Chapter 6

The contact problem for linear systems

6.1 Introduction

In the previous chapter it has been investigated under which conditions a unilateral constraint defines an invariant set for a dynamical system. It has been shown that linear mechanical systems subject to linear holonomic constraints do not possess such an invariance property, not even when applying linear state feedback. This situation is of course a well known phenomenon in mechanics. As a simple example consider a rigid ball that is falling to the ground. Two principal observations can be made. First, the ball can start at any position on or above the ground, and with any initial velocity. Second, due to the presence of gravity the ball will inevitably come into contact with the ground. Moreover, if the ground is also rigid, the velocity of a rigid ball will change sign instantaneously when the ball touches the rigid ground. The constraint set, i.e. the area on and above the ground, is not an invariant set if one remains within a general system theoretical framework per se. The physics of contact is needed to model the discontinuities in the velocity of the ball ([24, 44]). The very existence of these discontinuities makes control of a unilaterally constrained mechanical system much harder than control of the unconstrained mechanical system [24]. There are many ways to include the physics of contact in a model of a unilaterally constrained mechanical system and usually some a priori assumptions are incorporated to simplify analysis and controller design.

In this chapter, and also in the next one, our main goal will be to develop a system-theoretical framework that can handle unilaterally constrained dynamical systems. The results of this chapter have appeared in ’A.A. ten Dam, K.F. Dwarshuis and J.C. Willems, The contact problem for linear continuous-time dynamical systems: a geometric approach, IEEE Transactions on Automatic Control, Vol. 42, No. 4, pages 458-472, 1997.’
In the remainder of this thesis we will investigate continuous-time dynamical systems described by a combination of differential equations and static inequalities. In a general system theoretical framework inequalities may model interconnections between subsystems [144] or general restrictions on the system imposed by the environment [131]. The specification and investigation of the interaction of trajectories of the system with the boundaries of the sets defined by the inequalities will be referred to as the contact problem for unilaterally constrained dynamical systems. We investigate what will happen if systems interconnect while the states of these systems have not been (or can not be) prepared properly. And it is the latter that can be seen as the contact problem proper. In mechanics the contact problem is usually discussed in terms of collisions and collision avoidance. We will borrow this terminology.

We take the following approach to the contact problem. The novelty of our approach, and one of the issues in which we deviate from the approach that is usually taken in the literature, is the following. First we investigate what can be deduced already from the mathematical models of the unconstrained dynamical system and the unilateral constraint. Second, we show where so called collision maps can be introduced in a system theoretical setting in order to deal with the contact problem. To gain a clear understanding of the contact problem we start our investigations for linear time-invariant systems in state representation in the present chapter. The nonlinear case is presented in chapter 7. In this general setting the collision map is regarded as an external factor, i.e. an analytical expression of this map is considered to be part of the application that is modelled. The latter is not surprising because a general theoretical discussion on dynamical systems does not involve the notion of collisions. An analogue can be found in [131], where certain mechanical properties, such as the notion of energy, are used to make a link between the framework of [126, 143] and 'physical modelling'. Third, we show how the collision maps can be detailed further to deal with real world systems. In particular we will show in chapter 8 that our approach to the contact problem fits constrained mechanical system models found in the literature. The particular sequence of steps taken here leads to important insights in the contact problem, and to useful theoretical developments regarding modelling and control of dynamical systems subject to unilateral constraints. We believe that this approach contributes to a better understanding of the influence of obstacles on the behaviour of a physical system. As research in the field of positively invariant polyhedral sets does not include a discussion on the contact problem itself, our approach generalizes, for restricted systems, the classic notion of positively invariant sets of dynamical systems.

The remainder of this chapter is organized as follows: In section 6.2 the problem formulation is presented. A constrained mechanical system is discussed and the assumptions are stated. In the linear case we will investigate polyhedral restrictions. Since an arbitrary convex polyhedral set can be represented as the intersection of a finite system of closed affine half-spaces [125], it will prove fruitful to first consider the case where the state trajectory is restricted to be in one closed half-space. When a trajectory of a dynamical system makes contact with the boundary set of the region modelled by an inequality this is referred to
as activation of the associated equality constraint. A first, and basic, subdivision of the state-space, based on activation and deactivation of a boundary set is given in section 6.3. In section 6.4 we complete our subdivision of the state-space by examining the behaviour of the unconstrained system on the boundary of the constraint set. Sections 6.5 and 6.6 are the core of the present chapter; a detailed description of the restricted behaviour of continuous-time constrained linear systems is given. The main results will be the allocation of the specific place that is reserved in our framework for modelling the laws of collision and a definition of the constrained state-space system in terms of its restricted behaviour. In section 6.7 relaxation of some of the assumptions is treated and the results will be extended to cover continuous-time dynamical systems restricted by an arbitrary convex polyhedral set. Finally, in section 6.8 conclusions are stated.

Relations with some other work on constrained systems is discussed at the end of chapter 7, after the extension to the nonlinear case.

6.2 Problem formulation

Recall from chapter 5 the following multi-input/multi-output linear second-order differential equation as a model for a mechanical system:

\[ M \frac{d^2 y}{dt^2}(t) + D \frac{dy}{dt}(t) + Ky(t) = Lu(t), \]  

for all \( t \in \mathbb{R}_+ \), where \( y \in (\mathbb{R}^d)^{\mathbb{R}_+} \), \( y \) is a generalized system coordinate vector, \( u \in (\mathbb{R}^m)^{\mathbb{R}_+} \), \( u \) is the generalized force vector, \( M \in \mathbb{R}^{d \times d} \) is the generalized positive definite inertia matrix, \( D \in \mathbb{R}^{d \times d} \) the generalized structural damping matrix, \( K \in \mathbb{R}^{d \times d} \) is the generalized structural stiffness matrix, and \( L \in \mathbb{R}^{d \times m} \) is the actuator force distribution matrix. In the remainder we usually suppress the argument \( t \).

Assume that the holonomic restriction implied by the presence of an object in the environment can be represented by a finite system of linear inequalities:

\[ Py \geq d, \]  

with \( P \in \mathbb{R}^{p \times d}, d \in \mathbb{R}^p \). By convention, inequalities between vectors are componentwise. Restriction (6.2) determines a convex polyhedral set.

It is well known that the combination of (6.1) and (6.2) does not specify completely the constrained mechanical system; the interaction with the boundary set of the constraint is yet to be accounted for. We will however first investigate what can be deduced from (6.1) and (6.2) before the physics of this interaction is taken into account.

It can be seen that if \( Py(t) > d \) for \( t \in [t_1, t_2] \), the behaviour of the constrained system in that time-interval is described by (6.1). In that case the boundary set of the allowed
6.2: Problem formulation

region, i.e. the equality constraints $P_iy = d_i$, associated with the inequalities $P_iy \geq d_i$, are called passive [94]. (Here subscript $i$ denotes the $i$th row of a matrix.) Activation and deactivation of a constraint, and the consequences of the addition and deletion of equations have been studied for instance in [36, 69, 94]. On the other hand, (part of) a boundary set is called active (at time $t$) when $P_i y(t) = d_i$ for some $i \in 1, \ldots, p$ [94]. In the latter case (6.1) and (part of) (6.2) reduce to a Differential/Algebraic Equation (DAE) on the time-interval $[t_1, t_2]$ and may be combined to yield a differential equation with fewer generalized variables than the one in (6.1) [69]. We will not follow this approach. For instantaneous collisions one has that a constraint can be active at a discrete time point only. Another difference compared to DAE systems is that one can not differentiate (6.2) to obtain an inequality constraint on velocity level as the differential operator does not preserve sign [41]. On the other hand for DAE’s one also has $P_i \frac{dy}{dt}(t) = 0$, a fact that is referred to as a hidden constraint in constrained mechanical systems [21].

For system (6.1) define the state $x := [y^T, (\frac{dy}{dt})^T]^T$ (as in chapter 5). Then (6.1) and (6.2) can be written equivalently as:

$$
\begin{aligned}
\frac{dx}{dt} &= \dot{A}x + Bu := \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1}L \end{bmatrix} u, \\
\dot{C}x &:= [P \ 0] x \geq d.
\end{aligned}
$$

(6.3)

Note that for constrained mechanical system (6.3) one has $\dot{C}B = 0$, i.e. $\text{im}(\dot{B}) \subseteq \ker(\dot{C})$.

Motivated by representation (6.3) we will investigate linear time-invariant dynamical systems:

$$
\Sigma : \frac{dx}{dt} = Ax + Bu,
$$

(6.4)

subject to inequality constraints

$$
Cx \geq d.
$$

(6.5)

Here $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}$, and $d \in \mathbb{R}^p$. Throughout we will assume that polyhedral set (6.5) is nonempty, an assumption which is trivially satisfied if $d = 0$. The combination of (6.4) and (6.5) gives rise to a constrained system $\Sigma^c(A, B, C, d)$ which for the moment is denoted as

$$
\Sigma^c(A, B, C, d) : \begin{cases}
\frac{dx}{dt} = Ax + Bu, \\
0 \leq Cx - d.
\end{cases}
$$

(6.6)

Whereas in chapter 5 we established restrictive conditions under which the set defined by the constraint (6.5) is an invariant set for system (6.4), we now look upon this combination of equations from a different perspective.
Problem 6.2.1 Consider the system $\Sigma$ in (6.4) subject to constraint (6.5). Given the set of matrices $A$, $B$, $C$ and $d$, how to make the set defined by the unilateral constraint an invariant set for the system $\Sigma$?

We already know from chapter 5 that positive invariance or controlled holdability holds under restrictive conditions only. So additional modelling is in general necessary to solve problem 6.2.1. Solving problem 6.2.1 will lead to a description of the unilaterally constrained dynamical system (6.6), where the interaction with the boundary set of the inequality constraint is explicitly taken into account. To reduce notation we usually suppress the arguments, for instance $\Sigma^c(A, B, C, d)$ will be denoted by $\Sigma^c$. In the remainder $x$ will denote a trajectory of a dynamical system.

We will discuss problem 6.2.1 under the following assumptions, which hold throughout this chapter for system matrices $A$ and $B$, and constraint matrices $C$ and $d$, unless stated otherwise:

**Assumption 6.2.2**

(i) $\text{rank}[B \ AB \ldots \ A^{n-1} B] = n$;

(ii) $\text{im}(B) \subseteq \ker(C)$;

(iii) $C \neq 0$;

(iv) $\{x \in \mathbb{R}^n \mid Cx \geq d\} \neq \emptyset$.

Assumption (i) is equivalent to controllability of the unconstrained system (6.4) [150]. Assumption (ii) is motivated by representation (6.3), and is a natural one to make (see also [131] for a discussion on equality constraints in case of Hamiltonian or gradient systems). Assumption (ii) does cover mechanical systems subject to holonomic inequality constraints, but is not limited to this case. Assumptions (iii) and (iv) are made to exclude the trivial cases $\Sigma^c = \Sigma$, or $\Sigma^c = \emptyset$ (depending on whether or not $0 \leq d$). Note that assumption (iv) is trivially satisfied for polyhedral cones. We will use the phrase '$\mathcal{X}_n(A, B, C, d)$ satisfies the assumptions' to indicate that in the state-space $\mathcal{X}$ with $\dim(\mathcal{X}) = n$, system matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ of the unconstrained system, and the constraint matrices $C \in \mathbb{R}^{1 \times n}$ and $d \in \mathbb{R}$, used to model a single unilateral constraint, satisfy the assumptions above. In the remainder of this thesis we will assume that the controls are piecewise $C^\infty$ signals, which is denoted by $u \in \mathcal{U}(\mathbb{R}_+, \mathbb{R}^m)$, or for short, $u \in \mathcal{U}$, whenever the dimension $m$ in $\mathbb{R}^m$ is clear from the context. Relaxation of the assumptions is discussed in section 6.7.

We will show that interactions of trajectories of a system (6.4) with the boundary set of the constraint (6.5) can be brought into the study of linear dynamical systems by using geometric terms, i.e. the use of particular structures of vector spaces. The geometric approach to systems is well developed, and we refer to [150] for an overview.

**Definition 6.2.3** Let $\mathcal{V}$ denote a subspace of $\mathbb{R}^n$. Then $\mathcal{V}$ is said to be a controlled invariant subspace if for all $x_0 \in \mathcal{V}$ there exist a trajectory of (6.4) such that $x(0) = x_0$ and $x(t) \in \mathcal{V}$ for all $t \in \mathbb{R}_+$. 


An important result in the geometric theory is the following theorem.

Theorem 6.2.4 ([150]) The following conditions are equivalent:

(i) $V$ is a controlled invariant subspace for system (6.4).
(ii) $V$ is $A$-invariant modulo $\text{im}(B)$: $AV \subseteq V + \text{im}(B)$.
(iii) There exists a feedback gain matrix $F$ such that $(A + BF)V \subseteq V$, i.e. the set $V$ is holdable by closed-loop control.

Of interest is the largest controlled invariant subspace contained in a given subspace. In our case this will almost always be the subspace $\ker(C)$. Let $\mathcal{V}^*$ denote this largest controlled invariant subspace in $\ker(C)$. The Invariant Subspace Algorithm (ISA) is a recursive algorithm for computing $\mathcal{V}^*$. For future reference we state this algorithm. Let matrices $A$, $B$ and $C$ be given and be of appropriate dimensions. Let $A^{-1}\mathcal{V} := \{x \in \mathbb{R}^n \mid Ax \in \mathcal{V}\}$. (Note that it is not required that $A$ is invertible.) Define $\mathcal{V}^0 := \mathbb{X}$ and

$$\mathcal{V}^{k+1} = \ker(C) \cap A^{-1}(\mathcal{V}^k + \text{im}(B)), \ k = 0, 1, \ldots \quad (6.7)$$

This defines a non-increasing sequence of subspaces of $\mathbb{X}$. Since $\dim(\mathbb{X})$ is finite there exists a value of $k$ such that $\mathcal{V}^{k+1} = \mathcal{V}^k$. This limit is $\mathcal{V}^*$. By construction one has $\mathcal{V}^* \subseteq \ker(C)$.

6.3 Contact and release sets

In this section we will start our subdivision of the state-space $\mathbb{X}$ by investigating the behaviour of an unconstrained system in a convex polyhedral cone. Any convex polyhedral cone can be represented as the intersection of a finite system of closed half-spaces [125]:

$$\{x \in \mathbb{R}^n \mid Cx \geq 0\} = \cap_{i=1}^p \{x \in \mathbb{R}^n \mid C_ix \geq 0\},$$

where $C_i$ denotes the $i$th row of matrix $C \in \mathbb{R}^{p \times n}$. It will prove fruitful to first consider in detail the case of a single inequality constraint:

$$Cx \geq 0, \text{ with } C \in \mathbb{R}^{1 \times n}. \quad (6.8)$$

The constrained system $\Sigma^c(A, B, C) := \Sigma^c(A, B, C, 0)$ with $C \in \mathbb{R}^{1 \times n}$ will be denoted by $\Sigma^c$. It is not until a full description is derived of the restricted behaviour for $\Sigma^c$ in section 6.5 that we return to the case of multiple constraints, i.e. $C \in \mathbb{R}^{p \times n}$ ($p \geq 1$) in section 6.6.

A first, and basic, subdivision of the state-space $\mathbb{X}$ is based on inequality constraint (6.8) only. Define $\mathcal{X}_g$ ($g$ for good) as the collection of states where the inequality constraint is satisfied strictly and $\mathcal{X}_f$ ($f$ for false) as the collection of states where the inequality constraint is not satisfied. One has:

$$\mathcal{X}_g = \{x \in \mathbb{R}^n \mid Cx > 0\},$$

$$\mathcal{X}_f = \{x \in \mathbb{R}^n \mid Cx < 0\}. \quad (6.9)$$
\[
\ker(C) = \{ x \in \mathbb{R}^n \mid Cx = 0 \}.
\] (6.11)

A further subdivision of the subspace \(\ker(C)\) can be based on the interaction of trajectories of unconstrained system \(\Sigma\) (6.4) with inequality constraint (6.8). Let \(\mathcal{X}_{\text{con}}\) (\(\text{con}\) for contact) denote the sets of points where a trajectory of \(\Sigma\) that starts in \(\mathcal{X}_g\) can come into contact with \(\ker(C)\). Analogously, let \(\mathcal{X}_{\text{rel}}\) (\(\text{rel}\) for release) denote the set of points where a trajectory of \(\Sigma\) can leave the boundary set and remain (for some period of time) in \(\mathcal{X}_g\). These sets are defined formally as follows.

**Definition 6.3.1** The contact and release sets. Let \(\mathbf{z}\) denote a trajectory of system (6.4).

(i) The contact set \(\mathcal{X}_{\text{con}}\) is defined as \(\mathcal{X}_{\text{con}} := \{ x \in \ker(C) \mid \exists \mathbf{z} \in \Sigma \text{ and } \exists t^* < 0 \text{ such that } x(0) = x, \text{ and } \mathbf{z}(\tau) \in \mathcal{X}_g, \forall \tau : t^* < \tau < 0 \} \).

(ii) The release set \(\mathcal{X}_{\text{rel}}\) is defined as \(\mathcal{X}_{\text{rel}} := \{ x \in \ker(C) \mid \exists \mathbf{z} \in \Sigma \text{ and } \exists t^* > 0 \text{ such that } x(0) = x, \text{ and } \mathbf{z}(\tau) \in \mathcal{X}_g, \forall \tau : 0 < \tau < t^* \} \).

Formally \(\mathcal{X}_{\text{con}} = \mathcal{X}_{\text{con}}(A, B, C)\) but to shorten notation we will usually suppress the arguments. The (finite-time) trajectory piece \(\mathbf{z}|_{[\tau_0, \tau]}\) in the definition of \(\mathcal{X}_{\text{con}}\) is referred to as locally viable in [122], where the so called target problem, i.e. how to reach a specific target subset, is discussed for systems described by differential inclusions.

It is intuitively clear that \(\mathcal{X}_{\text{con}}\) and \(\mathcal{X}_{\text{rel}}\), defined for system \(\Sigma(A, B, C)\), switch roles for the time-reversed system \(\Sigma(-A, -B, C)\). This is shown in proposition 6.4.11 below.

**Example 6.3.2** A single cart and a rigid block. Let the position of (a point on) a cart be denoted by \(y\) (see figure 6.1). Consider the representation \(\frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = u\), obtained from (6.1) by setting \(M = D = K = L = 1\), subject to the inequality constraint \(y \geq 0\), obtained from (6.2) by setting \(P = 1, d = 0\). Define \(x := [y^T, \left(\frac{dy}{dt}\right)^T]^T\). The system matrices are: \(A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\) and \(B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\). The constraint matrix \(C\) reads: \(C = [1, 0]\).
6.4 Subdivision of the boundary set

It can be verified that \((A, B)\) is controllable, and that \(CB = 0\). Inspection shows that \(\mathcal{X}_{\text{con}} = \{x \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 \leq 0\}, \mathcal{X}_{\text{rel}} = \{x \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 \geq 0\}, \) and \(\mathcal{V}^* = \{0\}.\) It follows that \(\mathcal{X}_{\text{con}} \cup \mathcal{X}_{\text{rel}} = \ker(C),\) and \(\mathcal{X}_{\text{con}} \cap \mathcal{X}_{\text{rel}} = \mathcal{V}^*\).

Already on this basic level some interesting properties arise. It can be seen that on \(\mathcal{X}_{\text{rel}} \setminus \mathcal{V}^*\), the (trajectory of the) cart will leave the boundary of the constraint and will go to \(\mathcal{X}_{j}\). On \(\mathcal{X}_{\text{con}} \setminus \mathcal{V}^*\) however, all trajectories will proceed into \(\mathcal{X}_f\), whatever the applied control. In general the situation will be more complicated than the one given in the example above. In subsequent sections we will locate exactly the subset of \(\ker(C)\) where additional modelling is required to make the constraint set an invariant set for a dynamical system.

6.4 Subdivision of the boundary set

If a trajectory \(x \in \Sigma\) enters the boundary of the constraint set, i.e. if at some time \(t\) one has \(C x(t) = 0\), it depends on the characteristics of the state (and its derivatives) at this contact point whether or not continuation of a trajectory \(x\) of system (6.4) is possible. Introduce the fictitious output \(y = C x\). Then the value of the \(i\)th derivative of \(y\), denoted by \(y^{(i)}\), can be used to determine whether or not such a continuation is possible. Use of (6.4) gives

\[
y^{(i)}(t) = CA^i x + CA^{i-1} Bu(t) + \ldots + CBu^{(i-1)}(t).
\]

To shorten notation some definitions are given that will enable us to present alternative representations of the subsets defined in the previous section. Let \(u \in \mathbb{U}^\mathbb{N}\); i.e. \(u\) is a sequence of vectors which take their values in \(\mathbb{U}\). Define the maps \(h_i : \ker(C) \times \mathbb{U}^\mathbb{N} \to \mathbb{R}, i \in \mathbb{Z}_+,\) by

\[
h_i(x, u) := CA^i x + \sum_{j=1}^{i} CA^{i-j} Bu_{j-1},
\]

with \(h_0(x, u) = C x\). If we take \(u_j = u^{(i)}(t)\) and \(x(t) = x\) it follows that \(h_i(x, u) = y^{(i)}(t)\).

The following integers will play an important role in the sequel.

**Definition 6.4.1** Let matrices \(A, B\) and \(C\) be given. Define:

(i) \(r : \ker(C) \times \mathbb{U}^\mathbb{N} \to \mathbb{N} \cup \{\infty\}\) as \(r(x, u) := \min\{i \in \mathbb{N} \mid h_i(x, u) \neq 0\}\) with \(r(x, u) := \infty\) if \(h_i(x, u) = 0, \forall i \in \mathbb{N}\).

(ii) \(r_C : \ker(C) \to \mathbb{N}\) as \(r_C(x) := \min\{i \in \mathbb{N} \mid \exists u \in \mathbb{U}^\mathbb{N} : h_i(x, u) \neq 0\}\).

(iii) \(r_0 := \min\{i \in \mathbb{N} \mid CA^{i-1} B \neq 0\}\).

A sufficient condition for both \(r_0\) and \(r_C(x)\) to be finite is that the pair \((A, B)\) is controllable \([150]\) and \(C \neq 0\). It is easy to see that for all \(x \in \ker(C)\) and for all \(u \in \mathbb{U}^\mathbb{N}\) one has: \(r_C(x) \leq r(x, u)\). Observe that \(r_0\) is the smallest integer \(i \in \mathbb{N}\) for which the function \(h_i(x, u)\) depends on the control. The integer \(r_0\) is sometimes referred to as the relative degree of a system.
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\( \frac{dx}{dt} = Ax + Bu, y = Cx \). In an input/output setting the integer \( r_0 \) can be used to derive, for instance, an input/output decoupling control law [58], and is also known as the order of the infinite zero in case \( p = 1 \), and the least of the order at infinity in the general case [49]. From the assumptions it is easy to see that \( r_0 \geq 2 \) and \( r_C(x) \geq 1 \). The following corollary is straightforward from the definitions and the assumption \( CB = 0 \).

**Corollary 6.4.2** Some useful properties. Let \( x \in \ker(C) \). Then:

(i) \( r_C(x) = \min \{ r_0, \inf \{ i \in \mathbb{N} | CA^i x \neq 0 \} \} \).

(ii) \( \forall \mathbf{u} \in \mathbb{U} \cap : 1 \leq i < r_C(x) \Rightarrow h_i(x, \mathbf{u}) = 0 \).

(iii) \( r_C(x) < r_0 \Rightarrow \forall \mathbf{u} \in \mathbb{U} \cap : h_i(x, \mathbf{u}) = h_i(x), i < r_0 \).

(iv) \( \exists \mathbf{u} \in \mathbb{U} \cap : i \geq r_0 \Rightarrow h_i(x, \mathbf{u}) = 0 \).

The integers \( r_0 \) and \( r_C(x) \) also have another important property.

**Lemma 6.4.3** Let \( \mathcal{X}_n(A, B, C) \) satisfy the assumptions. Then the integers \( r_0 \) and \( r_C(x) \) are invariant under the linear state feedback \( u = Fx + v \), with \( v \) the new control.

It is remarked that lemma 6.4.3 also holds if \( CB \neq 0 \), i.e. if \( r_0 = 1 \). Next we give alternative representations of the subsets \( \mathcal{X}_{con}, \mathcal{X}_{rel} \) and \( \mathcal{V}^* \).

**Lemma 6.4.4** The sets \( \mathcal{X}_{con}, \mathcal{X}_{rel} \), and \( \mathcal{V}^* \).

(i) \( \mathcal{X}_{con} = \{ x \in \ker(C) | \exists \mathbf{u} \in \mathbb{U} \cap \text{ such that: } \{ r(x, \mathbf{u}) < \infty \text{ and even, and } \} h_{r(x, \mathbf{u})}(x, \mathbf{u}) > 0 \}, \text{ or } \{ r(x, \mathbf{u}) < \infty \text{ and odd, and } h_{r(x, \mathbf{u})}(x, \mathbf{u}) < 0 \} \} \).

(ii) \( \mathcal{X}_{rel} = \{ x \in \ker(C) | \exists \mathbf{u} \in \mathbb{U} \cap \text{ such that } r(x, \mathbf{u}) < \infty \text{ and } h_{r(x, \mathbf{u})}(x, \mathbf{u}) > 0 \} \).

(iii) \( \mathcal{V}^* = \{ x \in \ker(C) | \exists \mathbf{u} \in \mathbb{U} \cap \text{ such that } r(x, \mathbf{u}) = \infty \} \).

It is important to note that lemma 6.4.4 does not say that, for instance, \( \mathcal{X}_{con} \) and \( \mathcal{X}_{rel} \) are disjoint. This is discussed in more detail in the remainder.

It is clear that the integers \( r_0, r(x, \mathbf{u}), \) and \( r_C(x) \) play an important role. And the decision whether or not a point \( x \in \ker C \) belongs to, for instance, \( \mathcal{X}_{rel} \) can be based on the value of \( r_C(x) \). It is easy to see that if \( \exists \mathbf{u} \) such that \( h_{r(x, \mathbf{u})}(x, \mathbf{u}) > 0 \) for \( r_0 < r_C(x) < \infty \) then there is also a control \( \mathbf{u} \) such that \( h_{r_0}(x, \mathbf{u}) > 0 \). If \( i \geq r_0 \) it follows from corollary 6.4.2 (iv) that it is basically a controller design problem to keep the derivatives of the state along the manifold \( C \) equal to zero (or not). This motivates the definitions in the remainder of this section.

Let \( \mathcal{V}_g \) denote all \( x \in \ker(C) \) for which all trajectories of \( \Sigma \) passing through \( x \) do so coming from \( \mathcal{X}_g \) and going to \( \mathcal{X}_g \). Similarly let \( \mathcal{V}_f \) denote all \( x \in \ker(C) \) for which all trajectories of \( \Sigma \) passing through \( x \) do so coming from \( \mathcal{X}_f \) and going to \( \mathcal{X}_f \). Let \( \mathcal{V}_e \) (c for control) denote the collection of states that belong to \( \ker(C) \) of which the smallest \( i \in \mathbb{N} \) for which \( h_i \) can be unequal to zero depends on the control. These sets are defined formally by:
Definition 6.4.5 The sets $\mathcal{V}_g$, $\mathcal{V}_f$ and $\mathcal{V}_c$.

(i) $\mathcal{V}_g := \{ x \in \ker(C) : \exists y \in \mathbb{F}^j : r_C(x) \text{ is even, and } h_{r_C(x)}(x, y) > 0 \}$.

(ii) $\mathcal{V}_f := \{ x \in \ker(C) : \exists y \in \mathbb{F}^j : r_C(x) \text{ is even, and } h_{r_C(x)}(x, y) < 0 \}$.

(iii) $\mathcal{V}_c := \{ x \in \ker(C) : r_C(x) = 0 \}$. \hfill \triangleright$

Lemma 6.4.6 Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions. Then $\mathcal{V}_c = \mathcal{V}_e$.

Based on our motivation of mechanical systems we make a further subdivision in contact (or release) with first derivative equal to zero and first derivative unequal to zero.

Definition 6.4.7 The sets $\mathcal{X}_{con, v}$, $\mathcal{X}_{rel, v}$ ($v$ for velocity), $\mathcal{X}_{con, h}$ and $\mathcal{X}_{rel, h}$ ($h$ for higher derivatives).

(i) $\mathcal{X}_{con, v} := \{ x \in \mathcal{X}_{con} : r_C(x) = 1 \}$.

(ii) $\mathcal{X}_{rel, v} := \{ x \in \mathcal{X}_{rel} : r_C(x) = 1 \}$.

(iii) $\mathcal{X}_{con, h} := \{ x \in \mathcal{X}_{con} : \exists y \in \mathbb{F}^j : 1 < r_C(x), r_C(x) \text{ is odd and } h_{r_C(x)}(x, y) < 0 \}$.

(iv) $\mathcal{X}_{rel, h} := \{ x \in \mathcal{X}_{rel} : \exists y \in \mathbb{F}^j : 1 < r_C(x), r_C(x) \text{ is odd and } h_{r_C(x)}(x, y) > 0 \}$. \hfill \triangleright$

Finally, we prove that all subsets defined so far are invariant under linear state feedback (which does necessarily imply that they are controlled invariant sets). For instance, the set where contact can be made with ‘velocity’ component unequal to zero does not change if we apply linear state feedback.

Proposition 6.4.8 Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions. Then the subsets $\mathcal{V}_c$, $\mathcal{V}_g$, $\mathcal{V}_f$, $\mathcal{X}_{con, v}$, $\mathcal{X}_{con, h}$, $\mathcal{X}_{rel, v}$ and $\mathcal{X}_{rel, h}$ are invariant under the linear state feedback $u = Fx + v$, with $v$ the new control, for the system $\Sigma$. \hfill \triangleright$

It will be important for a description of the restricted behaviour to have available some of the relations that exist between the subsets defined so far. The following result will be important.

Theorem 6.4.9 Relations between subsets of the state-space. Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions. Then:

(i) $\ker(C) = \mathcal{V}_c \cup \mathcal{V}_g \cup \mathcal{V}_f \cup \mathcal{X}_{con, v} \cup \mathcal{X}_{con, h} \cup \mathcal{X}_{rel, v} \cup \mathcal{X}_{rel, h}$.

(ii) The subsets $\mathcal{X}_g$, $\mathcal{X}_f$, $\mathcal{V}_c$, $\mathcal{V}_g$, $\mathcal{X}_{con, v}$, $\mathcal{X}_{con, h}$, $\mathcal{X}_{rel, v}$ and $\mathcal{X}_{rel, h}$ are pairwise disjoint.

(iii) $\mathcal{X}_{con} = \mathcal{V}_c \cup \mathcal{V}_g \cup \mathcal{X}_{con, v} \cup \mathcal{X}_{con, h}$.

(iv) $\mathcal{X}_{rel} = \mathcal{V}_c \cup \mathcal{V}_g \cup \mathcal{X}_{rel, v} \cup \mathcal{X}_{rel, h}$. \hfill \triangleright$

Theorem 6.4.9 together with proposition 6.4.8 yields that we can make a complete subdivision of $\ker(C)$ in disjoint subsets that are invariant under linear state feedback. In appendix A algorithms are derived that calculate these subsets in a finite number of steps. The following result is now immediate.

Corollary 6.4.10 The subsets $\mathcal{X}_{con}$ and $\mathcal{X}_{rel}$ are invariant under the linear state feedback.
u = Fx + v, with v the new control.

Next, as a summary, an intuitive explanation is given of all subsets that have been defined so far. Let \( \mathcal{X}_n(A, B, C) \) satisfy the assumptions. We can make the following statements relating \( x \in \mathcal{X} \) with a trajectory of the unconstrained system \( \Sigma \). (Notice again that only smooth controls are allowed.)

(a) \( x \in \mathcal{X}_g \Leftrightarrow x \) satisfies the inequality constraint strictly.
(b) \( x \in \ker(C) \Rightarrow x \) belongs to the boundary set.
(c) \( x \in \mathcal{X}_f \Leftrightarrow x \) does not satisfy the inequality constraint.
(d) \( x \in \mathcal{X}_{con,v} \Leftrightarrow \) all trajectories \( \underline{x} \in \Sigma \) with \( \underline{x}(t) = x \) for some \( t \) go transversally through the boundary set from \( \mathcal{X}_g \) to \( \mathcal{X}_f \).
(e) \( x \in \mathcal{X}_{con,h} \Leftrightarrow \) all trajectories \( \underline{x} \in \Sigma \) with \( \underline{x}(t) = x \) for some \( t \) go tangentially 'through' the boundary set from \( \mathcal{X}_g \) to \( \mathcal{X}_f \).
(f) \( x \in \mathcal{V}_g \Leftrightarrow \) all trajectories \( \underline{x} \in \Sigma \) with \( \underline{x}(t) = x \) for some \( t \) go tangentially 'through' the boundary set from \( \mathcal{X}_g \) to \( \mathcal{X}_f \).
(g) \( x \in \mathcal{V}_c \Leftrightarrow \) all trajectories \( \underline{x} \in \Sigma \) with \( \underline{x}(t) = x \) for some \( t \) go tangentially through the boundary set, and there exists at least one trajectory that remains in \( \ker(C) \).
(h) \( x \in \mathcal{V}_f \Leftrightarrow \) all trajectories \( \underline{x} \in \Sigma \) with \( \underline{x}(t) = x \) for some \( t \) go tangentially 'through' the boundary set from \( \mathcal{X}_f \) to \( \mathcal{X}_g \).
(i) \( x \in \mathcal{X}_{rel,h} \Leftrightarrow \) all trajectories \( \underline{x} \in \Sigma \) with \( \underline{x}(t) = x \) for some \( t \) go tangentially through the boundary set from \( \mathcal{X}_f \) to \( \mathcal{X}_g \).
(j) \( x \in \mathcal{X}_{rel,v} \Leftrightarrow \) all trajectories \( \underline{x} \in \Sigma \) with \( \underline{x}(t) = x \) for some \( t \) go transversally through the boundary set from \( \mathcal{X}_f \) to \( \mathcal{X}_g \).

As remarked earlier, it is intuitively clear that the contact sets and the release sets switch roles for the time-reversed system, i.e. the system with system matrices \( -A \) and \( -B \). Note that \( \mathcal{X}_n(A, B, C) \) satisfies assumptions 6.2.2 if and only if \( \mathcal{X}_n(-A,-B, C) \) satisfies these assumptions.

**Proposition 6.4.11** Let \( \mathcal{X}_n(A, B, C) \) satisfy the assumptions. Then the following relations hold between subsets of system \( \Sigma(A,B,C) \) and the time-reversed system \( \Sigma(-A,-B,C) \):

\[
\begin{align*}
\mathcal{X}_g(A,B,C) &= \mathcal{X}_g(-A,-B,C), \\
\mathcal{X}_{con,v}(A,B,C) &= \mathcal{X}_{rel,h}(-A,-B,C), \\
\mathcal{X}_{rel,v}(A,B,C) &= \mathcal{X}_{con,v}(-A,-B,C), \\
\mathcal{V}_g(A,B,C) &= \mathcal{V}_g(-A,-B,C), \\
\mathcal{V}_f(A,B,C) &= \mathcal{V}_f(-A,-B,C), \\
\mathcal{X}_{con,h}(A,B,C) &= \mathcal{X}_{rel,h}(-A,-B,C), \\
\mathcal{X}_{rel,h}(A,B,C) &= \mathcal{X}_{con,h}(-A,-B,C), \\
\mathcal{V}_c(A,B,C) &= \mathcal{V}_c(-A,-B,C).
\end{align*}
\]

Up till now we have investigated how trajectories of an unconstrained dynamical system interact with a boundary set. If the boundary set is looked upon as a mathematical constraint rather than a hard environment constraint, it follows that in \( \mathcal{X}_{con} \cap \mathcal{V}_c \) application of a smooth control can not prevent a trajectory of system (6.4) to enter \( \mathcal{X}_f \). It is clear that this finding has consequences for (feedback) controller design if one of the objectives is smooth contact with the boundary set. Moreover, the control must be chosen appropriately in \( \mathcal{V}_c \), i.e. one should exclude controls that will drive the system into \( \mathcal{X}_f \). This leads to the
6.5: Restricted behaviours: a single constraint

Concepts of (locally) applicable and forbidden controls, which will prove useful later on.

Definition 6.4.12 Consider system (6.4). Let $x \in \mathcal{V}_c$. Then

(i) The set of locally applicable controls is defined as: $\mathbb{U}_b(x) = \{u \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^m) \mid \exists \varphi \in \Sigma, \exists t > 0 \text{ such that } \varphi(0) = x, \text{ and } \varphi(\tau) \in \mathcal{X}_g \cup \ker(C), \forall \tau : 0 < \tau < t^*\}$.

(ii) The set of locally applicable boundary controls is defined as: $\mathbb{U}_b(x) = \{u \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^m) \mid \exists \varphi \in \Sigma, \exists t > 0 \text{ such that } \varphi(0) = x, \text{ and } \varphi(\tau) \in \ker(C), \forall \tau : 0 < \tau < t^*\}$.

(iii) The set of locally forbidden controls is defined as: $\mathbb{U}_f(x) = \{u \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^m) \mid \exists \varphi \in \Sigma, \exists t > 0 \text{ such that } \varphi(0) = x, \text{ and } \varphi(\tau) \in \mathcal{X}_f, \forall \tau : 0 < \tau < t^*\}$.

Definition 6.4.12 can be seen as a further subdivision of the boundary set. However, as far as the points $x \in \mathcal{V}_b$ are concerned, these sets are not disjoint. From the definitions it follows that for system (6.4) the set of forbidden controls is given by $\mathbb{U}_f(x) = \cup_i \mathbb{U}_f^i(x)$, $i \geq r_0$, where $\mathbb{U}_f^i(x) := \{u \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^m) \mid h_i(x, u) < 0\}$. The set $\mathbb{U}_b(x)$ is important also in case of bilaterally constrained systems, where representation (6.6) reduces to a DAE. For the set $\mathbb{U}_b(x)$ an explicit expression can be derived. In fact, it is easy to see that for system (6.4),

$$\mathbb{U}_b(x) = \{u \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^m) \mid CA^0x + CA^{a-1}Bu = 0\}.$$  \hspace{1cm} (6.13)

The above sets, and some implications of our findings for controller synthesis will be discussed in chapter 7, section 7.6.

6.5 Restricted behaviours: a single constraint

In this section we will define what we mean by a constrained linear system $\Sigma^c_i$, i.e. how an inequality constraint affects the behaviour of a dynamical system. Recall from section 6.2 that the constrained behaviour (with $C \in \mathbb{R}^{1 \times n}$ and $d = 0$) is given by:

$$\Sigma^c_i : \begin{cases} \frac{dy}{dt} = Ax + Bu, \\ 0 \leq Cx. \end{cases}$$ \hspace{1cm} (6.14)

A more detailed description of $\Sigma^c$ is based on the subsets defined in the previous sections. We need two more notions. First we introduce the notion of consistent and inconsistent initial conditions.

Definition 6.5.1 Initial condition sets. Consider system $\Sigma^c$ in (6.6). The set of consistent initial conditions for $\Sigma^c$ is defined as $\mathcal{I}^c := \{x \in \mathcal{X} \mid \exists \varphi \in \Sigma^c \text{ with } \varphi(0) = x\}$. The set of inconsistent initial conditions for $\Sigma^c$ is defined as $\mathcal{I}_f^c := \{x \in \mathcal{X} \mid \mathcal{X} \notin \Sigma^c \text{ with } \varphi(0) = x\}$. Clearly $\mathcal{I}^c \cap \mathcal{I}_f^c = \emptyset$, and $\mathcal{I}^c \cup \mathcal{I}_f^c = \mathcal{X}$. In the remainder of this section we again concentrate on the single constraint case with $C \in \mathbb{R}^{1 \times n}$ and $d = 0$. 

Recall that if contact is made in $X_{\text{con}} \setminus V_c$ then whatever (smooth) control is used all trajectories of $\Sigma$ go to $X_f$. In $V_c$, it depends also on the control whether a trajectory will proceed to $X_g$ or will remain in $\ker(C)$. The set $V_f$ is a special set; for $t > 0$ no trajectory of $\Sigma_t$ will ever enter this set. The set does play a role at the initial time $t = 0$. This motivates the following definition, in which we introduce some terminology that is used in the remainder.

First recall from definition 6.3.1 that if $x \in X_{\text{con}}$, the contact set, then there exists a trajectory $x$ such that $x(t^*) = x$ for some $t^*$, and $x(t) \in X_g$ on a (small) time-interval $(t^* - \alpha, t^*)$, $\alpha > 0$. In this sense, if $x \in X_{\text{con}}$, we can talk about contact at $x$ of a trajectory with the boundary set, or for short, contact at $x$. Similarly, we can talk about release at $x$ if $x \in X_{\text{rel}}$, the release set.

**Definition 6.5.2** Let $x \in \ker(C)$. If $x \in (X_{\text{con}} \setminus V_c) \cup V_f$ we will call contact at $x$ an uncontrolled collision. Likewise, if $x \in X_{\text{rel}} \setminus V_c$ we will call release at $x$ an uncontrolled release. If $x \in V_c$ then we will say that contact, release, at $x$ is a controlled collision, controlled release, respectively.

For constrained mechanical systems one often has that $r_0 = 2$, which gives that $X_{\text{con},v} \neq \emptyset$. This can be seen from (6.3). If each (sub)system is actuated then in general $\hat{C} \hat{A} \hat{B} = PM^{-1}L \neq 0$. Now if contact is made in $X_{\text{con},v}$ a problem arises since all trajectories will proceed to $X_f$. Consequently, for mechanical systems subject to unilateral constraints the contact problem arises: collisions do happen and a discussion on this subject should thus be an integral part of a general theory on constrained mechanical systems. Now in mechanics, a collision will not change the position, but will affect the velocity component. Note that this change will in general depend in a unique way on the state at the moment of collision.

In our framework contact with the boundary set of a single inequality constraint is modelled by a special map.

**Definition 6.5.3** Consider system $\Sigma_t(A, B, C)$. Let $X_n(A, B, C)$ satisfy the assumptions. Then we will call a map $T_u : (X_{\text{con}} \setminus V_c) \cup V_f \rightarrow X_{\text{rel}} \setminus V_c$ an uncontrolled collision map for system $\Sigma_t$. We will call a map $T_c : V_c \times \bar{U} \rightarrow \bar{V}_c \times (\bar{U} \cup \bar{U}_f)$ a controlled collision map for system $\Sigma_t$.

For a trajectory of $\Sigma$ to cross a boundary set, contact must be made in $X_{\text{con}} \cup V_f$. By choice of a collision map the trajectory will proceed from $X_{\text{rel}}$, where the new state acts as an initial condition for the system $\Sigma_t$. Thus, in our framework, trajectories of the constrained system will consist of concatenated pieces of the unconstrained system.

Definition 6.5.3 focuses on the local behaviour upon contact. For the global behaviour one also needs to consider an infinite number of collisions in a finite period of time, which is a modelling topic in itself (see [3] for a related discussion). We will sometimes use the notation $T$ without a subscript, to denote a collision map.
There are two ways to introduce some more detail to these maps also on the present level of abstraction. Firstly, as in mechanics we will make a distinction between elastic and inelastic collisions. (For a falling ball, in case of elastic collisions with the ground a bounce happens, for inelastic collisions not.) Secondly, we will use the subdivision of the contact set $\mathcal{X}_{\text{con},v}$ and the release set $\mathcal{X}_{\text{rel},v}$ from theorem 6.4.9 to decompose the uncontrolled collision map on its domain. We need to introduce some notation. Based on the definition of the contact and release sets we can write

\[ \mathcal{V}_g := \bigcup_{1 \leq i < \frac{1}{2} r_0} \mathcal{V}^i_g, \quad \text{with} \quad \mathcal{V}^i_g := \{ x \in \mathbb{R}^n \mid r_C(x) = 2i, h_2(x, u) > 0 \}, \]

\[ \mathcal{V}_f := \bigcup_{1 \leq i < \frac{1}{2} r_0} \mathcal{V}^i_f, \quad \text{with} \quad \mathcal{V}^i_f := \{ x \in \mathbb{R}^n \mid r_C(x) = 2i, h_2(x, u) < 0 \}, \]

\[ \mathcal{X}_{\text{con},h} := \bigcup_{1 \leq i < \frac{1}{2} (r_0 - 1)} \mathcal{X}^i_{\text{con},h}, \quad \text{with} \quad \mathcal{X}^i_{\text{con},h} := \{ x \in \mathbb{R}^n \mid r_C(x) = 2i + 1, h_{2i+1}(x, u) < 0 \}, \]

\[ \mathcal{X}_{\text{rel},h} := \bigcup_{1 \leq i < \frac{1}{2} (r_0 - 1)} \mathcal{X}^i_{\text{rel},h}, \quad \text{with} \quad \mathcal{X}^i_{\text{rel},h} := \{ x \in \mathbb{R}^n \mid r_C(x) = 2i + 1, h_{2i+1}(x, u) > 0 \}. \]

Now observe that for the sets $\mathcal{X}_{\text{con},v}$ and $\mathcal{X}_{\text{rel},v}$ the value of $r_C(x)$ is the same, as it is for the sets $\mathcal{X}_{\text{con},h}$ and $\mathcal{X}_{\text{rel},h}$, and for the sets $\mathcal{V}^i_g$ and $\mathcal{V}^i_f$. For the set $\mathcal{V}_c$ one has $r_0 = r_C(x)$, and once contact is made, a control can be chosen such that the trajectory remains in $\mathcal{V}_c$.

Since all subsets are disjoint (see theorem 6.4.9), and following the line of reasoning above, we introduce the following detailing of the collision maps.

**Definition 6.5.4 Collision maps for elastic collisions.** Uncontrolled elastic collision maps are defined as maps

(i) $T_u : \mathcal{X}_{\text{con},v} \to \mathcal{X}_{\text{rel},v}$, for collisions with first derivative unequal to zero.

(ii) $T^i_h : \mathcal{X}^i_{\text{con},h} \to \mathcal{X}^i_{\text{rel},h}$, for collisions with a higher odd derivative unequal to zero.

(iii) $T^i_f : \mathcal{V}^i_f \to \mathcal{V}^i_g$, for collisions with a higher even derivative unequal to zero.

(iv) $T^i_g : \mathcal{V}^i_g \to \mathcal{V}^i_f$, for collisions with a higher even derivative unequal to zero.

Controlled elastic collisions are modelled by a controlled collision map $T_c : \mathcal{V}_c \times \mathcal{U} \to \mathcal{V}_c \times (\mathcal{U} \setminus \mathcal{U}_f)$, for collisions in $\mathcal{V}_c$.

The map $T^i_{f,g}$ is only needed during initialization since for $t > 0$ no trajectory of $\Sigma^i_v$ will ever enter $\mathcal{V}_f$ again. The map $T^i_g$ is defined for completeness reasons.

Definition 6.5.4 provides us with a number of different collision maps to deal with the contact problem. The decomposition of the uncontrolled collision map $T_u$ is based on the (mathematical) result in theorem 6.4.9. Of course, whether or not collision modelling of a physical system allows such a decomposition should ultimately be based on the application at hand.

In case of inelastic collisions, in mechanical systems part of the velocity component of the state is set to zero. This can be captured in our framework, using the subsets of the boundary set, by mapping the state $x$ into $\hat{x}$, with $\hat{x}$ in the first nonempty subset in
belongs to $T$. Hence the state satisfies the unconstrained system equations. Furthermore, if contact is possible since we can always map $x$ into the set $\mathcal{V}_c$ (for which $r_C(x) = r_0$). Again, if originally $r_C(x) = r_0$ then control enters the formulation.

**Definition 6.5.5** *Inelastic collisions.* Let $x \in \ker(C)$ denote the contact point. Define $r_T := \min\{r_C(\hat{x}) \mid \exists \hat{x} \in \mathcal{X}_{rel} \text{ such that } r_C(x) < r_C(\hat{x}) \leq r_0\}$ with $r_T = r_0$ if $r_C(x) = r_0$.

Then inelastic collision maps are defined as maps:

(i) $T_r : (\mathcal{X}_{con} \setminus \mathcal{V}_c) \cup \mathcal{V}_f \to \mathcal{X}_{rel} \cap \{x \in \ker(C) \mid r_C(x) = r_T\}$ if $r_T < r_0$.

(ii) $T_p : (\mathcal{X}_{con} \setminus \mathcal{V}_c) \cup \mathcal{V}_f \to \mathcal{V}_c$ if $r_T = r_0$.

(iii) $T_e : \mathcal{V}_c \times \mathbb{U} \to \mathcal{V}_c \times (\mathbb{U} \setminus \mathcal{U}_f)$ if $r_T = r_0$.

The maps $T_r$ and $T_p$ are also referred to as uncontrolled inelastic collision maps. It can be seen that inelastic collisions differ in a number of ways from elastic collisions. Notably, if $x \in (\mathcal{X}_{con} \setminus \mathcal{V}_c) \cup \mathcal{V}_f$ is mapped into $\mathcal{V}_c$, then at the same time instance one must prevent the solution to enter into $\mathcal{X}_f$ by adapting the control (if necessary). This can be done by applying the map $T_r$ immediately after the map $T_p$. Of course, one could also combine the latter two maps in one controlled collision map, but the present definitions will allow us to unify elastic and inelastic collisions when discussing the behaviour of constrained dynamical systems below. Note also that the map $T_e$ in definition 6.5.5 is similar to the map $T_c$ in definition 6.5.4.

**Example 6.5.6** *Example 6.3.2 revisited I.* Consider again the system in example 6.3.2.

From $CB = 0$ and $CAB = 1$ it follows that $r_0 = 2$. From algorithm A.1.6 it follows that:

$\mathcal{X}_{con,v} = \{x \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 < 0\}$, $\mathcal{X}_{rel,v} = \{x \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 > 0\}$, $\mathcal{V}_c = \{0\}$. The sets $\mathcal{V}_g$, $\mathcal{V}_f$, $\mathcal{X}_{con,h}$ and $\mathcal{X}_{rel,h}$ are empty. From theorem 6.4.9 now follows that $\mathcal{X}_{con} = \{x \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 \leq 0\}$, and $\mathcal{X}_{rel} = \{x \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 \geq 0\}$, c.f. example 6.3.2. In the case of elastic collisions the intuitive ‘change of the sign of the velocity component’ follows by defining $T(0,x_2) = (0, -\delta x_2)$, $x_2 \neq 0$. Here $0 < \delta \leq 1$ is the elasticity parameter. The inelastic collisions correspond to $\delta = 0$.

We are now ready to present a more detailed description of a constrained linear system. For $\Sigma_\gamma$ it is clear that the initial conditions must be in the set $\mathcal{V}_g$. If a trajectory $\mathcal{x}$ at time $t$ belongs to $\mathcal{X}_g$ then the boundary of the constraint set does not influence the trajectory. Hence the state satisfies the unconstrained system equations. Furthermore, if contact is made, collision can take place. This is modelled by the uncontrolled and controlled collision maps from definitions 6.5.3, 6.5.4 and 6.5.5. We have arrived at the following definition of a unilaterally constrained dynamical systems $\Sigma_\gamma$.

**Definition 6.5.7** *A unilaterally constrained dynamical system.* Let $\mathcal{X}_u(A, B, C)$ satisfy the assumptions. Let $T_u$ be an uncontrolled (elastic or inelastic) collision map. Let $T_c$ be a controlled collision map. Then the constrained system $\Sigma_\gamma$ in (6.14) is defined as:

$\Sigma_\gamma = \{\mathcal{x} : \mathbb{R} \to \mathcal{X}\} \exists u \in \mathcal{U}(\mathbb{R}_+, \mathbb{R}^m)$ such that:

(i) $\mathcal{x}(0) \in \mathcal{V}_g^e$.
(ii) $x(t) \in X_g \cup X_{red}, t \geq 0 \Rightarrow \frac{dx}{dt}(t) = Ax(t) + Bu(t)$;

(iii) $x(t) \in (X_{con} \setminus \mathcal{V}_v) \cup \mathcal{V}_f, t \geq 0 \Rightarrow \lim_{t \rightarrow \infty} x(t) = T_u(x(t))$,

(iv) $x(t) \in \mathcal{V}_v, t \geq 0, x_{[t, t]} \in X_g, t < t \Rightarrow \lim_{t \rightarrow \infty} x(t) = T_v(x(t), u(t))$.

Application to a specific system amounts to specifying the matrices $A$, $B$, and $C$, and specifying the collision maps, where, if the application allows it, the decomposition of the collision map $T_u$ can be made according to the decompositions in definitions 6.5.4 and 6.5.5. General expressions for the collision maps needs involving the physical nature of the system and the constraint; we will not pursue such a general expression in the present chapter. For mechanical systems however, some general remarks are made in chapter 8.

**Lemma 6.5.8** Let $X_n(A, B, C)$ satisfy the assumptions. Consider system $\Sigma^e_1$ in definition 6.5.7. Then the consistent initial condition set is given by: $\mathcal{I}^c_g = X_g \cup X_{con} \cup X_{red} \cup \mathcal{V}_f$. The inconsistent initial condition set is given by: $\mathcal{I}^i_f = \emptyset$.

It is remarked that if one requires that all trajectories are smooth at $t = 0$, the condition $\lim_{t \rightarrow 0} x(t) = x(0)$ must be added to our definition of consistent initial conditions. In that case the set $X_{con,v} \cup X_{con,h} \cup \mathcal{V}_f$ can not be part of the set $\mathcal{I}^c_g$ and must be added to $\mathcal{I}^i_f$.

We give the following result.

**Corollary 6.5.9** $\Sigma^e_1(A, B, C) = \Sigma^c_1(A + BF, B, C)$, i.e. the constrained system $\Sigma^c_1$ in definition 6.5.7 is invariant under linear feedback $u = Fx + v$, with $v$ the new control.

To make a connection to previous chapters in this thesis, we note the following.

In a behavioural setting we obtain that a constrained input/state system $\Sigma^e_1$ is defined by $\Sigma^e_1 = (\mathbb{R}_+, \mathbb{U} \times X, \mathcal{B}, \mathcal{I})$, with $\mathcal{I}$ the collection of collision maps, and $\mathcal{B}$ the behaviour as specified in definition 6.5.7. Note that when we take $\mathcal{I}$ to be the empty set, we are again in the convex conical setting of chapter 4. In that case $\mathcal{B} = \mathcal{B}_1 \cap \mathcal{B}_2$, with $\mathcal{B}_1$ the behaviour of the unconstrained system, and $\mathcal{B}_2$ the behaviour compatible with the restrictions only.

With respect to the classical invariance properties discussed in chapter 5 we are now in the position to present a characterization of controlled holdability of the set $Cx \geq 0$ for the linear dynamical system $\frac{dx}{dt} = Ax + Bu$. The proof follows from the analysis in this section, and is therefore omitted.

**Proposition 6.5.10** Let $X_n(A, B, C)$ satisfy the assumptions. Then the set $\{x \in \mathbb{R}^n \mid Cx \geq 0\}$ is a controlled holdable set for the linear dynamical system $\frac{dx}{dt} = Ax + Bu$ if and only if $X_{con,v} \cup X_{con,h} \cup \mathcal{V}_f = \emptyset$.

It can be seen that the set $X_{con,v}$ plays a crucial role in a discussion on classical invariance properties. For mechanical systems the following result is now immediate.
**Corollary 6.5.11** Consider the mechanical system represented by (6.1). Then the set \( \{ y \in \mathbb{R}^d \mid Py \geq 0 \} \) is not a controlled holdable set for this system.

The statement in corollary 6.5.11 is a reformulation of the result in corollary 5.6.2 in chapter 5, but it has been derived by investigating unilaterally constrained dynamical systems in a totally different way.

**Example 6.5.12** *Two carts running on the same track.* Suppose that we want to model two carts riding on the same track, where the second cart is initially to the right of the first cart. Denote \( y = [y_1^T, y_2^T]^T \), with \( y_1, y_2 \), the position of the first cart and the second cart, respectively (see figure 6.2). The position constraint reads: \( y_2 - y_1 \geq 0 \). If we further assume that \( m_i = d_i = k_i = l_i = 1 \) it follows that \( r_0 = 2 \). From algorithm A.1.6 we obtain that \( \mathcal{X}_{\text{con},v} = \{ y_2 - y_1 = 0, \, \frac{dy_2}{dt} - \frac{dy_1}{dt} < 0 \} \). So, uncontrolled collisions occur when the carts make contact and the second cart is moving faster to the left, or slower to the right than the first cart. Assuming rigid carts, the elastic collision map \( T_v \), based on conservation of momentum, can read: \( T_v(y_1, y_2, \frac{dy_1}{dt}, \frac{dy_2}{dt}) = (y_1, y_2, \frac{dy_2}{dt}, \frac{dy_1}{dt}) \).

**6.6 Restricted behaviours: multiple constraints**

In this section we will extend the results of section 6.5 to the multiple constraints case, i.e.:

\[
\Sigma^r : \left\{ \begin{array}{l}
\frac{dx}{dt} = Ax + Bu \\
0 \leq Cx.
\end{array} \right.
\quad (6.16)
\]

with \( C \in \mathbb{R}^{p \times n} \), \( p \geq 1 \). For later reference we denote \( \mathcal{C} := \{ x \in \mathbb{R}^n \mid Cx \geq 0 \} \). To make full use of the results of the previous sections we make the following observations. Note that \( \mathcal{C} = \cap_{i=1}^p \{ x \in \mathbb{R}^n \mid C_i x \geq 0 \} \). (In this section the subscript \( i \) is used to denote subsets and
matrices of subsystem $i$.) Next, define the constrained systems $\Sigma_i^c$, $i \in \mathbb{P}$, by:

$$\Sigma_i^c : \begin{cases} 
\frac{dx}{dt} = Ax + Bu \\
0 \leq C_ix.
\end{cases} \quad (6.17)$$

It follows that $\Sigma^c$ is the interconnection of systems $\Sigma_i^c$, i.e. $\Sigma^c = \bigwedge_{i=1}^{p} \Sigma_i^c$. For the polyhedral cone $\mathcal{C}$ the boundary set $\mathcal{C}_b$, and the regions $\mathcal{X}_g$ and $\mathcal{X}_f$ are given by:

$$\mathcal{C}_b = \mathcal{C} \cap \{ \bigcup_{i=1}^{p} \{ x \in \mathbb{R}^n \mid C_ix = 0 \} \}. \quad (6.18)$$

$$\mathcal{X}_g = \bigcap_{i=1}^{p} \{ x \in \mathbb{R}^n \mid C_ix > 0 \}. \quad (6.19)$$

$$\mathcal{X}_f = \bigcup_{i=1}^{p} \{ x \in \mathbb{R}^n \mid C_ix < 0 \}. \quad (6.20)$$

For system $\Sigma^c$ contact or release with the boundary set $\mathcal{C}_b$ must be understood as contact or release with at least one boundary set $\mathcal{C}_{b,i}$, with

$$\mathcal{C}_{b,i} = \mathcal{C}_b \cap \ker(C_i) \quad (6.21)$$

It can be seen that $\mathcal{C}_b = \mathcal{C} \cap (\bigcup_{i=1}^{p} \mathcal{C}_{b,i})$, $\mathcal{X}_g = \bigcap_{i=1}^{p} \mathcal{X}_{g,i}$, and $\mathcal{X}_f = \bigcup_{i=1}^{p} \mathcal{X}_{f,i}$. For each subsystem $\Sigma_i^c$, ($i = 1 \ldots p$), the subsets of interest can be computed from algorithm A.1.6. The (in)consistent initial conditions sets are given by: $\mathcal{T}_g = \bigcap_{i=1}^{p} \mathcal{T}_{g,i} = \mathcal{C}$, and $\mathcal{T}_f = \bigcup_{i=1}^{p} \mathcal{T}_{f,i} = \mathcal{X}_f$.

Taking the sets $\mathcal{T}_g$ and $\mathcal{T}_f$ into account the following is obtained for the multiple constraints case:

$$\mathcal{V}_c := (\bigcup_{i=1}^{p} \mathcal{V}_{c,i}) \cap \mathcal{C}_b. \quad (6.22)$$

$$\mathcal{V}^* := \bigcap_{i=1}^{p} \mathcal{V}_{c,i}. \quad (6.23)$$

$$\mathcal{V}_g := (\bigcup_{i=1}^{p} \mathcal{V}_{g,i}) \cap \mathcal{C}_b. \quad (6.24)$$

$$\mathcal{V}_f := (\bigcup_{i=1}^{p} \mathcal{V}_{f,i}) \cap \mathcal{C}_b. \quad (6.25)$$

$$\mathcal{X}_{con,v} := (\bigcup_{i=1}^{p} \mathcal{X}_{con,v,i}) \cap \mathcal{C}_b. \quad (6.26)$$

$$\mathcal{X}_{rel,v} := (\bigcup_{i=1}^{p} \mathcal{X}_{rel,v,i}) \cap \mathcal{C}_b. \quad (6.27)$$

$$\mathcal{X}_{con,h} := (\bigcup_{i=1}^{p} \mathcal{X}_{con,h,i}) \cap \mathcal{C}_b. \quad (6.28)$$

$$\mathcal{X}_{rel,h} := (\bigcup_{i=1}^{p} \mathcal{X}_{rel,h,i}) \cap \mathcal{C}_b. \quad (6.29)$$

There are a number of things that make multiple constraints notably different in character from single constraints. Firstly, the subsets in (6.22)-(6.29) need not to be disjoint on the intersection of the constraints, as for each individual boundary set different characteristics can hold. Nevertheless, all subsets can still be computed. The multiple number of combinations of subsets that is possible also reveals that for dynamical systems subject to multiple constraints the organization of all the subsets is a problem in itself, which we will not tackle here.

Secondly, in contrast to the single constraint case, it is now possible that, for instance, $\mathcal{V}_{c,i}$ is not entirely in $\mathcal{C}$, or that $\mathcal{X}_{con,v,i} \in \mathcal{C}$, but $\mathcal{X}_{rel,v,i} \notin \mathcal{C}$. This is shown in the following
Chapter 6: The contact problem for linear systems

Example 6.6.1 Let \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \) and \( C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \). It is readily verified that \( X_n(A, B, C) \) satisfies the assumptions. From algorithm A.1.6 follows that \( X_{con,v,1} = \{ x \in \mathbb{R}^3 | x_1 = 0, -x_2 < 0 \} \), \( X_{rel,v,1} = \{ x \in \mathbb{R}^3 | x_1 = 0, x_2 < 0 \} \), \( X_{con,v,2} = \{ x \in \mathbb{R}^3 | x_2 = 0, x_3 < 0 \} \), and \( X_{rel,v,2} = \{ x \in \mathbb{R}^3 | x_2 = 0, x_3 > 0 \} \). From (6.26) and (6.27) it follows that \( X_{con,v,1} \in X_{con,v} \) and \( \overline{X}_{rel,v,1} \notin X_{rel,v} \). As a consequence an uncontrolled collision map \( \{ u, i \} \) for a system \( \Sigma_0 \) may not be provide us with the desired continuation of trajectories of the system \( \Sigma_0 \) when contact is made at \( x \in X_{con,v,i} \). Two solutions can be proposed to deal with this problem. The first is to treat the problem as a control problem: the system should be controlled such that contact is not made in \( X_{con,v,1} \) for system (6.16), since no jump to \( X_{rel,v,1} \) is possible. In the light of research in the field of positive invariance this seems to be a difficult problem. Another solution is to view this as an inelastic collision: map the contact point in the first subset (with higher value of \( r_C(x) \)) such that release can take place. Such a subset always exists since \( V^o \) is nonempty. Fortunately, symmetry is preserved in case of mechanical systems (with \( r_0 = 2 \)) subject to holonomic constraints, as in (6.2). This is immediate from the observation that the contact and release sets yield inequality constraints on the velocities, which can never be in conflict with the original position constraints. (They can be empty if the original position constraints are in fact implicit equalities [41].) Obviously example 6.6.1 does not deal with a mechanical system.

The third difficulty with multiple constraints is that for all intersections of boundary sets, new collision maps may need to be defined even if the collision maps for the single boundary sets \( \{ x \in X | C_i x = 0 \} \backslash \cup_{j=1, j \neq i}^p \{ x \in X | C_j x = 0 \} \), \( i \in \mathbb{P}, \) have already been specified. However, this requires specific knowledge of the application, which can be seen as follows. Suppose that there are two different inequality constraints, numbered 1 and 2, respectively, leading to two boundary sets. On the intersection of the boundary sets the combinations of all subsets introduce subsets with new characteristics. For instance, there may be a subset where for one constraint we enter the boundary set in \( X_{con,v,1} \) whereas for the other constraint we enter the boundary set in \( X_{con,h,2} \). Based on section 6.5, the obvious choice of the collision map \( T \) would be to map \( (X_{con,v,1} \cap X_{con,h,2}) \) onto \( (X_{rel,v,1} \cap X_{rel,h,2}) \). It is unclear however, how the original collision maps \( T_{v,1} \) and \( T_{h,2} \) should be combined. And a composition of \( T_{v,1} \) and \( T_{h,2} \) may also not be a correct expression as the map \( T_{h,2} \) may result in an (intermediate) state that violates the first inequality constraint.

Example 6.6.2 Example 6.5.12 revisited. Consider the system in figure 6.3. There are two position constraints. The first constraint is given by the position of the rigid block. The second position constraint states that the right cart must remain to right of the left cart. The collision maps from examples 6.5.6 and 6.5.12 can be used to deal with uncontrolled
collisions between the left cart and the rigid block, and with uncontrolled collisions between the two carts, respectively. However, if simultaneously the left cart collides with the rigid block, and the right cart collides with the left cart, it is not clear how these collisions should be modelled. This problem is similar to the problem of modelling the so-called striking balls system, which is discussed in detail in [24].

For multiple constraints a further complication arises if the boundary sets have different elasticity properties. Clearly the physics of the problem should not only specify (expressly) the collision maps on the boundary set of each individual constraint, but also on the intersection of the boundary sets. For the present chapter it suffices to remark that we have identified the places where collisions can occur, and where the collision maps should be defined to deal with these collisions. Note that the results presented here can also be used as a starting point for controller design such that simultaneously a state trajectory remains on one constraint and makes contact with another constraint in a smooth manner, i.e., contact with velocity components equal to zero. For instance, if $\mathcal{V}_{c,1} \subseteq \mathcal{V}_{c}$ such a design is possible if and only if $\mathcal{V}_{c,1} \cap (\mathcal{X}_{con,2} \setminus \mathcal{X}_{con,v,2}) \neq \emptyset$.

It is clear that with the above definitions the analog of theorem 6.4.9 (iii) and (iv) still holds, with the obvious redefinition of the sets $\mathcal{X}_{con}$ and $\mathcal{X}_{red}$, keeping in mind that contact and release is to be understood as contact or release with at least one boundary set.

Based on the results of section 6.5 and the discussion above we can now give a definition of a dynamical system subject to multiple inequality constraints, with the obvious redefinition of the set $\mathcal{U}_f$ and the domains of the collision maps.

**Definition 6.6.3** A unilaterally constrained dynamical system: multiple constraints. Let $\mathcal{X}_n(A,B,C_i)$ satisfy the assumptions for all $i \in \mathcal{P}$. Let $T_u$ be an uncontrolled (elastic or inelastic) collision map. Let $T_c$ be a controlled collision map. Then the constrained system $\Sigma^c$ given in (6.16) is defined as: $\Sigma^c = \{ \tilde{\mathcal{G}} : \mathbb{R} \rightarrow \mathcal{X} | \exists u \in \mathbb{U}(\mathbb{R}_+, \mathbb{R}^m) \text{ such that:} \}

(i) $\tilde{\mathcal{G}}(0) \in \mathcal{P}$;
Application to a system, again, amounts to specifying the system matrices $A$ and $B$, and the constraint matrices $C_i$ ($i \in \mathbb{P}$), and specifying the collision maps. However, compared to the single constraint case the collision maps are now much more complicated, especially on intersections of boundary sets. As an example we will consider the case where the inequalities model a linear subspace, showing that systems subject to equality constraints can also be treated with the theory presented here.

Example 6.6.4 Example 6.3.2 extended. Consider again the system in example 6.3.2 but now subject to the constraint pair: $y \geq 0$, $-y \geq 0$. For system $\Sigma'_i$ (with constraint $y \geq 0$) we have (from example 6.5.6): $\mathcal{X}_{\text{con},v,1} = \{ x \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 < 0 \}$ and $\mathcal{X}_{\text{rel},v,1} = \{ x \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 > 0 \}$. From algorithm A.1.6 follows for system $\Sigma_2$ (with constraint $-y \geq 0$): $\mathcal{X}_{\text{con},v,2} = \{ x \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 > 0 \}$ and $\mathcal{X}_{\text{rel},v,2} = \{ x \in \mathbb{R}^2 \mid x_1 = 0 \land x_2 < 0 \}$. Clearly, $\mathcal{C}_b = \{ y = 0 \}$, $\mathcal{J}_f = \{ y < 0 \} \cup \{ y > 0 \}$, $\mathcal{J}_g = \emptyset$. Since for $\Sigma'_i$ and $\Sigma_2$ the boundary set is the same it is obvious that also the intersection of the contact sets becomes important for trajectories of $\Sigma'_i \land \Sigma_2$. From $\mathcal{X}_{\text{con},v,1} \cap \mathcal{X}_{\text{con},v,2} = \emptyset$ and $\mathcal{X}_{\text{rel},v,1} \cap \mathcal{X}_{\text{rel},v,2} = \emptyset$ it follows that a trajectory can not leave $\mathcal{C}_b$, nor make contact with it. Consequently $y = 0$ is the only solution, as expected.

Corollary 6.6.5 $\Sigma'(A,B,C) = \Sigma'(A + BF,B,C)$, i.e. the constrained dynamical system $\Sigma'$ in definition 6.6.3 is invariant under linear feedback $u = Fx + v$, with $v$ the new control. <

The results in proposition 6.5.10 and corollary 6.5.11 can be generalized in a straightforward manner to the multiple constraints case.

6.7 Generalizations

In this section we will first discuss the relaxation of one or more of the assumptions. Secondly, we will extend the results to cover arbitrary convex polyhedral sets.

First consider the case where the assumption $\text{im}(B) \subseteq \ker(C)$ does not hold, i.e. $CB \neq 0$. The following result is valid independent of the controllability of $(A,B)$, and states that all collisions are controlled collisions.

Lemma 6.7.1 Let $C \in \mathbb{R}^{1 \times n}$, $C \neq 0$ and $d = 0$. If $\text{im}(B) \nsubseteq \ker(C)$ then $\ker(C) = \mathcal{V}' = \mathcal{V}_c = \mathcal{X}_{\text{con},v} = \mathcal{X}_{\text{rel},v}$, and $\mathcal{X}_{\text{con},h} = \mathcal{X}_{\text{rel},h} = \mathcal{V}_f = \mathcal{V}_g = \emptyset$. <

Next suppose that $(A,B)$ is not controllable, and $CB = 0$. Note that in case $B = 0$, i.e. in case of an autonomous system, obviously $(A,B)$ is not controllable. For simplicity
assume that we are dealing with a single inequality constraint. Two cases are distinguished: \(r_0 < \infty\) and \(r_0 = \infty\). In case \(r_0 < \infty\) the analysis of the previous sections still holds. The interesting case is when \(r_0 = \infty\). Obviously, observability of the pair \((C, A)\) has an influence on the set \(V^*\). For this, it is useful to define two more integers.

**Definition 6.7.2** Define the integers \(r_1\) and \(r_{\min}\) as: \(r_1 := \min\{i \in \mathbb{N} | \cap_{0 \leq j < i} A^{-j} \ker(C) \subseteq A^{-i} \ker(C)\}\), and \(r_{\min} := \min(r_0, r_1)\).

Note that the set \(N := \cap_{0 \leq j < r_1} A^{-j} \ker(C)\) is the unobservable subspace, and if \(N = \{0\}\), \((C, A)\) is observable and integer \(r_1\) equals the observability index [150].

If \(r_0 = \infty\), it is not difficult to show that \(r_{\min} = r_1\) (see also appendix A). It is now straightforward to show that \(V^* = V^+\) and \(V_c = \emptyset\). So, for instance step \((x)\) in algorithm A.1.6 should be adapted to \(V^+ = V^*\). Furthermore, from \(r_0 = \infty\) follows that \(CA^j B = 0\) \((j \geq 1)\). It follows that, in the notation of [150]: \(\langle A[\text{im}(B)] \rangle \subseteq \ker(C)\). Consequently, the state trajectories of the controllable part of the system are entirely in \(\ker(C)\). It depends on the characteristics of the uncontrollable part of the system whether collisions can happen.

**Example 6.7.3** Let \(A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\), \(B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\), and \(C = [1 \ 0 \ 0]\). It follows that \(\langle A[\text{im}(B)] \rangle = V^* = \{x \in \mathbb{R}^n | x_1 = 0 = x_2\}\). Algorithm A.1.6 yields \(X_{\text{con},v} = \{x \in \mathbb{R}^n | x_1 = 0, x_2 < 0\}\). These collisions are uncontrolled. On the other hand if \(A = I\), then \(X_{\text{con},v} = \emptyset\) since for the uncontrolled system we have \(\tilde{x}_1(t) = \tilde{e}\tilde{x}_1(0)\), which never comes in contact with the boundary \(x_1 = 0\) if \(\tilde{x}_1(0) \neq 0\).

We have now set the stage to extend our results to linear systems where the state trajectory is constrained to an arbitrary convex polyhedral set. It is well known that any polyhedral set in \(\mathbb{R}^n\) can be written as a convex polyhedral cone in \(\mathbb{R}^{n+1}\) by introducing an auxiliary variable [125]. Let \(\alpha\) denote this auxiliary variable. Now define the extended polyhedral cone \(\mathcal{C}^+ := \{(y, \alpha) \in \mathbb{R}^{n+1} | Cy - d\alpha \geq 0, \alpha \geq 0\}\). Clearly, taking \(\alpha = \alpha_0 > 0\) constant, by means of the projection map \(\Pi\), i.e. \(\Pi : (y, \alpha) \mapsto y\), we obtain the original polyhedral set \(\mathcal{C}\) again.

This idea for static cones can be extended to the dynamical system \(\Sigma^e(A, B, C, d)\) in (6.6). To avoid some technicalities we assume without loss of generality that there are no redundant inequalities [41]. It can be seen that contact and release takes place on the boundary set \(\mathcal{C}_b := \mathcal{C} \cap \{\cup_{i=1}^p \{x \in \mathbb{X} | C_i x = d_i\}\}\). For system (6.6) the integer \(r_0(x) := \min\{r(x, u) | u \in \mathcal{U}\}\) is of interest. (Compare with \(r_C(x)\) from definition 6.4.1).
Define the extended system $\Sigma^+$ as:

$$
\Sigma^+ : \begin{cases}
\frac{dz_1}{dt} = Az_1 + Bu z_2, \\
\frac{dz_2}{dt} = 0, \\
0 \leq Cz_1 - dz_2, \\
0 < z_2.
\end{cases}
$$

This system can be obtained from (6.6) in the following way. First define as new state variable: $z := [z_1^T, z_2^T]^T = [(x_0)^T, \alpha_0]^T$. Note that the auxiliary variable $\alpha$ is used as a scaling variable and is kept to its initial value $\alpha_0$ by taking $\frac{d\alpha}{dt} = 0$. In that case the original state $x$ can be obtained from the new state since $x = \frac{z_1}{\alpha_0}$. Clearly there is a one-to-one correspondence between the old state-variable $x$ and the new state-variable $z_1$. We obtain: $\frac{dx}{dt} = \frac{d(x_0 \alpha)^T}{dt} = \frac{dx}{dt} \alpha + \frac{d\alpha}{dt} x$. Substitution of this equation into (6.6), using $\frac{d\alpha}{dt} = 0$, yields: $\frac{dx}{dt} = (Ax + Bu) \alpha = Az_1 + Bu z_2$. This gives system (6.30). Note that $z_2(0) = 1$ is an appropriate initial condition for $z_2$ in this system.

Observe that in representation (6.30) the dynamics are (in part) represented by a nonlinear differential equation. Also note that the restricting cone is not closed. In order to put (6.30) in the standard form discussed in section 6.6 we first observe that $z_2 > 0$ is trivially satisfied if we take as initial condition $z(0) := [(\alpha_0 x(0))^T, \alpha_0^T]^T$, with $\alpha_0 > 0$. This follows from $\frac{dz}{dt} = 0$. Next, and again using the fact $\alpha > 0$ is constant, we make a change of the basis of $U$, i.e. $v := \alpha u$, with $v$ the new control. Since $\alpha \neq 0$, we can always recover the original input $u$. Substitution of $v = \alpha u$ into (6.30) gives:

$$
\Sigma^+ : \begin{cases}
\frac{dz_1}{dt} = Az_1 + Bv \\
\frac{dz_2}{dt} = 0 \\
0 \leq Cz_1 - dz_2.
\end{cases}
$$

with $z(0) := [(\alpha_0 x(0))^T, \alpha_0^T]^T$, $\alpha_0 > 0$. As a last step, define $\overline{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $\overline{B} := \begin{bmatrix} B & 0 \end{bmatrix}$, and $\overline{C} := [C - d]$. The resulting system equations now read:

$$
\Sigma^+ : \begin{cases}
\frac{dz}{dt} = \overline{A} z + \overline{B} v \\
0 \leq \overline{C} z.
\end{cases}
$$

Clearly, system (6.32) is a unilaterally constrained linear dynamical system, and fits the framework of section 6.6.

With respect to the assumptions made in section 6.2 we remark the following. First observe that $\overline{C} \overline{B} = [C - d] \begin{bmatrix} B \\ 0 \end{bmatrix} = CB$. Hence $\{\text{im}(B) \subseteq \ker(C)\} \Leftrightarrow \{|\text{im}(\overline{B}) \subseteq \ker(\overline{C})\}$. Also: $\{C \neq 0\} \Rightarrow \{\overline{C} \neq 0\}$. However, controllability of the pair $(A, B)$ is not preserved by
introduction of the auxiliary variable $\alpha$ the pair $(\overline{A}, \overline{B})$ is obviously not controllable. It can be seen however that $(\overline{A}\text{lim}(\overline{B})) = \mathbb{X} \cup \{\alpha_0\}$, and $\Pi_x(\overline{A}\text{lim}(\overline{B})) = \mathbb{X}$. We state the following result.

**Proposition 6.7.4** Let $C \in \mathbb{R}^{1 \times n}, d \in \mathbb{R}$ and $\alpha > 0$. Consider systems $\Sigma^c$ and $\Sigma^c+$. Let $\ell \in r_0$. Then the following relations hold:

(i) $r_0(A, B, C, d) = r_0(\overline{A}, \overline{B}, C)$.
(ii) $\{[(\alpha x)^T, \alpha^T]^T \in \mathbb{R}^{n+1} | r_x(\alpha x, \alpha) = \ell\} = \{x \in \mathbb{R}^n | r_0(x) = \ell\}$.

Clearly, for $\Sigma^c+$ we can use the results presented in sections 6.3 till 6.6, with the notable exception that we no longer have that $\mathcal{V}^u = \mathcal{V}_c$ for the constrained system $\Sigma^c$ if $d \neq 0$.

### 6.8 Conclusions

In this chapter we have studied the contact problem in a linear setting: the effect (of the boundary) of inequality constraints on the behaviour of linear continuous-time dynamical systems. A number of (pairwise disjoint) subsets of the state-space have been introduced. It was shown that these subsets are invariant under linear state feedback. Our main results are a system theoretical framework in which we described how to deal with discontinuities that arise from contact, identified the specific places for modelling the laws of collision (which are regarded as external factors), introduced the concepts of controlled and uncontrolled collisions, and presented a definition of the constrained state-space system in terms of the restricted behaviour. It has been shown that trajectories of the constrained system consist of concatenated trajectory pieces of the unconstrained system.

Additional conclusions will be stated in the next chapter where we will discuss a class of non-linear dynamical systems subject to nonlinear inequality constraints.

### Appendix 6.A: Proofs

**Proof of lemma 6.4.3:**

The proof is by induction. Let $u := Fx + v$. Denote $A_F := A + BF$. First we proof that $CA_F = CA^j$ for all $j < r_0$. From $CB = 0$ follows that $CA_F = C(A + BF) = CA$. Now assume that $CA_F^j = CA^j$ for $j = i < r_0 - 1$. Then $CA_F^{j+1} = CA_F^j(A + BF) = CA^j(A + BF) = CA^jBF = CA^{j+1}$. If $j \leq r_0 - 1$ then $h_{j,F}(x, u) := CA_F^j x + \sum_{i=1}^{j} CA_F^{j-i} Bu_{j-i} = CA^j x + 0 = h_j(x, u)$. If $j = r_0$ then $h_{r_0,F}(x, u) := CA_F^{r_0} x + \sum_{i=1}^{r_0} CA_F^{r_0-i} Bu_{r_0-i} = CA^{r_0-1} FA_F x + \sum_{i=1}^{r_0-1} CA_F^{r_0-i} Bu_{r_0-i} + CA_F^{r_0-1} Bu_0 = (CA^{r_0} + CA^{r_0-1} BF)x + CA^{r_0-1} Bu_0$. The definition of the integers $r_C(x)$ and $r_0$ now gives the desired result.

**Proof of lemma 6.4.4:**

Let $x \in \Sigma$ and let $\varphi(0) = x$ with $x \in \ker(C)$. If $h_i(x, u) = 0$ for all $i$ then $r(x, u) > n$. It
follows that \( x \in \mathcal{V}^i \). This gives (iii). Now suppose that \( \exists i \) such that \( h_i(x,u) \neq 0 \). From a Taylor-series expansion it follows that it suffices to look at the first derivative that is not equal to zero. From the definition of the sets \( \mathcal{X}_{con} \) and \( \mathcal{X}_{rel} \) the statements in (i) and (ii) now follow.

**Proof of lemma 6.4.6:**

(\( \mathcal{V}_c \subseteq \mathcal{V}^i \)): Suppose \( x \in \mathcal{V}_c \). Then \( r_0 = r_C(x) \). From corollary 6.4.2 (ii) and (iv) follows that \( \exists \mathcal{V}^i \) such that \( h_i(x,u) = 0 \), \( \forall i \). This gives \( r(x,u) = \infty \). From lemma 6.4.4 it follows that \( x \in \mathcal{V}^i \). (\( \mathcal{V}^i \subseteq \mathcal{V}_c \)): Let \( x \in \mathcal{V}^i \). Now suppose that \( x \notin \mathcal{V}_c \). By definition \( r_C(x) \neq r_0 \). From corollary 6.4.2 (i) follows \( r_C(x) < r_0 \). Moreover, from corollary 6.4.2 (iii) follows that \( h_{r_C(x)}(x,u) \neq 0 \), \( \forall u \in \mathbb{U}^i \). This contradicts that \( \exists \mathcal{V}^i \) such that \( r(x,u) = \infty \), as follows from \( x \in \mathcal{V}^i \).

**Proof of proposition 6.4.8:**

For the subsets \( \mathcal{V}_f \), \( \mathcal{V}_g \), \( \mathcal{X}_{con,v} \), \( \mathcal{X}_{con,h} \), \( \mathcal{X}_{rel,v} \) and \( \mathcal{X}_{rel,h} \) there holds: \( 1 \leq r_C(x) < r_0 \). From lemma 6.4.3 follows that these integers are invariant. It is easy to show that the sign of \( h_i \) is preserved. Finally, for \( \mathcal{V}_c \) one has \( r_C(x) = r_0 \) (without conditions on the sign of \( h_i \)).

**Proof of theorem 6.4.9:**

The first two statements are straightforward from the definitions and lemma 6.4.4. The last two statements then follow from lemma 6.4.6 and lemma A.1.1 (see appendix A) by straightforward argumentation.

**Proof of proposition 6.4.11:**

From definition 6.4.1 it follows that the sign of \( B \) is not important. The result now follows from algorithm A.1.6.

**Proof of lemma 6.5.8:**

Straightforward from the definitions of the subsets and the collision map \( T \).

**Proof of lemma 6.7.1:**

From \( \text{dim}(\ker(C)) = n-1 \) and \( \text{im}(B) \nsubseteq \ker(C) \) follows \( \ker(C) + \text{im}(B) = \mathbb{X} \). ISA now gives: \( \mathcal{V}^i = \ker(C) \). Moreover, since \( CB \neq 0 \) one has \( r_0 = 1 \). And from \( h_1(x,u) = CAx + CBu \) it follows that, depending on the value of \( u_0 \), \( h_1(x,u) < 0 \), \( h_1(x,u) = 0 \) or \( h_1(x,u) > 0 \).

**Proof of proposition 6.7.4:**

Straightforward computation gives:

\[
\begin{bmatrix}
A^i \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
= CA^j B = CA^i B.
\]

From the definitions it follows that \( h_i(z,u) = h_i([(\alpha x)^T, \alpha^T]^T, u_{j-i}) = \alpha C A^{i} \alpha x \alpha + \sum_{j=1}^{i} CA^{i-1} B u_{j-i} = \alpha h_i(x,u) \). The result in (ii) now follows from \( \alpha > 0 \).