $\vec{k}$. This means that all waves damp exponentially in time. Furthermore it shows that for very small wave numbers the square root, which represents the elastic shear velocity, becomes imaginary. In this case there is only convection along the streamlines and the elastic velocities add to the imaginary damping term. Throughout this discussion it was assumed that $\mathbf{A}$ is symmetric and consequently possesses real eigenvalues, however numerically this need not be the case. In fact, for many finite difference/volume schemes the discrete form of $\mathbf{A}$ is not symmetric, which may cause the numerical scheme to become unstable for a certain Weissenberg number, although the tensor $\mathbf{A}$ is pointwise symmetric and positive definite. This problem will be addressed in chapter 5.

**Remark 1** A similar analysis may be carried out for the LCM-model or the generalized Maxwell model, containing the slip parameter $a$, considered in the previous chapter. Since only attention will be paid to the UCM-model these calculations are not given here. The interested reader will find these calculations in Joseph [30].

**Remark 2** Analogous results may be obtained for the Maxwell models with more than one relaxation time.

**Remark 3** For infinitely small waves, or equivalently $||\vec{k}|| \to \infty$, the non-linear coefficients in the set of PDEs may be taken constant. This procedure is known as the *frozen coefficient* approach. However, if finite wavelengths are used, then the coefficients may not be taken as constants. The coefficients may also be written as a sum of Fourier modes and the problem of interacting waves appears (aliasing), which may violate the results above. Furthermore, if the problem is well-posed in terms of its frozen coefficient approximation,
this does not imply that the equation with variable coefficients is also well-posed. An example is given in Kreiss & Lorenz [35].

The change of type in the stationary equations, which occurs when the flow changes from subcritical to supercritical is contained in the wave equation. The product of the last two eigenvalues $\chi$ multiplied by $\rho$, in which $\|\vec{\kappa}\|$ is supposed to be very large, is given by

$$\rho(\vec{\kappa} \cdot \vec{u})^2 - \vec{\kappa}^T \tau_A \vec{\kappa} = \vec{\kappa}^T (\rho \vec{u} \otimes \vec{u} - \tau_A) \vec{\kappa}.$$ 

If this quantity is negative for all $\vec{\kappa}$ then the two eigenvalues have different sign for all values of $\vec{\kappa}$. This is the case when $\tau_B = \rho \vec{u} \otimes \vec{u} - \tau_A$ is negative definite. In this case a point $\vec{x}$ is contained in the interior of its domain of dependence and the flow is called subcritical, the value of the characteristic variable at $\vec{x}$ is influenced by all its surrounding points. If $\tau_B$ becomes indefinite there exist directions $\vec{\kappa}$ for which both eigenvalues $\chi$ have the same sign, either both positive or both negative. In this case the characteristic variables belonging to these characteristic velocities are influenced from one direction, since there is no information coming from the opposite direction. The flow is called supercritical for that particular direction $\vec{\kappa}$. If $\tau_B$ is positive definite then the flow is supercritical for all directions $\vec{\kappa}$.

The characteristic variables are obtained by computing the left eigenvectors belonging to the eigenvalues $\chi$. The two eigenvectors belonging to the convective speeds along the streamlines are

$$\vec{v}_1 = [0, 0, 0, l_4, l_5, l_6],$$

where

$$l_4 = -(\tau_{11} + \frac{\eta}{\lambda})(\tau_{22} + \frac{\eta}{\lambda})\kappa_1\kappa_2,$$

$$l_5 = 2(\tau_{11} + \frac{\eta}{\lambda})^2\kappa_1^2 + 2\tau_{12}\tau_{11} + \frac{\eta}{\lambda})\kappa_1\kappa_2,$$

and

$$l_6 = -4[(\tau_{11} + \frac{\eta}{\lambda})\tau_{12}\kappa_1^2 + (\tau_{11} + \frac{\eta}{\lambda})(\tau_{22} + \frac{\eta}{\lambda})\kappa_1\kappa_2 + \tau_{12}\kappa_1\kappa_2 + (\tau_{22} + \frac{\eta}{\lambda})\tau_{12}\kappa_2^2].$$

And the other left eigenvector belonging to $\chi = \vec{\kappa} \cdot \vec{u}$ is

$$\vec{v}_2 = [l_1, 0, 0, l_4, 0, l_6],$$

in which

$$l_1 = 2(\tau_{11} + \frac{\eta}{\lambda})^2\kappa_1^2 + 2\tau_{12}\tau_{11} + \frac{\eta}{\lambda})\kappa_1\kappa_2,$$

$$l_4 = (\tau_{11} + \frac{\eta}{\lambda})\kappa_1^2,$$

and

$$l_6 = 2(\tau_{11} + \frac{\eta}{\lambda})^2\kappa_1\kappa_2 + 2\tau_{12}\kappa_2^2.$$

Multiplying these eigenvectors from the right by the column vector $\vec{\phi}$ yields the two characteristic variables $R_1^{\vec{a}}$ and $R_2^{\vec{a}}$ which are convected along the streamlines.
Example 5 For arbitrary \( \vec{\kappa} \) the characteristic variables \( R_1^{\vec{\kappa}} \) and \( R_2^{\vec{\kappa}} \) do not give a lucid picture of what is essentially convected along the streamlines. A more transparent idea is obtained if one chooses \( \kappa_1 = 1 \) and \( \kappa_2 = 0 \). The first left eigenvector in this case is

\[
\vec{\phi}_1 = [0, 0, 0, 0, (\tau_{11} + \frac{\eta}{\lambda}), -2\tau_{12}],
\]

where \( \vec{\phi} \) is

\[
\vec{\phi} = [\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{\tau}_{11}, \tilde{\tau}_{22}, \tilde{\tau}_{12}]^T.
\]

So the first characteristic variable is given by

\[
R_1^{(1,0)} = (\tau_{11} + \frac{\eta}{\lambda})\tau_{22} - 2\tau_{12}\tilde{\tau}_{12}.
\]

Similarly the second characteristic variable by

\[
R_2^{(1,0)} = 2(\tau_{11} + \frac{\eta}{\lambda})\tilde{\rho} + \tilde{\tau}_{11}.
\]

These characteristic variables should be interpreted as, for the case \( R_1 \), multiply the equation for \( \tau_{22} \) by \( (\tau_{11} + \frac{\eta}{\lambda}) \) and the equation for \( \tau_{12} \) by \( -2\tau_{12} \), set all derivatives with respect to \( y \) equal to zero (since \( \kappa_2 = 0 \)) and add these two equations. One, in fact, takes a linear combination of some partial differential equations. Performing this operation yields

\[
(\tau_{11} + \frac{\eta}{\lambda}) \left[ \frac{\partial \tau_{22}}{\partial t} + u \frac{\partial \tau_{22}}{\partial x} - 2\tau_{12} \frac{\partial v}{\partial x} + \frac{\tau_{22}}{\lambda} \right] - 2\tau_{12} \left[ \frac{\partial \tau_{12}}{\partial t} + u \frac{\partial \tau_{12}}{\partial x} - \frac{\partial v}{\partial x} + \frac{\tau_{12}}{\lambda} \right] =
\]

\[
\left[ (\tau_{11} + \frac{\eta}{\lambda}) \frac{\partial \tau_{22}}{\partial t} - 2\tau_{12} \frac{\partial \tau_{12}}{\partial t} \right] + u \left[ (\tau_{11} + \frac{\eta}{\lambda}) \frac{\partial \tau_{22}}{\partial x} - 2\tau_{12} \frac{\partial \tau_{12}}{\partial x} \right] + (\tau_{11} + \frac{\eta}{\lambda}) \frac{\tau_{22}}{\lambda} - 2\tau_{12} \frac{\tau_{12}}{\lambda} = 0.
\]

Apart from the last two terms which constitute lower order terms in this PDE, the expression between brackets could be interpreted as

\[
\frac{\delta R_1^{(1,0)}}{\delta t} + u \frac{\delta R_1^{(1,0)}}{\delta x} = 1, \text{o.t.}
\]

In this equation \( \delta R_1 \) expresses the increment as a linear combination of the equations for increments of the primitive variables \( \delta \vec{\phi} \). A similar expression may be found for \( R_2^{(1,0)} \), giving

\[
\frac{D\tau_{11}}{Dt} + \frac{\tau_{11}}{\lambda} = 0,
\]

in which the term \(-2(\tau_{11} + \eta/\lambda)u_x\) cancels against the term appearing in the continuity equation. Now every linear combination of \( R_1 \) and \( R_2 \) is also convected along the streamlines, in particular \( R_1 + (\tau_{22} + \frac{\eta}{\lambda})R_2 \) yields

\[
(\tau_{22} + \frac{\eta}{\lambda}) \frac{D\tau_{11}}{Dt} + (\tau_{11} + \frac{\eta}{\lambda}) \frac{D\tau_{22}}{Dt} - 2\tau_{12} \frac{D\tau_{12}}{Dt} + \frac{1}{\lambda} \left( (\tau_{22} + \frac{\eta}{\lambda})\tau_{11} + (\tau_{11} + \frac{\eta}{\lambda})\tau_{22} - 2\tau_{12}^2 \right) = 0.
\]

This may be written as

\[
\frac{D(\det \tau_A)}{Dt} + 2\frac{\det \tau_A}{\lambda} = \frac{\eta}{\lambda^2} \text{tr} \tau_A.
\]

(3.3)
This characteristic equation does not only hold in the case \( \vec{\kappa} = (1, 0) \), but for arbitrary \( \vec{\kappa} \), so the determinant of \( \tau_A \) is a characteristic variable along the streamlines. Note that the incremental form \( \delta / \delta t \) may be replaced by the partial derivative \( \partial / \partial t \) and \( \delta / \delta x \) by \( \partial / \partial x \).

If \( \vec{\kappa} = (1, 0) \), then the eigenvalues for the wave part of the UCM-model are given by

\[
\chi = u \pm \sqrt{\frac{(\tau_{11} + \frac{\eta}{\lambda})}{\rho}} = u \pm c_x ,
\]

and the left eigenvectors belonging to these eigenvalues by

\[
l_{\pm c_x} = [0, 0, \rho c_x, 0, 0, \mp 1] ,
\]

which gives the two characteristic variables

\[
\rho c_x \delta v = \delta \tau_{12} .
\]

It is straightforward to set in both the \( v \)-momentum equation and the shear stress equation all derivatives with respect to \( y \) equal to zero to see that

\[
\begin{bmatrix}
\rho c_x \frac{\partial v}{\partial t} + \frac{\partial \tau_{12}}{\partial t} \\
(u \pm c_x) \left( \frac{\rho c_x}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial \tau_{12}}{\partial x} \right) \mp \frac{\tau_{12}}{\lambda} = 0 .
\end{bmatrix}
\]

In chapter 5 these two characteristic equations will be used to discretize \( \partial v / \partial x \) and \( \partial \tau_{12} / \partial x \) which appear both in the \( v \)-momentum and shear stress equation, such that when the above linear combination is performed numerically it will result numerically in these two characteristic equations. Instead of choosing \( \vec{\kappa} = (1, 0) \) one may take \( \vec{\kappa} = (0, 1) \) which leads to

\[
\begin{bmatrix}
\rho c_y \frac{\partial u}{\partial t} + \frac{\partial \tau_{12}}{\partial t} \\
(v \pm c_y) \left( \frac{\rho c_y}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial \tau_{12}}{\partial y} \right) \mp \frac{\tau_{12}}{\lambda} = 0 .
\end{bmatrix}
\]

The characteristic equations may also be written in an arbitrary direction \( \vec{\kappa} \), but for the numerical treatment, to be discussed in chapter 5, \( \vec{\kappa} = (1, 0) \) and \( (0, 1) \) are sufficient.

If one assumes that the solution has to be sought in the set of velocity vectors satisfying \( \text{div} \vec{u} = 0 \), then the type of the equations follows from

\[
\begin{vmatrix}
\rho v & 0 & -\kappa_1 & 0 & -\kappa_2 \\
0 & \rho v & 0 & -\kappa_2 & -\kappa_1 \\
-2\kappa_1(\tau_{11} + \eta/\lambda) - 2\kappa_2 \tau_{12} & 0 & \nu & 0 & 0 \\
0 & -2\kappa_2(\tau_{22} + \eta/\lambda) - 2\kappa_1 \tau_{12} & 0 & \nu & 0 \\
-\kappa_2(\tau_{22} + \eta/\lambda) & -\kappa_1(\tau_{11} + \eta/\lambda) & 0 & 0 & \nu
\end{vmatrix} = 0 .
\]

For \( \vec{\kappa} = (1, 0) \) this yields

- \( \nu = 0 \),
- \( \nu = \pm \sqrt{\frac{2(\tau_{11} + \eta)}{\rho}} \),
This shows that the elliptic part has been removed and the system has become totally hyperbolic. The left eigenvector for the eigenvalue \( \nu = 0 \) in the case \( \kappa = (1, 0) \) is
\[
l_{\nu=0} = [0, 0, 0, (\tau_{11} + G), -2\tau_{12}].
\]
This is the same eigenvector as obtained in the complete elliptic/hyperbolic case. The eigenvectors belonging to \( \nu = \pm c_x \), in which \( c_x = \sqrt{\frac{\tau_{11} + G}{\rho}} \) are
\[
l_{\nu=\pm c_x} = [0, c_x, 0, 0, \mp 1].
\]
Also this characteristic equation remains unchanged. The only new eigenvectors belonging to \( \nu = \pm \sqrt{2}c_x \) establish another wave-like connection between the constitutive equation and the momentum equation. Their eigenvectors are given by
\[
l_{\nu=\pm \sqrt{2}c_x} = [\sqrt{2}c_x, 0, \mp 1, 0, 0].
\]
This complete set of left eigenvectors forms the matrix \( l \).

\[
l = \begin{pmatrix}
0 & 0 & 0 & (\tau_{11} + G) & -2\tau_{12} \\
0 & c_x & 0 & 0 & -1 \\
0 & c_x & 0 & 0 & 1 \\
\sqrt{2}c_x & 0 & -1 & 0 & 0 \\
\sqrt{2}c_x & 0 & 1 & 0 & 0
\end{pmatrix}.
\]

Multiplying the 5 partial differential equations, in which all derivatives with respect to \( y \) are set to zero \((\kappa_2 = 0)\) from the left by \( l \) leads to
\[
(\tau_{11} + G) \frac{\partial \tau_{22}}{\partial t} - 2\tau_{12} \frac{\partial \tau_{12}}{\partial t} + u \left( (\tau_{11} + G) \frac{\partial \tau_{22}}{\partial x} - 2\tau_{12} \frac{\partial \tau_{12}}{\partial x} \right) = 0,
\]
\[
(\rho c_x \frac{\partial v}{\partial t} \pm \frac{\partial \tau_{12}}{\partial t}) + (u \pm c_x) \left( \rho c_x \frac{\partial v}{\partial x} \pm \frac{\partial \tau_{12}}{\partial x} \right) = 0,
\]
and
\[
(\rho \sqrt{2}c_x \frac{\partial u}{\partial t} \pm \frac{\partial \tau_{11}}{\partial t}) + (u \pm \sqrt{2}c_x) \left( \rho \sqrt{2}c_x \frac{\partial u}{\partial x} \pm \frac{\partial \tau_{11}}{\partial x} \right) = 0.
\]

Similar equations may be derived for the \( y \)-direction.

In numerical calculations these characteristic equations are discretized in the direction of these characteristic velocities, i.e. upwind, whereas the pressure term ensures that the discrete velocity field remains divergence free. This would be the way to treat the set of partial differential equations on a non-staggered grid, however, in chapter 5 it will be shown that on a staggered grid no upwinding is necessary for all the unknowns in order to prevent uncoupling of the unknowns.
3.3 The type of the Oldroyd-B model

The Oldroyd-B model is equal to the UCM-model, but contains an extra term \( \eta_s \Delta \ddot{u} \) in the right hand side of the momentum equation. \( \eta_s \) is called the solvent viscosity. This extra term either models the viscosity of the solvent or takes account of the very small relaxation times in the viscoelastic fluid. Simply taking the highest derivatives in the resulting set of PDEs turns out to be unsatisfactory to determine the behaviour for very small waves as was shown in example 4. The easiest way to circumvent this difficulty is to determine the principal part of the symbol. The symbol of the differential operator for this Jeffrey type model is given by

\[
P(\ddot{\phi}, i\ddot{\xi}) = i\begin{pmatrix}
0 & \xi_1 & & & & \\
\xi_1 & \rho \nu - i(\xi_1^2 + \xi_2^2) & \xi_2 & 0 & 0 & 0 \\
0 & 0 & \rho \nu - i(\xi_1^2 + \xi_2^2) & 0 & -\xi_1 & 0 \\
0 & m_1 & 0 & \rho \nu - i/\lambda & 0 & -\xi_2 \\
0 & 0 & m_2 & 0 & \rho \nu - i/\lambda & 0 \\
0 & m_3 & m_4 & 0 & 0 & \rho \nu - i/\lambda
\end{pmatrix},
\]

in which

- \( \nu = \xi_0 + u \xi_1 + v \xi_2 \),
- \( m_1 = -2(\tau_1 + \eta/\lambda) \xi_1 - 2\tau_1 \xi_2 \),
- \( m_2 = -2(\tau_2 + \eta/\lambda) \xi_2 - 2\tau_2 \xi_1 \),
- \( m_3 = -(\tau_2 + \eta/\lambda) \xi_2 \)
- \( m_4 = -(\tau_1 + \eta/\lambda) \xi_1 \).

The determinant of the symbol is

\[
\text{det} P(\ddot{\phi}, i\ddot{\xi}) = -\left( \xi_1^2 + \xi_2^2 \right) (\nu - i/\lambda)^2. \\
\left[ \left( \rho \nu - i\eta_s (\xi_1^2 + \xi_2^2) \right) (\nu - i/\lambda) - \xi_1^3 (\tau_1 + \eta/\lambda) - 2\xi_1 \xi_2 \tau_1 \xi_2 - \xi_2^3 (\tau_2 + \eta/\lambda) \right].
\]

The principal part of this polynomial is

\[
\text{det} P_p(\ddot{\phi}, i\ddot{\xi}) = i\nu^3 \eta_s \left( \xi_1^2 + \xi_2^2 \right)^2.
\]

The hyperbolic part of the set of equation consists of \( \nu = 0 \) (3x), which represents convection along the streamlines, whereas the last factor represents an elliptic part as in the Maxwell type equations and a parabolic part. (This may be seen by computing the principal part of the symbol of the diffusion equation \( \ddot{\phi} = \eta_s \Delta \ddot{\phi} \). It seems that the wave-like part, which was present in the Maxwell type equations, has completely disappeared. This may be illustrated by the following simple example: Consider the hyperbolic wave equation

\[
\epsilon^2 \dddot{\phi}_{tt} + \dddot{\phi}_t - \phi_{xxr} = 0 \quad (\epsilon > 0).
\]
This wave equation possesses two characteristic velocities given \( dx/dt = \pm 1/\epsilon. \) If \( \epsilon \to 0 \) then the characteristic velocities tend to \( \pm \infty. \) In the limit \( \epsilon = 0 \) the wave equation reduces to a diffusion equation. This may also be seen by remembering that the addition of a solvent viscosity mimics the presence of very small relaxation times.

**Remark 4** The type of a set of PDEs describes how this set reacts to infinitely small waves, but (fortunately) in practical situations the solution may be approximated by taking only a finite number of wave numbers into account, unless discontinuities appear in the solution. This fact allows the set of PDEs to be approximated by discrete (numerical) algorithms. If this is the case then one has to evaluate the roots of the symbol itself and not only its principal part in order to determine whether diffusion really dominates the wave like part which is present in the symbol as a lower order term. Possibly one can identify regions in which the additional viscosity term dominates over wavelike solution and regions where the wavelike structure is more dominant. One may compare this with the Navier-Stokes equations which are elliptic/parabolic, but in numerical schemes where the wave numbers are limited, the solution can be dominated by the hyperbolic convection terms. Since these terms are contained in the lower order part of the symbol it is desirable to find the roots of the symbol itself.

The term between brackets in the symbol of the Jeffrey type model may be expanded to yield

\[
-\imath \| \tilde{\kappa} \|^4 \eta_s \nu + \rho \nu^2 - \tilde{\kappa}^T \cdot \boldsymbol{\tau}_A \tilde{\kappa} - \frac{\eta_s}{\lambda} \| \tilde{\kappa} \|^4 \eta_s - \frac{\rho}{\lambda} \nu = 0.
\]

Suppose that the \( \| \tilde{\kappa} \| \) is bounded and solve this equation for the characteristic velocity in the direction \( \tilde{\kappa} \), called \( \alpha \). The fourth term may contracted in the definition of \( \boldsymbol{\tau}_A \) which in this calculation will be denoted by \( \tilde{\tau}_A = \tau + (\eta + \eta_s \| \tilde{\kappa} \|^2)I/\lambda. \)

\[
\nu^2 - \imath \left( \frac{\| \tilde{\kappa} \|^4 \eta_s}{\rho} + \frac{1}{\lambda} \right) \nu - \tilde{\kappa}^T \cdot \frac{\tilde{\tau}_A \tilde{\kappa}}{\rho} = 0.
\]

Solving this equation for \( \nu \) gives

\[
\nu = \frac{1}{2} \left( \frac{\| \tilde{\kappa} \|^4 \eta_s}{\rho} + \frac{1}{\lambda} \right) \pm \sqrt{\left( \frac{\| \tilde{\kappa} \|^4 \eta_s}{\rho} + \frac{1}{\lambda} \right)^2 - \frac{\tilde{\kappa}^T \cdot \frac{\tilde{\tau}_A \tilde{\kappa}}{\rho} - \frac{1}{4} \left( \| \tilde{\kappa} \|^4 \eta_s + \frac{1}{\lambda} \right)^2}{\frac{\tilde{\kappa}^T \cdot \frac{\tilde{\tau}_A \tilde{\kappa}}{\rho} - \frac{1}{4} \left( \| \tilde{\kappa} \|^4 \eta_s + \frac{1}{\lambda} \right)^2}{\frac{\tilde{\kappa}^T \cdot \frac{\tilde{\tau}_A \tilde{\kappa}}{\rho} - \frac{1}{4} \left( \| \tilde{\kappa} \|^4 \eta_s + \frac{1}{\lambda} \right)^2}}.}
\]

This equation tells something more than the roots based on the principal part of the symbol. If there is a region where

\[
\tilde{\kappa}^T \cdot \frac{\tilde{\tau}_A \tilde{\kappa}}{\rho} - \frac{1}{4} \left( \| \tilde{\kappa} \|^4 \eta_s + \frac{1}{\lambda} \right)^2 > 0,
\]

then there is a wave-like solution. If on the other hand

\[
\tilde{\kappa}^T \cdot \frac{\tilde{\tau}_A \tilde{\kappa}}{\rho} - \frac{1}{4} \left( \| \tilde{\kappa} \|^4 \eta_s + \frac{1}{\lambda} \right)^2 < 0,
\]

then the solution is convected and damped along the streamlines. The imaginary part of \( \nu \) is always larger or equal to zero if \( \tilde{\tau}_A \) is positive definite. However if \( \tilde{\tau}_A \) becomes indefinite
Im(\nu) may become less than zero, which means that the solution grows exponentially in
time proportional to Im(\nu). So the addition of solvent viscosity in the case where the wave
number is bounded still means that the tensor $\tau_A$ should be positive definite. If $||\kappa|| \to \infty$
the two solutions for $\nu$ are $\nu = 0$ and $\nu = i\infty$ and the positive definiteness requirement
on $\tau_A$ may be dropped. The first solution, $\nu = 0$, is the additional convection along the
streamlines, whereas the second, $\nu = i\infty$, represents the infinite velocity of the parabolic
part. Note that when $\eta_s = 0$ then this equation reduces to the damped wavelike velocities
obtained by the simple wave method for the Maxwell type model in the previous section.

It is interesting to determine what is convected along the streamlines. It turns out that
the three constitutive equations for the stress components (3n in case of n relaxation
times) are convected along the streamlines, in which the extra convected terms $L\tau + \tau L^T$
become lower order terms. The conservation laws contain the elliptic/parabolic part just
like in the Navier-Stokes equations.

Just like the incompressible Euler equations and the UCM fluid the incompressibility and
the pressure term may be dropped to yield a mixed hyperbolic/parabolic set of PDEs. It
turns out that in the numerical approach presented in chapter 5 almost all terms which
contribute to the wave like part of the viscoelastic fluid equations are positioned such that
only the convection part in the direction of the streamlines needs to be considered.

### 3.4 The elliptic part of the equations

All the viscoelastic fluid models combined with the conservation laws contain an elliptic
part which results from the incompressibility assumption. Conservation of mass requires
in this case that for all time $t \ \text{div}\ u = 0$ holds, so $(\partial / \partial t) \text{div}\ u \equiv 0$ for all $t$. This means
that in the momentum equation the pressure $p$ has to be chosen such that this constraint
is satisfied. Writing the momentum equation as

\[
\frac{\partial \vec{u}}{\partial t} + \nabla p = \vec{R},
\]

means that the pressure has to satisfy

\[
\Delta p = \text{div}\ \vec{R},
\]

since the operators $\partial / \partial t$ and $\text{div}$ commute. If the right hand side vector $\vec{R}$ is supposed to
be given this is a Poisson equation for the pressure, which is the inhomogeneous analogue
of the Laplace equation, which was considered in Example 2, and thus an elliptic equation.
This equation has to be provided with boundary conditions. In the section on boundary
conditions it will be shown that the natural boundary conditions are Neumann boundary
conditions given by

\[
\frac{\partial p}{\partial \vec{n}}(\vec{x}) = h(\vec{x}), \quad \vec{x} \in \partial \Omega,
\]

where $\partial \Omega$ is the boundary of the open connected set where the elliptic equation is solved
and $\vec{n}$ is the outward unit normal on $\partial \Omega$. This type of boundary conditions imposes
restrictions on the right hand side of the Poisson equation; the right hand side has to be in the range of the Laplace operator, which is equivalent to saying that the right hand side has to be orthogonal to the null space of the Laplace operator, where the null space is defined by
\[ \text{Null}(\Delta) = \{ p \mid \Delta p = 0 \} . \]
The null space of the Laplace operator is the set \( p = \text{constant} \), so consistency requires that
\[ \int \int_{\Omega} \text{div} \vec{R} \, d\Omega = \int_{\partial \Omega} \vec{R} \cdot \vec{n} \, dS = 0 . \]
If the right hand side satisfies this compatibility relation then there exists a solution \( p^* \) for the Poisson equation and if \( p^* \) is a solution then also \( p^* + p^0 \), with \( p^0 \in \text{Null}(\Delta) \).
The compatibility relation merely states that \( \text{div} \vec{u} = 0 \) can only be guaranteed locally if this holds globally. In the introduction it was mentioned that elliptic problems can be regarded as hyperbolic problems with an infinite wave speed where the change of an elliptic variable in the interior of \( \Omega \) is directly related to its value at the boundary. The above compatibility relation illustrates this fact.

### 3.5 The viscoelastic wave equation

Since the pressure is contained in the elliptic part of the equations the momentum equation may be reduced to
\[ \rho \frac{D\vec{u}}{Dt} = \text{div} \tau , \]
and the constitutive equation reads
\[ \frac{D\tau}{Dt} - L\tau - \tau^T \frac{\tau}{\lambda} = 2\frac{\eta}{\lambda} D . \]
If one applies the divergence operator to the constitutive equation and the material time derivative to the reduced momentum equation one obtains
\[ \rho \frac{D^2\vec{u}}{Dt^2} = \frac{D\vec{f}}{Dt} , \]
and
\[ \text{div} \left( \frac{D\tau}{Dt} - L\tau - \tau^T \frac{\tau}{\lambda} \right) + \frac{\vec{f}}{\lambda} = 2\frac{\eta}{\lambda} \Delta\vec{u} , \]
in which \( \vec{f} = \text{div} \tau \). Application of the divergence operator to the upper convected time derivative gives
\[ \frac{D\vec{f}}{Dt} = \left( \nabla \cdot \tau_A \nabla \right) \vec{u} + \frac{\vec{f}}{\lambda} = \vec{0} , \]
where \( \tau_A = \tau + \eta/\lambda I \). This may also be written as
\[ \frac{D\vec{f}}{Dt} - \text{div} [L\tau_A] + \frac{\vec{f}}{\lambda} = \vec{0} , \]
so the momentum and constitutive equation may be recombined to give

\[ \rho \frac{D^2 \vec{u}}{Dt^2} - \left( \vec{\nabla} \cdot \vec{\tau}_A \vec{\nabla} \right) \vec{u} + \rho \frac{D\vec{u}}{\lambda Dt} = \vec{0}. \]  

(3.4)

This suggests that there are two wave equations, one for \( u \) and one for \( v \), but the elliptic connection \( \text{div} \vec{u} \) and \( \text{grad} p \) reduces these two equations to one wave equation. This leads to the wave equation for the vorticity [30]. If non dimensional variables are introduced by

\[
\vec{u} = U \vec{u}^*, \quad t = \frac{L}{U} t^*, \quad x = L x^*,
\]

\[
y = L y^*, \quad \vec{\tau} = \frac{\eta}{\lambda} \vec{\tau}^*,
\]

then, omitting the superscripts, this wave equation may be written as

\[ \frac{D^2 \vec{u}}{Dt^2} - \frac{\eta}{\rho \lambda U^2} \left( \vec{\nabla} \cdot \vec{\tau}_A \vec{\nabla} \right) \vec{u} + \frac{L}{\lambda U} \frac{D\vec{u}}{Dt} = \vec{0}. \]

Introduce the dimensionless numbers

\[ De = \frac{\lambda U}{L}, \quad Re = \frac{\rho UL}{\eta} \quad \text{and} \quad M_0^2 = De Re = \frac{\rho \lambda U^2}{\eta}, \]

in which \( De \) is the ratio of the characteristic time of the fluid constituents to the characteristic time of the flow as mentioned in the chapter 1 and \( Re \) is the ratio of the inertial terms in the momentum equation to the influence of the total stress tensor. \( M_0 \) is called the elastic Mach number, which gives the ratio of the fluid velocity to the elastic shear wave velocity at rest, i.e. when \( \vec{\tau} = \vec{0} \).

The equation may also be written as

\[ \frac{D^2 \vec{u}}{Dt^2} - \frac{1}{M_0^2} \left( \vec{\nabla} \cdot \vec{\tau}_A \vec{\nabla} \right) \vec{u} + \frac{1}{De} \frac{D\vec{u}}{Dt} = \vec{0}, \]

or

\[ De \frac{D^2 \vec{u}}{Dt^2} - \frac{1}{Re} \left( \vec{\nabla} \cdot \vec{\tau}_A \vec{\nabla} \right) \vec{u} + \frac{D\vec{u}}{Dt} = \vec{0}. \]  

(3.5)

Based on these dimensionless numbers one can say that if \( De \) is small compared to unity, \( De \ll 1 \), and the shear wave velocity is virtually zero compared to the fluid velocity, \( M_0 \to \infty \), then the wave equation reduces to a convection equation along the streamlines, i.e.

\[ \text{if } De \ll 1 \text{ and } M_0 \to \infty \text{ then } \frac{D\vec{u}}{Dt} = \vec{0}. \]

If \( De \ll 1 \) and \( 0 < M_0 < \infty \) then the wave equation reduces to a parabolic convection-diffusion equation with a non isotropic diffusion tensor, i.e.

\[ \text{if } De \ll 1 \text{ and } 0 < M_0 < \infty \text{ then } Re \frac{D\vec{u}}{Dt} = \left( \vec{\nabla} \cdot \vec{\tau}_A \vec{\nabla} \right) \vec{u}. \]

If both \( De \approx 0 \) and \( M_0 \approx 0 \) then the dimensionless number \( Re \) determines the relative influence, since \( Re = M_0^2/De \).
If $De \gg 1$ then the first order material time derivative in the material equation is not very influential, so in this case the damped wave equation reduces to a pure non-linear wave equation if $M_0 < \infty$ and to a degenerate wave equation if $M_0 \to \infty$, i.e.

$$\text{If } De \gg 1 \text{ and } M_0 \to \infty \text{ then } \frac{D^2 \ddot{u}}{Dt^2} = 0.$$ 

In this case all the shear waves travel in the direction of the local velocity $\ddot{u}$ and this case presents the total supercritical case. If on the other side $M_0 \to 0$, which is equivalent to $Re \to 0$, then the shear wave velocity goes to infinity. This situation will be explored in the next section.

In numerical calculations converged solutions may be obtained for small Deborah numbers, but when this dimensionless number increases many algorithms fail to converge. This failure may be attributed to a poor approximation of the wave equation numerically. This issue will be addressed in chapter 5. Joseph, Renardy and Saut [29] suggest that the problems may be due to the transition from subcritical to supercritical flow, which is a phenomenon which is absent for small Deborah numbers. In order to show how this behaviour is contained in the wave equation it is useful to expand the second material time derivative, giving

$$\frac{D^2 \ddot{u}}{Dt^2} = \frac{\partial}{\partial t} \left( \frac{D\ddot{u}}{Dt} \right) + \frac{\partial}{\partial t} \left[ \left( \ddot{u} \cdot \nabla \right) \ddot{u} + \left( \nabla \cdot (\ddot{u} \otimes \ddot{u}) \right) \ddot{u} \right].$$

Inserting this expansion in the wave equation yields

$$\left( De \frac{\partial}{\partial t} + 1 \right) \frac{D\ddot{u}}{Dt} + \frac{1}{Re} \left( \nabla \cdot [Re \ddot{u} \otimes \ddot{u} - \tau_A] \nabla \right) \ddot{u} + De \frac{\partial}{\partial t} \left[ \left( \ddot{u} \cdot \nabla \right) \ddot{u} \right] = 0.$$

The expression between brackets in the second term is exactly $\tau_B$ of which the eigenvalues determine whether the flow is sub- or supercritical in a certain direction.

### 3.6 Stokes approximation and low Mach numbers

In many viscoelastic fluid calculations one assumes that the momentum equation is dominated by the total stress tensor, such that the inertia terms in the momentum, i.e. the term in $\rho D\ddot{u}/Dt$, may be neglected. In terms of the Reynolds number this amounts to setting $Re = 0$, when $Re$ is very small. This approximation is called the Stokes approximation. The momentum equation reduces to

$$\text{div} \sigma = -\nabla p + \text{div} \tau = 0.$$ 

The momentum equation may now be considered as a constraint on the stress tensor, just like the conservation of mass is a constraint on the velocity vector. This constraint may also be written as

$$\text{rot} \text{div} \tau = 0 \ \forall t.$$ 

This means that in the constitutive equation there have to be Lagrange multipliers (constraint forces) to enforce this constraint. It turns out that the velocity components should
be chosen such that this constraint is satisfied. Taking the divergence of the constitutive equation gives

$$\frac{D\vec{f}}{Dt} - (\nabla \cdot \tau_A \nabla) \vec{u} + \frac{\vec{f}}{\lambda} = \vec{0},$$

in which $\vec{f} = \text{div} \, \tau$. A short way to write this equation is

$$\frac{\partial \vec{f}}{\partial t} - (\nabla \cdot \tau_A \nabla) \vec{u} = \vec{0},$$

in which $\vec{R}$ contains the convective and source term of $\vec{f}$. The requirement that $\text{rot} \, \vec{f} = \vec{0}$ for all $t$ leads to the equation

$$-\nabla \times \left( (\nabla \cdot \tau_A \nabla) \vec{u} \right) = \nabla \times \vec{R}.$$  

Neglecting the lower order derivatives in the left hand side this gives

$$- (\nabla \cdot \tau_A \nabla) \vec{\omega} = \nabla \times \vec{R},$$

where $\vec{\omega} = \text{rot} \, \vec{u}$, which is called the *vorticity vector*. If $\tau_A$ is positive definite then (3,6) is an elliptic equation for $\vec{\omega}$. The symbol of the elliptic operator $\nabla \cdot \tau_A \nabla$ is given by

$$(\tau_{11} + \frac{\eta}{\lambda})\xi_1^2 + 2\tau_{12}\xi_1 \xi_2 + (\tau_{22} + \frac{\eta}{\lambda})\xi_2^2.$$  

This term also appears when the determinant of the symbol of the UCM-model is computed (for $Re = 0$). The symbol is given by

$$P(\vec{\phi}, i\vec{\xi}) = i\begin{pmatrix} 0 & \xi_1 & \xi_2 & 0 & 0 & 0 \\ \xi_1 & 0 & 0 & -\xi_1 & 0 & -\xi_2 \\ \xi_2 & 0 & 0 & -\xi_2 & -\xi_1 & \xi_1 \\ 0 & m_1 & 0 & \nu - i/\lambda & 0 & 0 \\ 0 & 0 & m_2 & 0 & \nu - i/\lambda & 0 \\ 0 & m_3 & m_4 & 0 & 0 & \nu - i/\lambda \end{pmatrix},$$

and $\text{det} P(\vec{\phi}, i\vec{\xi})$ by

$$\text{det} P(\vec{\phi}, i\vec{\xi}) = (\xi_1^2 + \xi_2^2)\nu^2 \left( (\tau_{11} + \frac{\eta}{\lambda})\xi_1^2 + 2\tau_{12}\xi_1 \xi_2 + (\tau_{22} + \frac{\eta}{\lambda})\xi_2^2 \right).$$

The first factor indicates the elliptic part due to the incompressibility constraint, the second the hyperbolic part, i.e. convection along the streamlines with multiplicity two and the last factor shows the appearance of an additional constraint on the stress tensor, which yields an elliptic equation for the vorticity vector. Since $2\vec{\omega} \times d\vec{x} = W \, d\vec{x}$, in which $W$ is the spin tensor, the constraint on the stress tensor determines the asymmetric part of the velocity gradient tensor. This means that the stress tensor not only depends on the deformation, but in the case $Re = 0$, also poses restrictions on the deformation! For an incompressible flow the equation for the pressure, conservation of mass, does not contain the pressure. This unknown has to be solved from the momentum equation.
before the velocity values are updated in order to satisfy the incompressibility constraint. In the case $Re = 0$ for the UCM-model the equation for the velocity components, the momentum equation, does not contain the velocity components. These unknowns have to be solved from the constitutive equation before the stress variables are updated in order to satisfy the constraint imposed by the momentum equation. In numerical computations one converts the set of PDEs to a set of (non)linear algebraic equations from which the unknowns have to be solved. The result of the constraints is that the matrix of the linearized equation contains zeros on the main diagonal for the continuity equation and the momentum equation. This is highly undesirable since many nice properties of the algebraic system result from the fact that the matrix is diagonal dominant, which means that the absolute value of the diagonal term is larger than the sum of the absolute values of the off-diagonal terms. So one has to rewrite the system of algebraic equations in order to remedy this deficiency. The incompressibility constraint is replaced by the Poisson equation for the pressure and the momentum equation should be replaced by an elliptic equation for the vorticity vector and the relation between the velocity vector and the vorticity vector. So, although it seems advantageous to remove the non-linear convective terms in the momentum equation by assuming $Re = 0$, this introduces nasty couplings between the various equations.

However, refraining from the Stokes approximation does not prevent the unconsidered calculator from getting into trouble. Suppose $Re \neq 0$, then there may be regions in the flow where the inertial terms dominate and region in the flow where the stresses dominate the momentum equation. Symbolically this may be written as

$$Re \ U = T \ \text{model momentum equation.}$$

$$De \ T = U \ \text{model constitutive equation.}$$

Suppose a decoupled solution scheme is used in which the velocity field is solved from the momentum equation based on known stress values, which are then inserted in the constitutive equation to solve for the stress variables as a function of the known velocity values. It is assumed that incompressibility is taken care of. Now suppose that $Re$ is small and $De$ is given, then the following error propagation is feasible. $T(0)$ is given with a small error $\epsilon$, then the velocity vector $U(1)$ will contain an error equal to $\epsilon/Re$ if $Re \neq 0$. This value $U(1)$ will then be used in the constitutive equation from which $T(1)$ follows with an error equal to $\epsilon/ReDe = \epsilon/M_0$. This shows that when $M_0 > 1$ the initial error in $T(0)$ will be reduced, whereas in the case $M_0 < 1$ the initial error is amplified. So in the latter case it seems appropriate to solve the stresses from the momentum equation and the velocity vector from the constitutive equation (in principle). So if $T(0)$ is given in the constitutive equation with error $\epsilon$ then the value $U(1)$ obtained from this equation contains an error $\epsilon De$ and the new value of the stress tensor obtained from the momentum equation will contain an error $\epsilon M_0$. For flows in which $M_0 < 1$ this method reduces the initial error, whereas the opposite occurs in flow where $M_0 > 1$. The case $Re = 0$ in which the momentum equation reduces to a constraint on the stresses corresponds to the latter 'algorithm' in which the velocity components are computed from the constitutive equation in order to satisfy a relation among the stresses in the momentum equation, but even when $Re \neq 0$ it may be beneficial to solve the velocity
components from the constitutive equation and the stresses from the momentum equation. If in a certain flow $Re$ and $De$ are interpreted as the local Reynolds number and the local Deborah number, then there may be regions in which $M_0 < 1$ and regions where $M_0 > 1$ in which case one of the above schemes will diverge in a certain region. The remedy will be to couple both solutions instead of solving the variables from the equations alternatively. In the next chapter an overview will be given of some numerical techniques and results found in the literature. This overview indeed suggests that simultaneously solving the momentum and constitutive equation produces more stable algorithms than the decoupled approach. Also in the literature it may be found that instabilities occur in regions where the velocity is small (e.g. the centerline in a time dependent Couette flow or along walls) and the stress high. This type of (numerical) instability may be explained by the low Mach number problem, whereas problems associated with the transition from subcritical to supercritical flow may be associated with the high Mach number problem. The problems associated with low Mach numbers also appear in the compressible Navier-Stokes equation where an implicit treatment of the pressure and the continuity equation ameliorates the numerical solutions.

However the use of a totally coupled algorithm requires solving a non-linear set of algebraic equations at every time step, which is rather time consuming. In chapter 5 the issue of the coupling between the various equations will be reconsidered.

The dimensionless numbers introduced in the previous section contain $U$ and $L$, which may be defined locally and the constants $\rho$, $\eta$ and $\lambda$. This means that the velocity is scaled with the local characteristic velocity $U$, time with $L/U$ and the coordinate direction with $L$, whereas the stress tensor is scaled with a constant! Especially in those regions where the stresses are very large $M_0$ may deviate significantly from the true elastic Mach number in a certain direction. By introducing the MCSH-approximation, discussed in the previous chapter and the transformation $\tau \mapsto \tilde{\tau}$ the constitutive equations are given by

$$\frac{D\tau}{Dt} + \frac{\tau}{\lambda} = 2\frac{\eta}{\lambda} D .$$

If instead of the constitutive equation for the $\tau$ its approximated associated relative form is used, the type of the equations remains unchanged, but the elastic shear wave velocity is given by

$$c^2 = \frac{\eta}{\rho \lambda} ,$$

which is independent of the direction $\vec{\kappa}$. So in terms of $\tilde{\tau}$ the elastic Mach number is given by $M_0$.

In chapter 5 it will be shown that the elastic shear wave velocity in the direction $\vec{\kappa}$, given by $\vec{\kappa}_\tau \cdot \tau_A \vec{\kappa}$, plays a role in the transformation $\tau \mapsto \tilde{\tau}$.

### 3.7 Stability and well-posedness of the set of PDEs

It has been mentioned throughout this chapter that the positive definiteness of the tensor $\tau_A$ ensures (analytically) well-posedness of the set of PDEs. This section will briefly review what is meant by well-posedness.
The two concepts, well-posedness and stability, are closely related but physically mean something different. In words, well-posedness expresses two properties one likes the solution of a set of PDEs to possess

i) there exists a solution and it is unique, and

ii) the solution depends continuously on its parameters (initial conditions, right hand side, etc.)

The first assumption is very hard to establish in general non-linear problems and the second requirement supposes that the solution is known for different initial and boundary values. Another way to fulfill the second condition is the following: Suppose the initial solution may be decomposed in spatial Fourier modes to yield \( \tilde{\phi}(\omega, 0) \) and there exists a mapping \( E(\omega, t) \) such that

\[
\tilde{\phi}(\omega, t) = E(\omega, t) \tilde{\phi}(\omega, 0) ,
\]

then the following two definitions may be introduced

**Definition 1** The initial value problem is well-posed (in time) if constants \( \alpha \) and \( K \) exist, independent of \( \omega \), such that

\[
|E(\omega, t)| \leq K e^{\alpha t} \quad \forall t \geq 0 .
\]

Well-posedness allows the solution to go to infinity, but not in finite time. This implies that the difference of two nearby solutions with initial conditions which are arbitrarily close, remains bounded in any closed time interval.

**Example 6** Suppose the initial value problem is given by

\[
u_t = P(\partial / \partial x) u \quad u(x, 0) = \hat{f}(\omega) e^{i \omega x} ,
\]

in which \( P \) is a polynomial equation which acts only on the spatial coordinates. The solution in terms of \( \hat{u}(\omega, t) \) will then be given by

\[
\hat{u}(\omega, t) = e^{P(\omega)t} \hat{f}(\omega) .
\]

So in this case \( E(\omega, t) = \exp(P(\omega)t) \) and inspection of the eigenvalues of the spatial Fourier symbol will show whether the problem is well-posed or not.

A stronger requirement than well-posedness is that perturbations in the initial conditions and parameters remain bounded. If this condition is fulfilled then the set of PDEs is called stable.

**Definition 2** The initial value problem given above in term of its spatial Fourier components is called *stable* if there exits a \( K \), independent of \( \omega \), such that

\[
|E(\omega, t)| \leq K \quad \forall t \geq 0 .
\]

If

\[
|E(\omega, t)| \to 0 \quad \text{for} \quad t \to \infty ,
\]
then the initial value problem is called \textit{strongly stable}.

Since the operator $E$ can rarely be calculated, one has to use $\exp(P(t\omega)t)$ to determine whether the solution grows exponentially in time proportional to $\omega$ or not. If the well-posedness condition of the UCM-model is investigated then the characteristic velocities should have imaginary parts greater than zero and if the imaginary part is less than zero it should be independent of $\omega$. This is the case if the tensor $\tau_A$ is positive definite. Dupret and Marchal [21] have shown that if $\tau_A$ is positive definite initially then it will remain positive definite. The positive-definiteness requirement also follows from Eq. (3.2).

It can in fact be shown that if the problem is well-posed then the determinant of $\tau_A$ will eventually be greater or equal to $(\eta/\lambda)^2$, provided this inequality holds at the boundary.

### 3.8 A positive lower bound for $\det \tau_A$

The analytical system is well posed if $\tau_A$ is positive definite, so it is necessary that $\det \tau_A > 0$. In a two dimensional flow the condition $\det \tau_A > 0$ and $\text{tr} \tau_A > 0$ are necessary and sufficient conditions to ensure well-posedness. If one of the eigenvalues of $\tau_A$ goes through zero, then the determinant will be the first indicator to reveal indefiniteness. Therefore well posedness will be ensured if the determinant is bounded from below by a positive constant.

It has been shown in this chapter that $\det \tau_A$ satisfies

$$\frac{D\det \tau_A}{Dt} + \frac{2\det \tau_A}{\lambda} = \frac{\eta}{\lambda^2} \text{tr} \tau_A.$$

Suppose the material derivative of the determinant is equal to zero, then the following relation between the determinant and the trace of $\tau_A$ can be established

$$2\det \tau_A = \frac{\eta}{\lambda} \text{tr} \tau_A.$$

Or written in terms of the eigenvalues $\chi_1$ and $\chi_2$ of $\tau_A$

$$2\chi_1\chi_2 = \frac{\eta}{\lambda}(\chi_1 + \chi_2).$$

In the $\chi_1 - \chi_2$-plane the values satisfying this relation are depicted by the solid line in Fig (3.1). In the region denoted by $+$ the material derivative of the determinant increases, whereas in the region labeled by $-$ this derivative decreases. The dashed line in Fig (3.1) indicates the points where $\det \tau_A = \eta^2/\lambda^2$. The two curves touch in the point $\chi_1 = \chi_2 = \eta/\lambda$. At this point the extra stress tensor is zero.

If $\det \tau_A \leq \eta^2/\lambda^2$ then the determinant increases in the direction of the streamlines, until $D\det \tau_A/Dt = 0$. If this is the case, then $\det \tau_A \geq \eta^2/\lambda^2$.

If on the other hand the eigenvalues are in the region in which $D\det \tau_A/Dt < 0$, then $\det \tau_A \geq \eta^2/\lambda^2$. So if the lower bound, $\det \tau_A \geq \eta^2/\lambda^2$, holds at the boundary of the computational domain, then it will eventually hold throughout the computational domain, thus ensuring well-posedness. This lower bound on the determinant will also be encountered in the numerical calculations, performed in chapter 5.
Figure 3.1: The curve for which $2\det \tau_A = \text{tr} \tau_A$ holds (solid line), and the curve for which $\det \tau_A = \eta^2/\chi^2$ (dashed line). The solid line divides the $\chi_1 - \chi_2$ plane in a part where $D\det \tau_A/Dt > 0$, indicated by + and a part where $D\det \tau_A/Dt < 0$, indicated by −.

3.9 Elongational flow and dissipative stability

In the previous chapter it was shown that when $\det L < 0$ in the 2-dimensional case and $L = 0$ (MCSH) then the stress grows exponentially in time in a certain direction. It was shown in the second section of this chapter that the UCM model contains a damping term, which damps exponentially in time with exponent $-1/(2\lambda)$. If the growth due to the associated relative tensor exceeds this value, the stress will still grow exponentially and no steady solution exists. This phenomenon is called dissipative instability. Since the solution grows exponentially in time independent of the wavelength $\omega$ this growth is stable in the sense of Hadamard. The next example will illustrate this form of instability.

Example 5 Planar elongational flow (2-dimensional).

Suppose the flow is given by $u = \dot{\varepsilon}x$ and $v = -\dot{\varepsilon}y$, $\dot{\varepsilon} \in \mathbb{R}$, in which $\vec{u} = (u, v)$ and $\vec{x} = (x, y)$. This flow satisfies the continuity equation. The velocity gradient $L$ is given by

$$L = \begin{pmatrix} \dot{\varepsilon} & 0 \\ 0 & -\dot{\varepsilon} \end{pmatrix}.$$ 

For this type of flow the pressure is irrelevant and the velocity components are given. The stress may then be calculated from the constitutive equation. A steady state solution is given by

- $\tau_{11} = \frac{2\eta\dot{\varepsilon}}{1-2\lambda^2}$,
- $\tau_{12} = 0$. 


The Trouton or elongational viscosity is defined as

\[ \eta_T := \frac{\tau_{11} - \tau_{22}}{\dot{\epsilon}}. \]

So the UCM-model predicts an elongational viscosity

\[ \eta_T = \frac{4\eta}{(1 - 2\lambda\dot{\epsilon})(1 + 2\lambda\dot{\epsilon})}. \]

The graph of the Trouton viscosity vs. the elongation rate \( \dot{\epsilon} \) is given in Fig. (3.2). Both the individual stress components and the elongational viscosity show that a steady state solution only exists if \(-1/2\lambda < \dot{\epsilon} < 1/2\lambda\) and \(\eta_T \to \infty\) if \(\dot{\epsilon} \nearrow 1/2\lambda\) or \(\dot{\epsilon} \searrow -1/2\lambda\). Beyond these limits no steady state solution exists. In the neighborhood of these limit values of the elongation rate small changes in the elongation rate induce very large changes in the Trouton viscosity and \(\tau_{11}\) if \(\dot{\epsilon} > 0\), but the other way around is 'more stable', i.e. a small change in the Trouton viscosity or the stress tensor induces even smaller changes in the elongation rate. This is not only advantageous from the point of view of accuracy, but also to maintain the positive definiteness of the tensor \(\mathbf{T}_A\).

From a physical point of view it is strange to impose a velocity field and to expect that there must be a stationary stress tensor which is compatible with this flow. The least thing one can do is to impose an initial velocity field, supplemented by boundary conditions, and hope that this initial field remains unchanged. It seems that if the elongation rate is smaller than \(1/2\lambda\), then such a stationary stress solution exists, but if this condition is not fulfilled then there is no reason to assume that the initial velocity field remains the same. It may well be that some sort of 'viscoelastic buckling' occurs, in which the elongational flow starts to rotate or shear, and in consequence the pure elongational flow is lost.

Numerical flow simulations have to give the answer if unbounded stress components really occur in flows with an elongational character, such as contraction flow.
3.10 Boundary and initial conditions

At the boundary all the characteristic quantities which enter the computational domain have to be prescribed; either they are given a pre-assigned value or are expressed in terms of the outgoing characteristic variables. The boundary can be divided into an inflow boundary, an outflow boundary, a solid wall and a symmetry condition.

3.10.1 Boundary conditions for the elliptic part

It has been shown in this chapter that the Poisson equation has to satisfy a certain compatibility relation, which states that mass can be conserved locally if it is conserved globally. So for an incompressible fluid this means that

$$\int_{\partial\Omega} \vec{u} \cdot \vec{n} \, d\Omega = 0.$$ 

This relation requires that the normal velocity has to be prescribed, such that this relation holds. Along a wall and an axis of symmetry $\vec{u} \cdot \vec{n} = 0$, so one has to choose the normal velocity at the inflow and outflow boundary condition such that the amount of mass entering the domain is equal to the amount of mass leaving the domain.

3.10.2 Inflow boundary conditions

The inflow boundary condition is characterized by $\vec{u} \cdot \vec{n} < 0$, in which $\vec{n}$ is the outward unit normal to the boundary. The number of boundary conditions depends on the number of characteristic velocities $\vec{a}$ satisfying $\vec{a} \cdot \vec{n} < 0$ and this in turn depends whether the flow is sub- or supercritical at the inflow boundary. If the flow is subcritical at the inflow boundary the characteristic quantity belonging to $\vec{a} \cdot \vec{n} = \vec{u} \cdot \vec{n}$ should be given and the characteristic velocity belonging to $\vec{a} \cdot \vec{n} = \vec{u} \cdot \vec{n} + c_n$. If $\vec{n}$ is given by $\vec{n} = (-1, 0)$ as depicted in Fig. 3.3, then the characteristic quantities to be given a value are

- $\delta \tau_{11}$ which is convected in the direction of the streamlines;
- $(\tau_{11} + G)\delta \tau_{22} - 2\tau_{12}\delta \tau_{12}$ which is also convected along the streamlines; and
- $\rho c_x \delta v - \delta \tau_{12}$ which belongs to the characteristic velocity $u + c_x$.

The last quantity can be expressed as a function of the outgoing characteristic quantity, $\rho c_x \delta v + \delta \tau_{12}$. The remaining unknowns at the boundary have to be prescribed by so-called numerical boundary conditions.

If the flow is supercritical at the inflow boundary the prescribed quantities at the this boundary are given by

- $\delta \tau_{11}$ which is convected along the streamlines;
- $(\tau_{11} + G)\delta \tau_{22} - 2\delta \tau_{12}$ which is also convected along the stream lines;
- $\rho c_x \delta v + \delta \tau_{12}$ which belongs to the characteristic velocity $u + c_x$; and
Figure 3.3: *Boundary conditions for a supercritical inlet and subcritical outlet perpendicular to the boundary*

- \( \rho \varepsilon_c \delta v - \delta \tau_{12} \) which belongs to the characteristic velocity \( u - c_x \).

In viscoelastic flow calculations one may also assume that a fully developed flow enters the computational domain. This allows one to prescribe all the stress and velocity unknowns at the inflow boundary. In the numerical scheme outlined in chapter 5 only the stress components and the normal velocity to the inflow boundary are prescribed and the tangential velocity is given by the numerical boundary condition, which is compatible with the number of in-going characteristics.

If the flow is supercritical at the inflow boundary condition both the extra stress components and the velocity unknowns should be given.

### 3.10.3 Outflow boundary condition

One way of treating the outflow boundary is to position it so far away that the flow has become fully developed. However, this may require very large exit lengths. In this work the normal velocity, associated with the elliptic part, is prescribed at the exit, such that the elliptic compatibility relation is satisfied and both the tangential velocity component and the shear stress component are extrapolated to the exit. If the flow is subcritical then one of these boundary conditions is a numerical boundary condition, and if the flow is supercritical then both boundary conditions are numerical, since no physical boundary conditions are to be prescribed in this case.

### 3.10.4 Boundary conditions at a wall

If the wall is only allowed to move in its own plane the normal velocity to the wall is equal to zero. Consequently, the flow near the wall is always subcritical and there is only
one characteristic quantity entering the domain. In this case the characteristic quantity \( \rho c_x \delta v - \delta \tau_{12} \) may be given as function of the outgoing characteristic quantity,

\[
\rho c_x \delta v - \delta \tau_{12} = A[\rho c_x \delta v + \delta \tau_{12}] + B .
\]

(3.7)

So a linear combination of \( \delta v \) and \( \tau_{12} \) may be prescribed in terms of the internal unknowns. If the no-slip condition for the velocity is used \( \delta v = 0 \), this leads to

\[
(1 + A) \delta \tau_{12} + B = 0 ,
\]

so \( \delta \tau_{12} = -B/(1 + A) \), where \( A \) and \( B \) are given, or this relation gives no value for \( \delta \tau_{12} \) if \( A = -1 \) and \( B \neq 0 \) or is unspecified if \( A = -1 \) and \( B = 0 \). The numerical calculations in chapter 5 show that imposition of the no-slip condition only leads to an unstable discretization, so another linear combination is used, compatible with the characteristic relation (3.7). The exact form will be given in chapter 5.

### 3.10.5 Symmetry boundary conditions

For channel flows one usually assumes that the flow is symmetric. This means that \( \vec{u} \cdot \vec{n} = 0 \), \( \partial \vec{u} / \partial \vec{n} = 0 \) and \( (\vec{l}, \vec{\tau} \vec{n}) = 0 \), in which \( \vec{l} \) is the unit tangential vector to the axis of symmetry.

### 3.11 Summary

In order to design stable numerical algorithms it is necessary to discretize the equations according to its type. In this chapter the type of the set of PDEs for the Maxwell and Jeffrey type models has been investigated. It was shown that the Maxwell type models constitute a set of equations of mixed elliptic/hyperbolic type. The hyperbolic part contains a wavelike part and one can distinguish regions in the flow where the fluid velocity is less than the shear wave velocity, the subcritical region, and parts in the flow where the fluid velocity exceeds the shear wave velocity, the supercritical part. The Jeffrey type models, on the other hand, do not contain a wavelike part, since the addition of Newtonian viscosity yields infinitely fast shear waves. These infinitely fast waves lead to a parabolic part and an extra hyperbolic part in the type of the Jeffrey type models.

It has been shown that the tensor \( \tau_A \) should be positive definite to ensure well-posedness for the Maxwell type equations. This requirement is not necessary for the Jeffrey type models, however, for finite wavelengths some positive definiteness requirement should be obeyed to ensure well-posedness. One may compare this with the situation which arises in the numerical simulation of the Navier-Stokes equations; the momentum equation is parabolic, but in numerical simulations one may have to use hyperbolic discretization methods in order to stabilize the numerical scheme. This is due to the fact that on a given grid the wave numbers are finite and if the mesh Reynolds number exceeds unity the hyperbolic convective terms dominate the diffusive terms.

For finite wavelengths \( \kappa \) all characteristic quantities damp exponentially. It has been noted that in numerical schemes \( \kappa \) is inversely proportional to the meshsize, so this exponential
damping is less influential on fine meshes, which may explain problems in the past due to mesh refinement.

The Stokes approximation or the creeping flow approximation leads to an additional elliptic part in both the Maxwell type models and the Jeffrey type models. The momentum equation reduces to a constraint on the extra stress tensor and the velocity vector acts as a constraint force to enforce this constraint on the extra stress tensor. It was argued that not only in the Stokes approximation the velocity vector has to be solved from the constitutive equations, but also in regions in the flow where the shear waves dominate over the fluid velocity, i.e. for very small viscoelastic Mach numbers. A similar treatment is recommended in regions where the elongational viscosity is very high, since a small change in the extension rates induces a large change in the stresses, whereas a small change in the stresses leads to an even smaller change in the extension rates.

The definitions of well-posedness and stability have been given in section 3.7 and it follows that the tensor $\tau_A$ has to be positive definite in order to have a well-posed set of PDEs. If one of the eigenvalues of the tensor $\tau_A$ changes sign it will immediately change the sign of the determinant of $\tau_A$, whereas the trace of this tensor may still be positive. The lower bound on the determinant of $\tau_A$, given in section 3.8, ensures that this will not be the case. Based on the sign of the characteristic velocities the number and form of the boundary conditions were discussed. The number of physical boundary conditions to be prescribed depends whether the flow is sub- or supercritical. Those values which may not be prescribed as physical boundary conditions may be given as numerical boundary conditions. The numerical boundary conditions will be discussed in chapter 5.