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Unilaterally constrained dynamical systems
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Chapter 7

The contact problem in a nonlinear setting

7.1 Introduction

In this chapter we will investigate the contact problem for a class of nonlinear systems. The results of this chapter will appear in 'A.A. ten Dam, An approach to the contact problem for unilaterally constrained nonlinear dynamical systems'.

The nonlinear analogue of system (6.6) is given by

\[
\Sigma^c(f, g, h) : \begin{cases} 
\frac{dx}{dt} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i, \\
h(x) &\geq 0,
\end{cases}
\]  

(7.1)

where \( x \in \mathcal{M} \), with \( \mathcal{M} \) an open subset of \( \mathbb{R}^n \), the controls \( u_i, \ i \in m \), take their values in \( \mathbb{R}, f : \mathcal{M} \to \mathbb{R}^n \), and \( g_i : \mathcal{M} \to \mathbb{R}^n, \ i \in m, \) and \( h(x) = \text{col}(h_1(x), \ldots, h_p(x)) \) with \( h_i : \mathcal{M} \to \mathbb{R}, \ i \in p \). We assume that \( f(x), g_i(x) \) and \( h_i(x) \) are analytical functions on \( \mathcal{M}, \) i.e. with continuous partial derivatives of any order, and expandable in a Taylor-series in its arguments about each point of \( \mathcal{M} \) [79, 113]. \( \Sigma^c(f, g, h) \) is an affine nonlinear control system subject to nonlinear inequality constraints. But it still not completely specified, i.e. it is yet to be defined what we mean by a trajectory of system \( \Sigma^c(f, g, h) \).

The geometric theory of nonlinear systems proved to be a very successful framework to discuss many issues in nonlinear dynamical systems [79, 113]. In many instances, nonlinear systems are studied when represented by

\[
\begin{align*}
\frac{dx}{dt} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i, \\
y &= h(x).
\end{align*}
\]  

(7.2)
Chapter 7: The contact problem in a nonlinear setting

i.e., an output equation is present.

If the inequality constraint \( h(x) \geq 0 \) in (7.1) is replaced by the equality constraint \( h(x) = 0 \) then this can be captured in (7.2) by imposing the equality constraint \( y = 0 \). This leads to the concept of output zeroing manifolds (see e.g. [79]). The resulting system can be described (locally) by differential and algebraic equations of which many fundamental issues have already been resolved (see e.g. [21, 101]). In case of unilaterally constrained dynamical systems there are less results available and contributions usually focus on robotic systems. (See chapter 8 for a discussion.) For instance, classical invariance properties for nonlinear dynamical systems analogous to the discussion in chapter 5 have so far only received limited attention. (An approach for discrete-time systems is presented in [19]).

In this chapter we closely follow the line of presentation in the previous chapter and the reader may experience a sense of déjà vu. This is caused by the fact that we treat the same ideas and concepts as in chapter 6 but in a different mathematical setting. The importance of this chapter lies in the fact that in applications the system equations are often nonlinear, which makes treatment of the contact problem in a nonlinear setting important. At first emphasis will be on single inequality constraints since much can be learned already from this case. In section 7.2 we will discuss a unilaterally constrained mechanical system to motivate the assumptions we make in the nonlinear case. In section 7.3 we look in detail to the interaction of a nonlinear system with a manifold. In section 7.4 we then show how a discussion of the contact problem gives rise to the concept of collision in the nonlinear case. Multiple constraints and relaxation of the assumptions are discussed briefly in section 7.5. Alternative representations of system \( \Sigma^c(f,g,h) \) will be presented in section 7.6. Finally, in section 7.7 conclusions are stated.

Since chapter 8 is devoted to a detailed discussion of a complex nonlinear mechanical system, the examples in the present chapter usually deal with simple (linear) systems.

7.2 Mathematical preliminaries and problem formulation

Much of our motivation to discuss the contact problem has already been given in chapter 6. Compared to the linear case, for the nonlinear case some additional issues arise, mainly of a technical nature. Our fundamental assumptions for the class of systems (7.1) are now introduced and motivated using concepts from the differential geometric approach to nonlinear systems which are taken from [79, 113].

Let \( x = [x_1, \ldots, x_n]^T \). Given a \((n\text{-vector})\)-valued function \( f(x) = \text{col}(f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)) \) and a real-valued function \( h(x) = h(x_1, \ldots, x_n) \) one can define a new real-valued function of \( x, L_f h(x) : \mathbb{M} \to \mathbb{R} \), in the following way:

\[
L_f h(x) = L_f h(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i(x_1, \ldots, x_n). \tag{7.3}
\]
If one sets \( \frac{\partial h}{\partial x} = [\frac{\partial h}{\partial x_1} \ldots \frac{\partial h}{\partial x_m}] \), the function \( L_f h(x) \) can be expressed as:

\[
L_f h(x) = \frac{\partial h}{\partial x} f(x),
\]

where the latter is an inner product. The function \( L_f h(x) \) is sometimes called the directional derivative of \( h(x) \) along \( f(x) \). Repeated use of this operation is possible: one can first differentiate \( h(x) \) along \( f(x) \) and then along \( g_i(x) \), giving \( L_{g_i} L_f h(x) = \frac{\partial L_{g_i} h}{\partial x} g_i(x) \). It can be seen that \( i \) times differentiation of \( h(x) \) along \( f(x) \) can be defined recursively as

\[
L_f^i h(x) = \frac{\partial L_f^{i-1} h}{\partial x} f(x),
\]

where \( L_f^0 h(x) := h(x) \).

Denote for system (7.1), \( u := \text{col}(u_1, \ldots, u_m) \) and \( g := \text{row}(g_1, \ldots, g_m) \). To shorten notation we define, for \( i \in \mathbb{N} \),

\[
L_{g_i} L_f^i h(x) := [L_{g_1} L_f^i h(x) \ldots L_{g_m} L_f^i h(x)].
\]

The notion of relative degree [79], usually defined for systems of the form (7.2), is often applied in controller synthesis. We will use this notion in the analysis phase of unilaterally constrained dynamical systems, where we take a single inequality constraint.

**Definition 7.2.1** Let \( f : \mathcal{M} \to \mathbb{R}^n \), \( g_i : \mathcal{M} \to \mathbb{R}^n \), \( i \in \mathbb{N} \), and \( h : \mathcal{M} \to \mathbb{R} \). Then system \( \Sigma^c(f, g, h) \) has relative degree\(^1\) \( r_0 \) at a point \( \hat{x} \in \mathcal{M} \) if:

(i) \( L_{g_i} L_f^i h(x) = 0 \) for all \( x \) in a neighbourhood of \( \hat{x} \) and all \( i < r_0 - 1 \),

(ii) \( L_{g_i} L_f^{r_0 - 1} h(\hat{x}) \) has full-row rank.

For points where the relative degree is finite one has that \( r_0 \leq n \) [79]. Contrary to the linear case, the concept of relative degree is now defined with respect to a point \( \hat{x} \), and is thus a local concept. However, if we consider the special case \( f(x) = Ax, g(x) = B \) and \( h(x) = Cx \) it is easy to show that \( L_f^i h(x) = CA^i x \) and \( L_{g_i} L_f^i h(x) = CA^i B \). Hence the integer \( r_0 \) is characterized by the conditions \( CA^i B = 0 \) for all \( i < r_0 \) and \( CA^{r_0 - 1} B \neq 0 \). These conditions are equal to the conditions used to define the relative degree in the linear case (see chapter 6). In case of multiple constraints, i.e. \( h : \mathcal{M} \to \mathbb{R}^p \), sometimes the assumption is made that the \( p \) systems \( \Sigma^c(f, g, h_i) \) all have the same relative degree, which is then referred to as uniform relative degree. We will return to this in section 7.5.

Let us see what this local concept of relative degree brings us in case of a unilaterally constrained mechanical system. The examples given in chapter 1 that deal with mechanical systems subject to holonomic inequalities can be cast in a generic form. The unconstrained

\[\text{\footnotesize\textsuperscript{1}Where possible we will use the same notation as in the previous chapter. This is justified by proposition 7.3.5 below.}\]
Chapter 7: The contact problem in a nonlinear setting

dynamics part can be represented by:

\[ M(y) \frac{d^2 y}{dt^2}(t) + N(y(t), \frac{dy}{dt}(t)) = u(t). \quad (7.7) \]

Analogously to chapter 6, \( y \in (\mathbb{R}^d)^\mathbb{R}_+ \), \( y \) denotes the generalized system coordinate vector, \( u \in (\mathbb{R}^n)^\mathbb{R}_+ \), \( u \) the generalized force vector, \( M(\cdot) \in \mathbb{R}^{d \times d} \) the generalized positive definite inertia matrix, and \( N(\cdot, \cdot) \in \mathbb{R}^{d \times 1} \) contains nonlinear terms such as Coriolis and centrifugal vectors.

Let the restrictions be modelled by a map \( \phi: \mathbb{R}^d \rightarrow \mathbb{R} \),

\[ \phi(y) \geq 0. \quad (7.8) \]

Take \( x = [x_1^T, x_2^T]^T := [y^T, (\frac{dy}{dt})^T]^T \). Then equations (7.7) and (7.8) can be written in first-order form as:

\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt}
\end{bmatrix}
= \begin{bmatrix}
x_2 \\
-M^{-1}(x_1)N(x_1, x_2)
\end{bmatrix} + \begin{bmatrix}
0 \\
M^{-1}(x_1)
\end{bmatrix} u
= f(x) + g(x)u,
\]

\[ h(x) := \phi(x_1) \geq 0. \quad (7.9) \]

It is not difficult to show that for system (7.9) there holds

\[ L_g h(x) = 0, \quad \text{and} \]

\[ L_g L_f h(x) = \frac{\partial \phi}{\partial x_1}(x_1) M^{-1}(x_1). \quad (7.10) \]

Since it is assumed that \( M(x_1) \) is a positive definite matrix (for all \( x_1 \)) it depends on the value of \( \frac{\partial \phi}{\partial x_1}(\hat{x}_1) \) whether system (7.9) has constrained relative degree equal to 2 or not at a point \( \hat{x} = [\hat{x}_1^T, \hat{x}_2^T]^T \). From [79] we have that if the relative degree at one point in \( \mathcal{M} \) is finite, then in almost all points in \( \mathcal{M} \) the relative degree has the same finite value: the collection of points where the relative degree is finite is an open and dense subset of \( \mathcal{M} \) (examples are given in [79]). The question arises for which points the relative degree does not equal 2 for system (7.9). From chapter 5 we know that for the linear case the boundary set is important. One can expect the same to hold in a discussion on invariance properties for the nonlinear case. If, for instance, \( \frac{\partial \phi}{\partial x_1}(x_1) \) has full row-rank, then it follows from (7.10) that \( L_g L_f h(\hat{x}) \neq 0 \) at almost any point \( \hat{x} \). We can therefore rephrase the question as: can it happen that \( L_g L_f h(\hat{x}) = 0 \) when \( h(\hat{x}) = 0 \) for system (7.9)? To answer this question we look at an example.
Let \( r \in \mathbb{R} \) be a fixed constant with \( r > 0 \). Consider the constraint in two dimensions,

\[
\phi(y) = r^2 - y_1^2 - y_2^2 \geq 0.
\]

This is the constraint that is used in example 1.2.3. Denote \( x_1 = [x_{11}, x_{12}]^T := [y_1, y_2]^T \). It is easy to see that

\[
\frac{\partial \phi}{\partial x}(x_1) = [-2x_1 - 2x_1_2].
\]

Since matrix \( M(x_1) \) is positive definite, it now follows from (7.10) that the relative degree at a point \( \hat{x} \) is larger than 2 if and only if \( \hat{x} = (0, \hat{x}_2)^T \). In that case, however, one is not on the boundary set since \( h(0, \hat{x}_2) = \phi(0) = r^2 \neq 0 \). In this case the relative degree has the same value for points on the manifold defined by the equation \( h(x) = 0 \). So, although it may well happen that for system (7.9) \( L_g L_f h(\hat{x}) = 0 \) when \( h(\hat{x}) = 0 \), there exist many interesting cases where this will not happen.

The above motivates the discussion on the nonlinear contact problem. The following assumptions hold throughout this chapter for system functions \( f(x) \) and \( g(x) \), and constraint function \( h(x) \), unless stated otherwise:

**Assumption 7.2.2**

(i) \( \Sigma^c(f, g, h) \) has finite relative degree \( r_0 \) for all \( x \) such that \( h(x) = 0 \);

(ii) \( L_g h(x) = [0 : \ldots : 0] \);

(iii) \( \{x \in \mathcal{M} \mid h(x) = 0\} \neq \emptyset \);

(iv) \( \{x \in \mathcal{M} \mid h(x) \geq 0\} \neq \mathcal{M} \).

Relaxation of these assumptions will be discussed in section 7.5. In the linear case we required controllability of the matrix pair \( (A, B) \), c.f. assumption 6.2.2 (i). That assumption implies that the relative degree in the linear case is finite, provided that matrix \( C \neq 0 \) (which is assumption 6.2.2 (iii)). Assumption 7.2.2(ii) is based on (7.9). It is easy to show that the assumption \( L_g h(x) = 0 \) becomes assumption 6.2.2 (ii) for linear systems. Assumptions 7.2.2 (iii) and (iv) are made to exclude the trivial cases from our analysis. For instance, a constraint \( x^2 + 1 \geq 0 \) is not very interesting in a discussion on the contact problem. We will use the phrase ‘\( \mathcal{M}_n(f, g, h) \) satisfies the assumptions’ to indicate that in the open subset \( \mathcal{M} \subseteq \mathbb{R}^n \), with \( \dim(\mathcal{M}) = n \), system functions \( f : \mathcal{M} \to \mathbb{R}^n \) and \( g_i : \mathcal{M} \to \mathbb{R}^n, i \in m \), of the unconstrained system, and single constraint function \( h : \mathcal{M} \to \mathbb{R} \) satisfy the above assumptions.

Our research will concentrate on the nonlinear analogue of problem 6.2.1. Consider the system

\[
\Sigma : \frac{dx}{dt} = f(x) + g(x)u,
\]
with \( f : M \to \mathbb{R}^n \), \( g := \text{col}(g_1, \ldots, g_m) \), \( g_i : M \to \mathbb{R}^n \), and \( u \) takes its values in \( \mathbb{R}^m \), with the inequality constraint defined by

\[
\begin{align*}
h(x) & \geq 0. \\
\end{align*}
\]  

Since linear systems are a special case of nonlinear systems, we already know from chapter 6 that the set \( \{ x \in M \mid h(x) \geq 0 \} \) need not be an invariant set of system (7.13).

**Problem 7.2.3** Consider the system \( \Sigma \) in (7.13) subject to constraint (7.14). Given the set of functions \( f \), \( g \), and \( h \), how to make the set defined by the unilateral constraint an invariant set for the system \( \Sigma \)?

As in the linear case, solving problem 7.2.3 will lead to a description of the unilateral constrained dynamical system (7.1), where the interaction with the boundary set of the inequality constraint is explicitly taken into account.

In the linear case, discussed in chapter 6, the largest controlled invariant subspace in \( \ker(C) \) is important. For the nonlinear case this becomes the largest locally controlled invariant manifold [79]. A manifold \( N \) is said to be locally controlled invariant if one can remain on this manifold, in a neighbourhood of some point \( \hat{x} \) with \( h(\hat{x}) = 0 \), by applying a static feedback of the form \( u = \alpha(x) + \beta(x)v \), with \( v \) the new input. If the relative degree \( r_0 \) is finite then it can be shown, see for instance [79], that the largest locally controlled invariant manifold in a neighbourhood \( M(\hat{x}) \) of \( \hat{x} \), assuming that \( h(\hat{x}) = 0 \), can be given by

\[
N^i = \{ x \in M \cap M(\hat{x}) \mid h(x) = 0, L_f h(x) = 0, \ldots, L_f^{r_0-1} h(x) = 0 \}.
\]

The use of \( N^i \) in our research of the contact problem is discussed in the sequel of this chapter.

### 7.3 Boundary set: a subdivision

In this section we will investigate how trajectories of the system

\[
\Sigma : \frac{dx}{dt} = f(x) + g(x)u,
\]

interact with the boundary set of the single inequality constraint defined by

\[
h(x) \geq 0.
\]

Trajectories of system (7.16) will be denoted by \( \bar{x} \).

We will make a subdivision of the open set \( M \) based on the interaction of trajectories of unconstrained system (7.16) with \( \{ x \in M \mid h(x) = 0 \} \). We will also discuss the relation
between the subsets that will be defined for the nonlinear case with the subsets defined for the linear case.

A global decomposition of the set $\mathcal{M}$ is given by

$$\mathcal{M}_g = \{ x \in \mathcal{M} \mid h(x) > 0 \},$$  \hspace{1cm} (7.18)

$$\mathcal{M}_b = \{ x \in \mathcal{M} \mid h(x) = 0 \},$$  \hspace{1cm} (7.19)

$$\mathcal{M}_r = \{ x \in \mathcal{M} \mid h(x) < 0 \}.$$  \hspace{1cm} (7.20)

As in the previous chapter, the subscript $g$ stands for 'good', the subscript $f$ stands for 'false', and the subscript $b$ stands for 'boundary'.

Let $\mathcal{M}_{con}$ (con for contact) denote the set of points where a trajectory of system (7.16) that starts in $\mathcal{M}_g$ can come into contact with the set $\mathcal{M}_b$. Analogously, let $\mathcal{M}_{rel}$ (rel for release) denote the set of points where a trajectory of system (7.16) can leave the set $\mathcal{M}_b$, and continue in $\mathcal{M}_g$. These sets are the nonlinear equivalent of the contact and release sets defined in definition 6.3.1, and are formally defined by:

**Definition 7.3.1** The contact and release sets. Let $x$ denote a trajectory of system (7.16).

(i) The contact set $\mathcal{M}_{con}$ is defined as $\mathcal{M}_{con} := \{ x \in \mathcal{M}_b \mid \exists \underline{x} \in \Sigma \text{ and } \exists t^+ < 0 \text{ such that } x(0) = x, \text{ and } x(\tau) \in \mathcal{M}_g, \forall \tau : t^+ < \tau < 0 \}$. 

(ii) The release set $\mathcal{M}_{rel}$ is defined as $\mathcal{M}_{rel} := \{ x \in \mathcal{M}_b \mid \exists \underline{x} \in \Sigma \text{ and } \exists t^- > 0 \text{ such that } x(0) = x, \text{ and } x(\tau) \in \mathcal{M}_g, \forall \tau : 0 < \tau < t^- \}$. 

The question arises how to characterize the contact and release sets in terms of the system functions $f$ and $g$, and the constraint function $h$. It will be shown that the notion of relative degree is as useful in the nonlinear case as it was in the linear case.

Define the fictitious output $y := h(x)$. Assume that the unconstrained system at time $t = 0$ is in state $x(0) = x_0$. Whether or not a trajectory leaves the boundary set, or remains in it, can be decided by looking at the derivatives $y^{(i)}(t), i \geq 0$. It can be seen that for the fictitious output $y(t)$, the first coefficient of the Taylor expansion around $t = 0$ that is unequal to zero is important. Let $u \in \mathbb{R}^m$; i.e. $u$ is a sequence of vectors which take their value in $U$. Define the map $h_i : \mathcal{M}_b \times \mathbb{R}^m \rightarrow \mathbb{R}$ as

$$h_i(x, u) := L_i^f h(x) + \sum_{j=1}^{i} L_a L_{f_j}^{-1} h(x) u_{i-j}.$$  \hspace{1cm} (7.21)

**Lemma 7.3.2** Let $\mathcal{M}_{a}(f,g,h)$ satisfy the assumptions. Define $y := h(x)$. Let $\underline{x} \in \Sigma^c(f,g,h)$ with $\underline{x}(0) = x_0$ and $h(x_0) = 0$. Assume that relative degree at $x_0$ equals $r_0$. Define $u_0(t) = u^{(i)}(t)$, with $u_0(t) := u(t)$. Then $y^{(i)}(t) = h_i(\underline{x}(t), u(t)) = L_i^f h(x)$ for all $i < r_0$ and all $t$ near 0, and $y^{(r_0)}(0) = h_{r_0} (x_0, u(0)) = L_{f_0}^r h(x_0) + L_a L_{f_0}^{-1} h(x_0) u(0)$. 

From lemma 7.3.2 follows that the relative degree \( r_0 \) at a point \( x_0 \) is exactly equal to the number of times the fictitious output \( y(t) \) has to be differentiated (with respect to time) at time \( t = 0 \) in order to have the value \( u(0) \) of the input appearing explicitly. For \( i > r_0 \) the map \( h_i \) does not have a direct relation to derivatives of the fictitious output \( y \), as derivatives of the map \( g \) also need to be taken into account. Lemma 7.3.2 provides the key observation to generalize the work done in the previous chapter on constrained linear dynamical systems to the nonlinear case.

**Definition 7.3.3** Define the integers \( r \) and \( r_0 \) as

(i) \( r : M_b \times U^\mathbb{R} \to \mathbb{N} \cup \{ \infty \} \) as \( r(x, u) := \min\{i \in \mathbb{N} \mid h_i(x, u) \neq 0\} \)

with \( r(x, u) := \infty \) if \( h_i(x, u) = 0, \forall i \in \mathbb{N} \).

(ii) \( r_0(x) := \min\{i \in \mathbb{N} \mid \exists u \in U^\mathbb{R} : h_i(x, u) \neq 0\} \).

Note that if \( M_n(f, g, h) \) satisfies the assumptions then \( 1 \leq r_0(x) \leq r_0 \leq n \), and \( 1 < r_0 \). Since the map \( h_i \) does not have a direct relation with the derivatives of the fictitious output \( y \) for \( i > r_0 \), a description of the contact set \( M_{\text{con}} \) and release set \( M_{\text{rel}} \) in terms of the map \( h_i \) is not feasible for \( i > r_0 \). The nonlinear analogue of lemma 6.4.4 must be stated in terms of \( y^{(0)} \). And similar to the linear case, a subdivision of the boundary set can be based on the finite integers \( r_0(x) \) and \( r_0 \). In analogy to the sets defined in section 6.4 we define the following sets.

**Definition 7.3.4** Define the following subsets of the boundary set \( M_b \):

(i) \( N_g := \{ x \in M_b \mid \forall u \in U^\mathbb{R} : h_b(x) \text{ is even, and } h_{r_b(x)}(x, u) > 0 \} \).

(ii) \( N_f := \{ x \in M_b \mid \forall u \in U^\mathbb{R} : h_b(x) \text{ is even, and } h_{r_b(x)}(x, u) < 0 \} \).

(iii) \( N_c := \{ x \in M_b \mid r_b(x) = r_0 \} \).

(iv) \( M_{\text{con},v} := \{ x \in M_{\text{con}} \mid r_b(x) = 1 \} \).

(v) \( M_{\text{rel},v} := \{ x \in M_{\text{rel}} \mid r_b(x) = 1 \} \).

(vi) \( M_{\text{con},h} := \{ x \in M_{\text{con}} \mid \forall u \in U^\mathbb{R} : 1 < r_b(x), h_b(x) \text{ is odd and } h_{r_b(x)}(x, u) < 0 \} \).

(vii) \( M_{\text{rel},h} := \{ x \in M_{\text{rel}} \mid \forall u \in U^\mathbb{R} : 1 < r_b(x), h_b(x) \text{ is odd and } h_{r_b(x)}(x, u) > 0 \} \).

In order to compare the contact and release sets defined so far for the nonlinear case with the contact and release sets defined for the linear case, let us consider the system

\[
\frac{dx}{dt} = Ax + Bu,
\]

\[
Cx \geq d.
\]  

(7.22)

**Proposition 7.3.5** Consider system (7.22) as a special case of a unilaterally constrained nonlinear system, i.e. take \( f(x) = Ax \), \( g(x) = B \), and \( h(x) = Cx - d \). Let \( M_n(f, g, h) \) satisfy the assumptions and let \( h : M \to \mathbb{R} \). Then the following relations hold: \( N_g = V_g \), \( N_f = V_f \), \( N_c = V_c \), \( M_{\text{con},v} = X_{\text{con},v} \), \( M_{\text{rel},v} = X_{\text{rel},v} \), \( M_{\text{con},h} = X_{\text{con},h} \), \( M_{\text{rel},h} = X_{\text{rel},h} \).

Proposition 7.3.5 provides us with the motivation to use the same notation in the linear and the nonlinear case where possible. Note that the constraint \( Cx = d \) is nonlinear (if
Finally, we prove that all subsets defined so far are invariant under regular static state feedback.

**Proposition 7.3.6** Let $\mathcal{M}_n(f, g, h)$ satisfy the assumptions. Then the subsets $\mathcal{N}_c, \mathcal{N}_g, \mathcal{N}_f, \mathcal{M}_{con,v}, \mathcal{M}_{con,h}, \mathcal{M}_{rel,v}$ and $\mathcal{M}_{rel,h}$ are invariant under the regular static state feedback $u = \alpha(x) + \beta(x)v$, with $v$ the new input and $\beta(x)$ nonsingular for all $x$, for system $\Sigma(f, g, h)$.

The following result will be important.

**Theorem 7.3.7** Relations between subsets of the state-space.

Let $\mathcal{M}_n(f, g, h)$ satisfy the assumptions. Then:

(i) $\mathcal{M}_0 = \mathcal{N}_c \cup \mathcal{N}_f \cup \mathcal{N}_g \cup \mathcal{M}_{con,v} \cup \mathcal{M}_{con,h} \cup \mathcal{M}_{rel,v} \cup \mathcal{M}_{rel,h}$.

(ii) The subsets $\mathcal{M}_g, \mathcal{M}_f, \mathcal{N}_c, \mathcal{N}_f, \mathcal{N}_g, \mathcal{M}_{con,v}, \mathcal{M}_{con,h}, \mathcal{M}_{rel,v}$ and $\mathcal{M}_{rel,h}$ are pairwise disjoint.

(iii) $\mathcal{M}_{con} = \mathcal{N}_c \cup \mathcal{N}_g \cup \mathcal{M}_{con,v} \cup \mathcal{M}_{con,h}$.

(iv) $\mathcal{M}_{rel} = \mathcal{N}_c \cup \mathcal{N}_g \cup \mathcal{M}_{rel,v} \cup \mathcal{M}_{rel,h}$.

The following result is now immediate from proposition 7.3.6.

**Corollary 7.3.8** The subsets $\mathcal{M}_{con}$ and $\mathcal{M}_{rel}$ are invariant under the regular static state feedback $u = \alpha(x) + \beta(x)v$, with $v$ the new input and $\beta(x)$ nonsingular for all $x$, for the system $\Sigma(f, g, h)$.

Next, as a summary, an intuitive explanation is given of all subsets that have been defined so far. Let $\mathcal{M}_n(f, g, h)$ satisfy the assumptions and let $h : \mathcal{M} \rightarrow \mathbb{R}$. We can make the following statements relating $x \in \mathcal{M}$ with a trajectory $\bar{x}$ of the unconstrained system $\frac{dx}{dt} = f(x) + g(x)u$:

(a) $x \in \mathcal{M}_g \Leftrightarrow x$ satisfies the inequality constraint strictly.

(b) $x \in \mathcal{M}_0 \Leftrightarrow x$ belongs to the boundary set.

(c) $x \in \mathcal{M}_f \Leftrightarrow x$ does not satisfy the inequality constraint.

(d) $x \in \mathcal{M}_{con,v} \Leftrightarrow$ all trajectories with $\bar{x}(t) = x$ for some $t$ go transversally through the boundary set from $\mathcal{M}_g$ to $\mathcal{M}_f$.

(e) $x \in \mathcal{M}_{con,h} \Leftrightarrow$ all trajectories with $\bar{x}(t) = x$ for some $t$ go tangentially through the boundary set from $\mathcal{M}_g$ to $\mathcal{M}_f$.

(f) $x \in \mathcal{N}_g \Leftrightarrow$ all trajectories with $\bar{x}(t) = x$ for some $t$ go tangentially through the boundary set from $\mathcal{M}_g$ to $\mathcal{M}_f$.

(g) $x \in \mathcal{N}_c \Leftrightarrow$ all trajectories with $\bar{x}(t) = x$ for some $t$ go tangentially through the boundary set, and there exists at least one trajectory that remains in $\mathcal{M}_0$.

(h) $x \in \mathcal{N}_f \Leftrightarrow$ all trajectories with $\bar{x}(t) = x$ for some $t$ go tangentially through the boundary set from $\mathcal{M}_f$ to $\mathcal{M}_f$. 

Chapter 7: The contact problem in a nonlinear setting

(i) \( x \in \mathcal{M}_{\text{rel},h} \iff \) all trajectories with \( x(t) = x \) for some \( t \) go tangentially through the boundary set from \( \mathcal{M}_f \) to \( \mathcal{M}_g \).

(j) \( x \in \mathcal{M}_{\text{rel},v} \iff \) all trajectories with \( x(t) = x \) for some \( t \) go transversally through the boundary set from \( \mathcal{M}_f \) to \( \mathcal{M}_g \).

In appendix A.2 expressions are derived for all subsets defined so far in terms of the original functions \( f, g \) and \( h \).

It is intuitively clear that the contact sets and the release sets switch roles for the time-reversed system i.e. the system with system functions \(-f(x)\) and \(-g(x)\).

**Proposition 7.3.9** Let \( \mathcal{M}_n(f, g, h) \) satisfy the assumptions. Then the following relations hold between subsets of system \( \Sigma(f, g, h) \) and the time-reversed system \( \Sigma(-f, -g, h) \):

\[
\begin{align*}
\mathcal{M}_g(f, g, h) &= \mathcal{M}_g(-f, -g, h), \\
\mathcal{M}_{\text{con},v}(f, g, h) &= \mathcal{M}_{\text{rel},v}(-f, -g, h), \\
\mathcal{M}_{\text{rel},v}(f, g, h) &= \mathcal{M}_{\text{con},v}(-f, -g, h), \\
\mathcal{N}_g(f, g, h) &= \mathcal{N}_g(-f, -g, h), \\
\mathcal{N}_f(f, g, h) &= \mathcal{N}_f(-f, -g, h), \\
\mathcal{M}_{\text{con},h}(f, g, h) &= \mathcal{M}_{\text{rel},h}(-f, -g, h), \\
\mathcal{M}_{\text{rel},h}(f, g, h) &= \mathcal{M}_{\text{con},h}(-f, -g, h), \\
N_c(f, g, h) &= N_c(-f, -g, h).
\end{align*}
\]

It can be seen that in the subset \( \mathcal{M}_{\text{con}} \setminus N_c \), application of a smooth control can not prevent a trajectory of the unconstrained system (7.16) to enter \( \mathcal{M}_f \). It is clear that this finding has consequences for (feedback) controller design if one of the objectives is smooth contact with the boundary set. Also, in \( N^a \), one should not apply a control that will drive trajectories of the system (7.16) into \( \mathcal{M}_f \). This leads to the concepts of locally applicable and forbidden controls for the nonlinear case.

**Definition 7.3.10** Consider system \( \Sigma^c = \Sigma^c(f, g, h) \). Let \( x \in N_c \). Then:

(i) The set of **locally applicable controls** is defined as: \( \mathcal{U}_g(x) = \{ u \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^m) \mid \exists x \in \Sigma^c, \exists ^t > 0 \text{ such that } x(0) = x, \text{ and } x(\tau) \in M_g \cup M_h, \forall \tau : 0 < \tau < ^t \} \).

(ii) The set of **locally applicable boundary controls** is defined as: \( \mathcal{U}_b(x) = \{ u \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^m) \mid \exists x \in \Sigma^c, \exists ^t > 0 \text{ such that } x(0) = x, \text{ and } x(\tau) \in M_b, \forall \tau : 0 < \tau < ^t \} \).

(iii) The set of **locally forbidden controls** is defined as: \( \mathcal{U}_f(x) = \{ u \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^m) \mid \exists x \in \Sigma^c, \exists ^t > 0 \text{ such that } x(0) = x, \text{ and } x(\tau) \in M_f, \forall \tau : 0 < \tau < ^t \} \).

Definition 7.3.10 provides a further subdivision of the boundary set \( h(x) = 0 \). As far as the points \( x \in N_c \) is concerned, these sets are not disjoint. The set of forbidden controls will prove useful when we introduce the collision maps in our framework.

In case of bilaterally constrained dynamical systems the set \( \mathcal{U}_b(x) \) models the set of applicable controls for a DAE system. In this case an explicit expression can be derived for \( \mathcal{U}_b(x) \); it can be shown that for system (7.16),

\[
\mathcal{U}_b(x) = \{ u \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^m) \mid L_f^p h(x) + L_q L_f^{p-1} h(x) u = 0 \}.
\]

The above sets play an important role also in finding alternate representations of \( \Sigma^c(f, g, h) \), which will be discussed in section 7.6 below.
7.4 Introducing the collision maps

In this section we will define what we mean by a constrained nonlinear system $\Sigma^c$, i.e. what we mean by a trajectory of a unilaterally constrained dynamical system.

Recall that if contact is made in $\mathcal{M}_{\text{con}} \setminus \mathcal{N}_c$ then application of a smooth control can not prevent a trajectory to enter $\mathcal{M}_f$. Hence, if $\mathcal{M}_{\text{con}} \setminus \mathcal{N}_c \neq \emptyset$ the set $\{ x \in \mathcal{M} \mid h(x) \geq 0 \}$ is not an invariant set in the sense of chapter 5 for the system $\frac{dx}{dt} = f(x) + g(x)u$. If we want the set $\{ x \in \mathcal{M} \mid h(x) \geq 0 \}$ to act as an invariant set we need additional modelling. In mechanical systems contact between subsystems is discussed in terms of collisions. This motivates the following definition.

**Definition 7.4.1** Let $x \in \mathcal{M}$. If $x \in (\mathcal{M}_{\text{con}} \setminus \mathcal{N}_c) \cup \mathcal{N}_f$ we will call contact at $x$ an *uncontrolled collision*. Likewise, if $x \in \mathcal{M}_{\text{rel}} \setminus \mathcal{N}_c$ we will call release at $x$ an *uncontrolled release*. If $x \in \mathcal{N}_c$ then we will say that contact, release, at $x$ is a *controlled collision, controlled release*, respectively.

The introduction and decomposition of the collision maps $T$ that deal with contact with a boundary set have already been motivated in chapter 6. In the nonlinear case we propose the following set up, motivated by the results in theorem 7.3.7 and definition 7.4.1.

**Definition 7.4.2** Consider system $\Sigma^c(f, g, h)$. Let $\mathcal{M}_n(f, g, h)$ satisfy the assumptions. Then we will call a map $T_u : (\mathcal{M}_{\text{con}} \setminus \mathcal{N}_c) \cup \mathcal{N}_f \to \mathcal{M}_{\text{rel}} \setminus \mathcal{N}_c$ an *uncontrolled collision map*. We will call a map $T_c : \mathcal{N}_c \times \mathbb{U} \to \mathcal{N}_c \times (\mathbb{U} \setminus \mathcal{F})$ a *controlled collision map*.

Due to the introduction of the collision maps, it can be seen that any point in the unilateral constraint set can be chosen as initial condition for the constrained nonlinear system: in order to cross the boundary set, contact must be made in $\mathcal{M}_{\text{con}} \cup \mathcal{N}_f$. By choice of a collision map the trajectory will proceed from $\mathcal{M}_{\text{rel}}$, where the new state acts as an initial condition for the system. Thus, in our framework, trajectories of a unilaterally constrained nonlinear system consist of concatenated pieces of the unconstrained nonlinear system.

Similar to the linear case, we will specify a number of collision maps by further detailing the domain of this map. For this we need to introduce some notation. Based on the definition of the contact and release sets we can write

$$
\begin{align*}
\mathcal{N}_g & := \bigcup_{1 \leq i < \frac{1}{2^m}} \mathcal{N}_g^i, \text{ with } \mathcal{N}_g^i := \{ x \in \mathcal{M}_b \mid r_b(x) = 2i, h_{2i}(x, u) > 0 \}, \\
\mathcal{N}_f & := \bigcup_{1 \leq i < \frac{1}{2^m}} \mathcal{N}_f^i, \text{ with } \mathcal{N}_f^i := \{ x \in \mathcal{M}_b \mid r_b(x) = 2i, h_{2i}(x, u) < 0 \}, \\
\mathcal{M}_{\text{con}, h} & := \bigcup_{1 \leq i < \frac{1}{2^m}} \mathcal{M}_{\text{con}, h}^i, \\
\text{ with } \mathcal{M}_{\text{con}, h}^i & := \{ x \in \mathcal{M}_b \mid r_b(x) = 2i + 1, h_{2i+1}(x, u) < 0 \}, \\
\mathcal{M}_{\text{rel}, h} & := \bigcup_{1 \leq i < \frac{1}{2^m}} \mathcal{M}_{\text{rel}, h}^i, \\
\text{ with } \mathcal{M}_{\text{rel}, h}^i & := \{ x \in \mathcal{M}_b \mid r_b(x) = 2i + 1, h_{2i+1}(x, u) > 0 \}. 
\end{align*}
$$

(7.24)
Now observe that for the sets $\mathcal{M}_{\text{con},v}$ and $\mathcal{M}_{\text{rel},v}$, the value of $r_b(x)$ is the same, as it is for the sets $\mathcal{M}_{\text{con},h}$ and $\mathcal{M}_{\text{rel},h}$, and for the sets $\mathcal{N}^i_g$ and $\mathcal{N}^i_f$.

**Definition 7.4.3** Uncontrolled elastic collision maps are maps:

(i) $T_u : \mathcal{M}_{\text{con},v} \rightarrow \mathcal{M}_{\text{rel},v}$, for collisions with first derivative unequal to zero.

(ii) $T^i_h : \mathcal{M}^i_{\text{con},h} \rightarrow \mathcal{M}^i_{\text{rel},h}$, for collisions with a higher odd derivative unequal to zero.

(iii) $T^i_{j,g} : \mathcal{N}^i_j \rightarrow \mathcal{N}^i_g$, for collisions with a higher even derivative unequal to zero.

(iv) $T^i_g : \mathcal{N}^i_g \rightarrow \mathcal{N}^i_g$, for collisions with a higher even derivative unequal to zero.

A controlled elastic collision map is defined as a map $T_c : \mathcal{N}_c \times \mathbb{U} \rightarrow \mathcal{N}_c \times (\mathbb{U} \setminus \mathbb{U}_f)$, for collisions in $\mathcal{N}_c$.

The map $T^i_{j,g}$ is only needed during initialization. The map $T^i_g$ is defined for completeness reasons.

**Definition 7.4.4** Inelastic collisions. Let $x \in \mathcal{M}_b$ denote the contact point. Define $r_T := \min \{ r_b(\hat{x}) | \exists \hat{x} \in \mathcal{M}_{\text{rel}} \text{ such that } r_b(x) < r_b(\hat{x}) \leq r_b(x) \}$ with $r_T = r_0$ if $r_b(x) = r_0$. Then inelastic collision maps are maps:

(i) $T_u : (\mathcal{M}_{\text{con}} \setminus \mathcal{N}_c) \cup \mathcal{N}_f \rightarrow \mathcal{M}_{\text{rel}} \cap \{ x \in \mathcal{M}_b | r_b(x) = r_T \}$ if $r_T < r_0$.

(ii) $T_p : (\mathcal{M}_{\text{con}} \setminus \mathcal{N}_c) \cup \mathcal{N}_f \rightarrow \mathcal{N}_c$ if $r_T = r_0$.

(iii) $T_c : \mathcal{N}_c \times \mathbb{U} \rightarrow \mathcal{N}_c \times (\mathbb{U} \setminus \mathbb{U}_f)$ if $r_T = r_0$.

The maps $T_r$ and $T_p$ are also referred to as uncontrolled inelastic collision maps. The introduction of the concept of collision maps leads to the following definition of a unilaterally constrained nonlinear system.

**Definition 7.4.5** Let $\mathcal{M}_b(f,g,h)$ satisfy the assumptions. Let $T_u$ be an uncontrolled (elastic or inelastic) collision map. Let $T_c$ be a controlled collision map. Then the constrained system $\Sigma_{i}^f$ in (7.1) is defined as:

$\Sigma_{i}^f = \{ \sigma : \mathbb{R} \rightarrow \mathcal{M}_b | \exists u \in \mathcal{U}(\mathbb{R}_+, \mathbb{R}^m) \text{ such that:} \}

(i) $\sigma(0) \in R^g$;

(ii) $\sigma(t) \in \mathcal{M}_b \cup \mathcal{M}_{\text{rel}}, t \geq 0 \Rightarrow d\sigma(t) = f(\sigma(t)) + g(\sigma(t))u(t);$

(iii) $\sigma(t) \in (\mathcal{M}_{\text{con}} \setminus \mathcal{N}_c) \cup \mathcal{N}_f, t \geq 0 \Rightarrow \lim_{t \rightarrow \infty} \sigma(t) = T_u(\sigma(t));$

(iv) $\sigma(t) \in \mathcal{N}_c, t \geq 0, \sigma(t) \in \mathcal{M}_b, t \leq t \Rightarrow \lim_{t \rightarrow \infty} \sigma(t) = T_c(\sigma(t), u(t)) \}$. <

Application to a specific system amounts to specifying the maps $f$, $g$, and $h$, and specifying the collision maps, where, if the application allows it, the decomposition of the collision map $T_u$ can be made according to the decompositions in definitions 7.4.3 and 7.4.4. It is of interest to find representations of $\Sigma_{i}^f$ that are useful to conduct e.g. simulations. This is discussed in section 7.6.
Formally, we have for the consistent and inconsistent initial condition sets $\mathcal{I}^c_g$ and $\mathcal{I}^c_f$:

$$
\begin{align*}
\mathcal{I}^c_g &= \mathcal{M}_g \cup \mathcal{M}_b = \mathcal{M}_g \cup \mathcal{M}_{\text{con}} \cup \mathcal{M}_{\text{rel}} \cup \mathcal{N}_f, \\
\mathcal{I}^c_f &= \mathcal{M}_f.
\end{align*}
$$

(7.25)

To illustrate our approach to unilaterally constrained dynamical systems we now present two simple examples. The first example treats the motion of a ball that is constrained by a circular basin (see also example 1.2.3).

**Example 7.4.6** Consider again the system in example 1.2.3 with the additional assumption that the motion of the ball can be controlled. Assume that the ball has mass $m = 1$. Let the position of the ball be denoted by $y = [y_1, y_2]^T \in \mathbb{R}^2$. Let $r \in \mathbb{R}$ be a fixed constant, with $r > 0$. Let the governing equations be given by:

$$
\begin{align*}
\frac{d^2 y_1}{dt^2}(t) &= u_1, \\
\frac{d^2 y_2}{dt^2}(t) &= g + u_2(t), \\
0 &\geq y_1^2(t) + y_2^2(t) - r^2.
\end{align*}
$$

An equivalent first-order formulation is obtained by setting $x = [x_1, x_2, x_3, x_4]^T := [y_1, y_2, \frac{dy_1}{dt}, \frac{dy_2}{dt}]^T$, $f(x) := [x_3, x_4, 0, -g]^T$, $g_1(x) := [0, 0, 1, 0]^T$, and $g_2(x) := [0, 0, 0, 1]^T$, and $h(x) := r^2 - y_1^2(t) - y_2^2(t)$. Consider system $\Sigma(f, g, h)$. The boundary set is given by

$$
\mathcal{M}_b = \{x \in \mathcal{M} | x_1^2 + x_2^2 - r^2 = 0\}. 
$$

(7.26)

It can be seen that $\mathcal{M}_b(f, g, h)$ satisfies the assumptions and that the relative degree on the boundary set equals $r_0 = 2$. From appendix A.2 it follows that:

$$
\begin{align*}
\mathcal{M}_{\text{con}, V} &= \{x \in \mathcal{M}_b | 2x_1x_3 + 2x_2x_4 > 0\}, \\
\mathcal{M}_{\text{rel}, V} &= \{x \in \mathcal{M}_b | 2x_1x_3 + 2x_2x_4 < 0\}, \\
\mathcal{N} &= \{x \in \mathcal{M}_b | 2x_1x_3 + 2x_2x_4 = 0\}. 
\end{align*}
$$

(7.27)

The sets $\mathcal{V}_g$, $\mathcal{V}_f$, $\mathcal{M}_{\text{con}, h}$ and $\mathcal{M}_{\text{rel}, h}$ are empty. If for instance contact is made at $(x_1, x_2) = (r, 0)$ then in the case of elastic collisions the intuitive 'change of the sign of the velocity component' follows by defining $T(r, 0, x_3, x_4) = (r, 0, -\delta x_3, x_4)$, $x_3 \neq 0$. Here $0 < \delta \leq 1$ is the elasticity parameter. Inelastic collisions correspond to $\delta = 0$ in this case.

So far, we have applied our approach to unilaterally constrained dynamical systems successfully to a number of constrained mechanical systems. The following example is taken from [129], where another promising approach to deal with difficulties in hybrid systems is presented. (A link to our work will also be established in section 7.6.) The example deals with a linear mechanical system that can not be handled completely within our framework:
additional physical modelling is still necessary. (A nonlinear example is discussed in detail in chapter 8.)

**Example 7.4.7** Suppose two carts are connected to each other by a spring, and suppose that the left cart is connected to a fixed wall by means of a spring. The motion of the left cart is restricted by a purely non-elastic stop, which is placed at the equilibrium position of the left cart. A control force can be exerted on the right cart. (See figure 7.1.) Let $x_1(t)$ and $x_2(t)$ represent the deviation from the equilibrium positions of the left and right cart respectively, and let $x_3(t)$ and $x_4(t)$ denote the corresponding velocities. The inequality constraint reads:

$$ x_1(t) \geq 0. \tag{7.28} $$

The unconstrained ‘mode’ of the system is the one where $x_1(t) > 0$ and is given by:

$$
\begin{align*}
\frac{dx_1}{dt}(t) &= x_3(t), \\
\frac{dx_2}{dt}(t) &= x_4(t), \\
\frac{dx_3}{dt}(t) &= -2x_1(t) + x_2(t), \\
\frac{dx_4}{dt}(t) &= x_1(t) - x_2(t) + u(t).
\end{align*} \tag{7.29}
$$

To find the contact and release sets we simply look at system (7.29) combined with the inequality constraint $x_1(t) \geq 0$. With the obvious definition of the system functions it is readily shown that the relative degree equals $r_0 = 4$. From algorithm A.1.6 in appendix A it follows that the set of points (at initial time $t = 0$) where the unconstrained system becomes constrained is given by:

\[
(M_{\text{con}} \cup N_f) \setminus N_g = \overline{M_{\text{con},v}} \cup N_f \cup \overline{M_{\text{con},h}} \cup N_c \\
= \{ x \in \mathbb{R}^4 \mid x_1 = 0 \land x_3 < 0 \} \cup \{ x \in \mathbb{R}^4 \mid x_1 = 0 \land x_3 = 0 \land x_2 < 0 \} \cup
\]

![Figure 7.1: Two connected carts constrained by a rigid block.](image-url)
Section 7.4: Introducing the collision maps

\[ \{ x \in \mathbb{R}^4 \mid x_1 = 0 \land x_3 = 0 \land x_2 = 0 \land x_4 < 0 \} \cup \]
\[ \{ x \in \mathbb{R}^4 \mid x_1 = 0 \land x_3 = 0 \land x_2 = 0 \land x_4 = 0 \} . \]

This is also the set given in [129] with the exception of the point \( x = 0 \). This exclusion of the set \( \mathcal{N}_c \) is not surprising because in [129] control is not part of the discussion. The set of points where the constrained system becomes unconstrained is given by:

\[ \mathcal{M}_{rel} = \mathcal{M}_{rel,v} \cup \mathcal{N}_g \cup \mathcal{M}_{rel,h} \cup \mathcal{N}_c \]
\[ = \{ x \in \mathbb{R}^4 \mid x_1 = 0 \land x_3 > 0 \} \cup \]
\[ \{ x \in \mathbb{R}^4 \mid x_1 = 0 \land x_3 = 0 \land x_2 > 0 \} \cup \]
\[ \{ x \in \mathbb{R}^4 \mid x_1 = 0 \land x_3 = 0 \land x_2 = 0 \land x_4 > 0 \} \cup \]
\[ \{ x \in \mathbb{R}^4 \mid x_1 = 0 \land x_3 = 0 \land x_2 = 0 \land x_4 = 0 \} . \]

Our approach gives the set of points where the constrained system becomes unconstrained, and vice versa. Use of the collision maps defined in this chapter provides continuation of trajectories. However, we have not obtained a model for the constrained system in figure 7.1. For elastic collisions a map \( T_v \) can be applied, but application of a map \( T_{f,g} \) does not have a physical meaning since this would mean a jump in the position, and not a jump in the velocity. The reason for this discrepancy is that although the constrained system has relative degree 4, the constrained mechanical system acts as if it has relative degree 2: in the set \( \{ x \in \mathbb{R}^4 \mid x_1 = 0 \land x_3 = 0 \land x_2 = 0 \land x_4 = 0 \} \) controlled contact with the boundary does happen, but physics tells us that the motion of the left cart remains in contact with the boundary set already in the set \( \{ x \in \mathbb{R}^4 \mid x_1 = 0 \land x_3 = 0 \} \). The latter set can not be obtained from our analysis of the contact and release sets, since the analysis has been based on the notion of relative degree. Our analysis seems to be able to identify when there is still a need for additional modelling. What is still missing is obviously the reaction force between the left cart and the rigid block. We will return to this issue in section 7.6. In [129] a direct link between the contact set and release set is not made since elastic collisions are not discussed there. An advantage of our approach is that the analysis has been done in the time-domain, which is often the domain in which numerical simulation studies are performed [21].

With respect to the classical invariance properties discussed in chapter 5 we are now in the position to present a characterization of controlled holdability of the set \( h(x) \geq 0 \) for the nonlinear dynamical system \( \frac{dx}{dt} = f(x) + g(x)u \). The proof follows from the analysis in this section, and is therefore omitted.

**Proposition 7.4.8** Let \( \mathcal{M}_n(f,g,h) \) satisfy the assumptions. Then the following conditions are equivalent:

(i) The set \( \{ x \in \mathcal{M} \mid h(x) \geq 0 \} \) is a controlled holdable set for the nonlinear dynamical
system \( \frac{dx}{dt} = f(x) + g(x)u; \)

(ii) \( M_{\text{con,v}} \cup M_{\text{con,h}} \cup N_f = \emptyset. \)

It can be seen that the set \( M_{\text{con,v}} \) plays a crucial role in a discussion on classical invariance properties. For mechanical systems the following result is now immediate.

**Corollary 7.4.9** Consider the system (7.7). Then the set \( \{ y \in M \mid \phi(y) \geq 0 \} \) is not a controlled holdable set for this system.

### 7.5 Generalizations

In this section we will briefly discuss some generalizations. First we investigate the multiple constraint case, and then we discuss relaxation of assumption 7.2.2.

We adapt the notation: in this section the subscript \( i \) is used to denote subsets and matrices of subsystem \( i \). Let \( \Sigma \) denote system (7.1) with \( h : M \rightarrow \mathbb{R}^p, \ p \geq 1 \). Clearly, one has \( h(x) \geq 0 \) if and only if \( h_i(x) \geq 0, \ \forall i \in p \). Denote \( \Sigma_i^c := \Sigma^c(f,g,h_i) \). Then \( \Sigma^c = \wedge_{i=1}^p \Sigma_i^c \), i.e. \( \Sigma^c \) is the interconnection of systems \( \Sigma_i^c \). We obtain for \( \Sigma^c \):

\[
M_b = \{ x \in M \mid h(x) \geq 0 \} \cap \{ \bigcup_{i=1}^p \{ x \in M \mid h_i(x) = 0 \} \}. \tag{7.30}
\]

\[
M_g = \cap_{i=1}^p \{ x \in M \mid h_i(x) > 0 \}. \tag{7.31}
\]

\[
M_f = \bigcup_{i=1}^p \{ x \in M \mid h_i(x) < 0 \}. \tag{7.32}
\]

For system \( \Sigma^c \) contact with the boundary set \( M_b \) must be understood as contact with at least one, but possibly more, boundary sets \( M_{b,i}, i \in p \), where

\[
M_{b,i} = M_b \cap \{ x \in M \mid h_i(x) = 0 \}. \tag{7.33}
\]

The boundary sets may exhibit different characteristics, both physically, e.g. with respect to elasticity properties, and mathematically. For instance, in [129] the assumption is made that system (7.2), with \( h : M \rightarrow \mathbb{R}^p \) has uniform relative degree. This assumption becomes in our case: all \( p \) systems \( \Sigma^c(f,g,h_i) \) have the same relative degree. This is a restrictive assumption to make a priori, although it seems to provide good inroads to tackle problems associated with multiple constraints in [129].

For system \( \Sigma^c(f,g,h) \) assumption 7.2.2 (i) becomes: each system \( \Sigma^c(f,g,h_i) \) has constrained relative degree \( r_{0,i} \) for all \( x \in M_b \). For each subsystem \( \Sigma_i^c, i \in p \), the contact and release subsets of interest can be found using proposition A.2.3 in appendix A. The (in)consistent initial conditions sets are given by: \( \mathcal{I}_g^c = \cap_{i=1}^p \mathcal{I}_{0,i} = \{ x \in M \mid h(x) \geq 0 \} \), and \( \mathcal{I}_{f}^c = \bigcup_{i=1}^p \mathcal{I}_{f,i} = M_f \). Taking the sets \( \mathcal{I}_g^c \) and \( \mathcal{I}_{f}^c \) into account the following is obtained for
the multiple constraints case:

\[
\begin{align*}
N_c & := (\cup_{i=1}^P N_{c,i}) \cap M_b. \\
N^* & := \cap_{i=1}^P N_{c,i}. \\
N_g & := (\cup_{i=1}^P N_{g,i}) \cap M_b. \\
N_f & := (\cup_{i=1}^P N_{f,i}) \cap M_b. \\
M_{con,v} & := (\cup_{i=1}^P M_{con,v,i}) \cap M_b. \\
M_{rel,v} & := (\cup_{i=1}^P M_{rel,v,i}) \cap M_b. \\
M_{rel,h} & := (\cup_{i=1}^P M_{rel,h,i}) \cap M_b. \\
M_{con,h} & := (\cup_{i=1}^P M_{con,h,i}) \cap M_b. \\
\end{align*}
\]

The difficulties that exist in the nonlinear multiple constraint case are similar to the difficulties in the linear case, and we refer to section 6.6 for details. It can be seen that the number of combinations of subsets that is possible in the nonlinear case equals the number of combinations in the linear case. This suggests that research with respect to efficient organization of all these combinations can be studied in a linear setting. Note that on the intersection of all boundaries, i.e. in \(N^*\), it seems logical to assume that all individual boundary sets have the same relative degree.

As in the single unilateral constraint case, the collision maps are used to provide continuation of trajectories that come or are in contact with the boundary sets: trajectories of \(\Sigma^e\) consist of concatenated pieces of the unconstrained system. We give the following result, based on the results of section 7.4 and the discussion above, with the obvious redefinition of the set \(U_f\) and the domains of the collision maps.

**Definition 7.5.1** Let \(M_n(f, g, h_i)\) \((i \in \mathbb{P})\) satisfy the assumptions. Let \(T_u\) be an uncontrolled (elastic or inelastic) collision map. Let \(T_c\) be a controlled collision map. Then the constrained system \(\Sigma^e(f, g, h)\) in (7.1) is defined as:

\[
\Sigma^e = \{ \underline{x} : \mathbb{R} \rightarrow M \mid \exists u \in U(\mathbb{R}^+, \mathbb{R}^m) \text{ such that:} \}
\]

\[
\begin{align*}
(i) & \quad \underline{x}(0) \in \mathbb{R}^n; \\
(ii) & \quad \underline{x}(t) \in M_g \cup M_{rel}, t \geq 0 \Rightarrow \frac{d\underline{x}}{dt}(t) = f(\underline{x}(t)) + g(\underline{x}(t))u(t); \\
(iii) & \quad \underline{x}(t) \in (M_{con} \cup M_f) \setminus N_c, t \geq 0 \Rightarrow \lim_{t \to \infty} \underline{x}(t^+) = T_u(\underline{x}(t)); \\
(iv) & \quad \underline{x}(t) \in N_c, t \geq 0, \underline{x}([t, t]) \in M_g, t \leq t \Rightarrow \lim_{t \to \infty} \underline{x}(u)(t^+) = T_c(\underline{x}(t), u(t)). 
\end{align*}
\]

It is now easy to see that proposition 7.4.8 is valid in the multiple constraint case also.

In the remainder of this section we will focus on relaxation of assumption 7.2.2. First suppose that assumption 7.2.2 \((i)\) is not valid. Then there is a subset of \(M_b\) (or even the complete boundary set) such that the relative degree \(r_0 = \infty\). In this case all collisions are uncontrolled. Consider the system in example 7.4.6 but now with \(r = 0\). In that case the inequality \(x_1^2 + x_2^2 \leq r^2\) can be written equivalently as \(x_1^2 + x_2^2 = 0\). Although the relative degree equals 2 for almost any point in \(M\), for the set of interest, i.e. \(x_1 = 0 = x_2\), however,
Chapter 7: The contact problem in a nonlinear setting

the constrained relative degree equals $\infty$. This example deals with a degenerate case, and also shows that in contrast to the linear case discussed in chapter 3, in the nonlinear case a single inequality can be an implicit equality.

As another example consider the inequality constraint

$$x_1 \cdot x_2 \geq 0.$$  \hspace{1cm} (7.42)

It is easy to show that $\frac{\partial h}{\partial x}(x) = [x_2 \ x_1]$, which loses rank precisely at the point $x_1 = 0 = x_2$. Our results can still be applied, locally, provided that $(x_1,x_2) \neq 0$. How serious is this singularity? The answer is that it will depend on the application. First, restriction

(7.42) may arise as a simplification when modelling a mechanical system. If $x_1$ and $x_2$ model position constraints of a mass in $\mathbb{R}^2$ then problems may arise when a circular mass is modelled as a point-mass (see figure 7.2 (a)). Restriction (7.42) can also arise in a different manner, i.e. when modelling the behaviour of an ideal diode in example 1.2.2 (see figure 7.2 (b)). In this case, for $V = 0 = I$ one can expect singularity problems in our framework. Clearly nonsmooth manifolds require further investigation: assumption 7.2.2 (i) is a smoothness condition on the manifold defined by $h(x) = 0$. It seems that in cases where the (constrained) relative degree is not finite, one can expect the same difficulties as in general differential geometric theories since many control laws explicitly use this assumption (locally).

Next, if assumption 7.2.2 (ii) does not hold then the relative degree equals 1 almost everywhere. All collisions are controlled collisions, provided we assume that assumption 7.2.2 (i) still holds. It can be seen that on the boundary set, any point belongs to a locally maximal controlled invariant integral manifold.

Finally, suppose that $g(x) = 0$. The notion of relative degree can not be defined anymore, and hence only uncontrolled collisions are possible.
7.6 Representations

In this section our goal will be to derive representations of the system in definitions 7.4.5 and 7.5.1 that are suitable for use in simulation (see chapter 9). In the analysis presented so far the starting points have been the unconstrained dynamics model, and a model of the unilateral constraints. In the present section we will follow a different approach by first looking at bilaterally constrained dynamical systems. In other words, the starting point will be a model of a dynamical system subject to equality constraints.

Consider the system

\[ \Sigma : \frac{dx}{dt} = f(x) + g(x)u, \]  

subject to the equality constraint

\[ h(x) = 0. \]

In order that a trajectory of the system remains in the largest locally controlled invariant manifold \( \mathcal{N}^* \), the control must be chosen appropriately. This can be achieved by chosen the control according to equation (7.23) for all \( x \in \mathcal{N}_c \) and gives

\[ L_f^0 h(x) + L_g L_f^{r_0 - 1} h(x) u = 0. \]

If we assume that each subsystem \( \Sigma' (f,g,h_i) \) satisfies assumption 7.2.2 (i) with the same integer \( r_0 \) then (7.45) is also valid for the multiple constraint case [79]. In the remainder we assume that \( L_g L_f^{r_0 - 1} h(x) \) has full row-rank. This is trivially satisfied, by definition of the relative degree, for the single constraint case. Equation (7.45) can be solved (but not uniquely) for the control \( u \) such that locally the trajectory remains on the boundary. This leads to the following result, in which we introduce a map \( Z \). We need this map in chapter 8. In the proposition below the map \( Z \) arises from the freedom of choice of a right-inverse of \( C \).

**Proposition 7.6.1** Let \( x_0 \) be such that \( h(x_0) = 0 \). Let \( \mathcal{M}(x_0) \) be a neighbourhood of \( x_0 \) such that \( \mathcal{N}^* = \{ x \in \mathcal{M} \cap \mathcal{M}(x_0) \mid L_f^i h(x) = 0, 0 \leq i < r_0 \} \). Let \( C(x) := L_g L_f^{r_0 - 1} h(x) \) have full row-rank for all \( x \in \mathcal{M}(x_0) \cap \{ x \in \mathbb{R}^n \mid h(x) = 0 \} \). Let \( Z(x) \) be a function such that \( \text{rank}(C(x) Z(x) C^T(x)) = \text{rank}(C(x) C^T(x)) \) for all \( x \in \mathcal{M}(x_0) \cap \{ x \in \mathbb{R}^n \mid h(x) = 0 \} \). Then the following are equivalent:

(i) \( \exists u \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^n) \) such that \( \tilde{x} \) is a trajectory of (7.43) and (7.44) in \( \mathcal{M}(x_0) \) with \( \tilde{x}(0) = x_0 \).

(ii) \( \exists v \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^n), \lambda \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^p) \) such that \( \tilde{x} \) is a trajectory of

\[
\begin{align*}
\frac{dx}{dt} &= f(x) + g(x)v + g(x)Z(x)C^T(x)\lambda, \\
h(x) &= 0,
\end{align*}
\]
Chapter 7: The contact problem in a nonlinear setting

in \( M(x_0) \) with \( \bar{x}(0) = x_0 \).

(iii) \( \exists v \in U(\mathbb{R}^+, \mathbb{R}^n) \) such that \( \bar{x} \) is a trajectory of

\[
\frac{dx}{dt} = f(x) + g(x)v + g(x)Z(x)C^T(x)\lambda,
\]

\[
\lambda = -(C(x)Z(x)C^T(x))^{-1}(\mathcal{L}_{\mathcal{C}}^{\mathcal{C}} h(x) + C(x)v),
\]

in \( M(x_0) \) with \( \bar{x}(0) = x_0 \).

The variable \( \lambda \) is referred to as a vector of **Lagrange multipliers**, or for short, Lagrange multiplier, after the terminology used in optimization theory and also in the theory of constrained robotic systems.

Proposition 7.6.1 leads to the following description of the unilaterally constrained dynamical system \( \Sigma^*(f, g, h) \) in definition 7.4.5: a trajectory \( x \), the control \( u \) and the auxiliary variable \( \lambda \) need to satisfy

\[
\frac{dx}{dt} = f(x) + g(x)v + g(x)Z(x)C^T(x)\lambda,
\]

\[
h(x) \begin{cases} 
    \geq 0 & \text{if the motion is in } \mathcal{N}^* \text{ c.f. (7.15),} \\
    = 0 & \text{elsewhere,}
\end{cases}
\]

(7.46)

together with an uncontrolled collision map.

Model (7.46) with the function \( Z(x) \) will be discussed in chapter 8.

Constrained mechanical systems are a special case of constrained dynamical systems. In case restrictions are imposed on subsystems that can not be controlled directly (as in example 7.4.7), then there is still a problem. The notion of relative degree alone does not provide us the desired description of a constrained mechanical system. The following heuristic approach can be followed. Let \( \phi(x) \geq 0 \) be the holonomic inequality constraint. Introduce fictitious controls \( u_i \) for those subsystems that are constrained, but not fully actuated. Now apply proposition 7.6.1 to arrive at a representation of the form (7.46). Subsequently set the fictitious controls \( v_i = 0 \). Then the system of equations thus obtained contains the same control entries as the original system, but now also terms are present that can be interpreted as Lagrange multipliers. This procedure is illustrated in the following example.

**Example 7.6.2** Example 7.4.7 revisited. Consider the combination of equations (7.28) and (7.29). Use of proposition 7.6.1, with \( Z(x) = I \), gives as system equations

\[
\begin{align*}
\frac{dx_1}{dt}(t) &= x_3(t), \\
\frac{dx_2}{dt}(t) &= x_4(t), \\
\frac{dx_3}{dt}(t) &= -2x_1(t) + x_2(t), \\
\frac{dx_4}{dt}(t) &= x_1(t) - x_2(t) + v(t) + \lambda_1.
\end{align*}
\]
Now introduce the fictitious control $u_1$ in the dynamics of the left cart to obtain $\frac{dx_3}{dt}(t) = -2x_1(t) + x_2(t) + u_1$. The relative degree of the modified system equals 2. Apply proposition 7.6.1 with $Z(x) = I$, to obtain $\frac{dx_3}{dt}(t) = -2x_1(t) + x_2(t) + v_1 + \lambda_2$, and set $v_1 = 0$. This gives

\[
\begin{align*}
\frac{dx_3}{dt}(t) &= x_3(t), \\
\frac{dx_2}{dt}(t) &= x_4(t), \\
\frac{dx_2}{dt}(t) &= -2x_1(t) + x_2(t) + \lambda_2, \\
\frac{dx_4}{dt}(t) &= x_1(t) - x_2(t) + v(t).
\end{align*}
\]

(7.48)

as a model. We have obtained two different representations, as expected. Representation (7.47) is obtained in a system theoretical setting using the notion of relative degree: four times differentiation is necessary. Representation (7.48) is obtained by the physical knowledge that twice differentiation of the equality constraint suffices to limit the behaviour of the left cart. In the sets $N_f \cup M_{comp} \cup N_c$ the constraint $x_1 = 0$ becomes active, and $\lambda_2$ is unequal to zero. This situation can not be captured in (7.47).

The Lagrange multiplier $\lambda_2$ is used to model uncontrolled contact, whereas the Lagrange multiplier $\lambda_1$ can be interpreted as part of a controlled contact map. For robotic systems one often has that $g(x)$ is nonsingular. Even in this case control is known to be difficult [25]. It is to be expected that when the Lagrange multiplier arises from uncontrolled contact, as the Lagrange multiplier $\lambda_2$, controller design issues will be even harder to solve. Although the difference between (7.47) and (7.48) is subtle, it identifies that physical modelling will never be replaced by system theoretical modelling only.

### 7.7 Concluding remarks

In the previous chapters we have discussed the contact problem for unilaterally constrained dynamical systems. Our framework combines general system theoretical considerations with ideas from physical modelling. The novelty of our approach, and one of the issues in which we deviate from the approach that is usually taken in the literature, is the following. First, we made a general discussion on the contact problem before the physics of contact is incorporated. Second, we have made precise in which (sub)sets of the state space additional (collision) modelling is necessary to make the unilateral constraint set act as an invariant set. Third, after introducing the concepts of controlled and uncontrolled collisions, collision maps have been defined. We have also introduced the notions of elastic and inelastic (or plastic) collisions in a general system theoretical setting. For a large class of systems we have presented a definition of the constrained state-space system in terms of the restricted behaviour. In this framework, the consistent initialization of a constrained (non)linear system has been discussed. For unilateral nonlinear constraints we have solved the problem of deriving the contact and release sets. These sets can be computed off-line. Difficulties
remain in modelling the laws of collision for multiple constraints in a general setting. These
difficulties seem to be similar to the difficulties that arise in constrained mechanical sys-
tems subject to multiple unilateral constraints as discussed in [24, 69, 94]. The ideas in
[94], using index sets of active and inactive equality constraints, might provide good in-
roads to the organization of the contact and release sets for dynamical systems subject to
multiple unilaterally constraints. A connection has been established with the research in
the field of classical invariance properties. A characterization of controlled holdability of a
unilateral constraint set has been derived. The results presented here can be considered as
an extension to the classic positive invariance theory for linear systems.

We have identified that use of the notion of relative degree provides an extension to the
classical notion of invariant sets, but still does not capture all constrained mechanical
systems. If in a mechanical system, the constraints are put on a subsystem that can be
controlled directly, our framework is capable of modelling such systems in terms of contact
and release sets, and in terms of collision laws. In other cases however, we have identified
that there is still a need for additional physical modelling. The reason for this is that certain
physical interaction forces can not be captured in a system theoretical framework that is
built around the notion of relative degree, since this explicitly uses control to deal with
restricted subsystems. A heuristic approach was suggested to deal with these situations.

The research presented in this chapter has been strongly motivated by the observation that
collision modelling is necessary for mechanical systems. It also motivated our analysis on
invariance properties of unilaterally constrained dynamical systems. We have discussed
the question in which way unilaterally constrained mechanical systems are different from
unilaterally constrained dynamical systems. Corollary 7.4.9 provides a part of the answer
in mathematical terms. The statement in corollary 7.4.9 is a reformulation of the result
in corollary 5.6.2 in chapter 5, but it has been derived by investigating unilaterally con-
strained dynamical systems in a totally different way. In fact, more can be concluded in
our framework when we further specialize to second-order systems in the next chapter.

Appendix 7.A: Proofs

Proof of lemma 7.3.2:
The proof is based on the statements in [79, page 147]. One has \( y(0) = h(x(0)) = h(x_0) \).
For the first derivative we obtain \( y^{(1)}(t) = \frac{dx}{dt} = x \frac{df}{dx} (f(x(t)) + g(x(t))u(t)) = L_fh(x(t)) + L_g h(x(t))u(t) = L_fh(x(t)) = h_1(x(t), u(t)) \) for all \( t \) such that \( x(t) \) is near \( x_0 \), where the
latter equalities follow from the definition of relative degree. Repeating this process it
can be shown that \( y^{(i)}(t) = L_f^i h(x(t)) = h_i(x(t), u(t)) \) for all \( i < r_0 \). Finally:
\( y^{(r_0)}(0) = L_f^{r_0} h(x_0) + L_g L_f^{r_0-1} h(x_0)u(0) = h_{r_0}(x_0, u(0)) \).

Proof of proposition 7.3.5:
First recall that now \( L_f^k h(x) = CA^k x \) for \( k > 0 \) and \( L_g L_f^k h(x) = CA^kB \). It follows that
if \( M_n(f, g, h) \) satisfy the assumptions then the linear system satisfies assumptions (6.2.2) (ii), (iii) and (iv). Since assumption (6.2.2) (i) ensures the finiteness of the integer \( r_0 \) in the linear case, we arrive at the situation of chapter 6. From equation (7.21) it follows that
\[ h_i(x, u) = L^i_j h(x) + \sum_{j=1}^{i} L_{g} L^j_{i-j} h(x) y_{i-j} = CA^i x + \sum_{j=1}^{i} CA^{j-1} B y_{i-j}, \]
which is exactly the map defined in definition 6.4.1. It follows that, by definition, the integer \( r_0(x) \) (c.f. definition 6.4.1) equals the value of \( r_0(x) \) (c.f. definition 7.3.3). The definition of the sets in section 6.4 and definition 7.3.4 now gives the desired result.

**Proof of proposition 7.3.6:**

We follow closely (part) of the proof of lemma 2.4 in [79]. Applying the regular state feedback \( u \) to system \( \Sigma'(f, g, h) \) gives \( \frac{dx}{dt} = f(x) + g(x) \alpha(x) + g(x) \beta(x) v \). First we proof that \( L^i_{j+g_0} h(x) = L^i_j h(x) \) for all \( 0 \leq i < r_0 \). This equality is trivially true for \( i = 0 \). By induction, suppose it is true for some \( 0 < i < r_0 - 1 \). Then:
\[ L^{i+1}_{j+g_0} h(x) = L^{i+1}_j h(x) + L_{g} L^{i}_j h(x) \alpha(x) = L^{i+1}_j h(x). \]
It can now be shown that \( L_{g \beta} L^{i+1}_{j+g_0} h(x) = 0 \) for all \( 0 \leq i < r_0 - 1 \), and that, if \( \beta(x_0) \neq 0 \) then \( L_{g \beta} L^{i+1}_{j+g_0} h(x_0) \neq 0 \). This shows that the integer \( r_0 \) is invariant under feedback. It is now easy to see that from equation (7.21) it follows that the sign of the map \( h_i \) is preserved. From definition 7.3.3 now follows that the integer \( r_0(x) \) is invariant under feedback. The result now follows from the definition of the subsets.

**Proof of theorem 7.3.7:**

The first two statements follow are straightforward from the definitions. The last two statements can be proven analogously to the ones in theorem 6.4.9. The details are omitted.

**Proof of proposition 7.3.9:**

From definition 7.2.1 it follows that the sign of \( g \) is not important. The result now follows from proposition A.2.3.

**Proof of proposition 7.6.1:**

First observe that from \( C(x) \) full row-rank it follows that \( (C(x) Z(x) C^T(x)) \) is nonsingular on \( M(x_0) \cap \{ x \in \mathbb{R}^n \mid h(x) = 0 \} \). We will prove the sequence \( (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i) \).

\((i) \Rightarrow (ii)\): Write \( u = v + w \) where \( v \) is the new control, and \( w \) must be chosen such that whatever \( v \), the control \( u \) is applicable. Now solve equation (7.45) for \( w \), using that \( (C(x) Z(x) C^T(x)) \) is nonsingular to obtain
\[ w = -Z(x) C^T(x) (C(x) Z(x) C^T(x))^{-1} (L^0_j h(x) + C(x) v). \]
Now write \( \lambda = -(C(x) Z(x) C^T(x))^{-1} (L^0_j h(x) + C(x) v). \) This gives (ii).

\((ii) \Rightarrow (iii)\): The expression for \( \lambda \) follows readily from the first part of the proof.

\((iii) \Rightarrow (i)\): It suffices to show that \( u := v + Z(x) C^T(x) \lambda \) is a locally applicable boundary control. For this, substitute the expression for \( u \) in equation (7.45). This gives:
\[ L^0_j h(x) + C(x) (v - Z(x) C^T(x) (C(x) Z(x) C^T(x))^{-1} (L^0_j h(x) + C(x) v)) = L^0_j h(x) - L^0_j h(x) = 0. \]
This gives (i).