Unilaterally constrained dynamical systems
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Chapter 5

On invariant polyhedral sets

5.1 Introduction

In this chapter we will discuss discrete-time and continuous-time dynamical systems in an input/state representation

$$\sigma x = Ax + Bu,$$  \hspace{1cm} (5.1)

subject to polyhedral restrictions on the state and/or the control

$$Cx \geq d,$$

$$Gu \geq g.$$  \hspace{1cm} (5.2)

Here $\sigma$ denotes the differential operator if $T = \mathbb{R}_+$ and the shift operator if $T = \mathbb{Z}_+$. Analogous to the previous chapter we will call the combination of (5.1) and (5.2) an input/state constrained linear dynamical system. The inequality constraint on the state may be due to a constraint on an output $y$ of system (5.1): the output constraint $Qy \geq q$ can be transformed to a state constraint with the aid of the output equation. (We will usually focus on strictly proper systems.) Some of the results presented here have appeared in 'A.A. ten Dam and J.W. Nieuwenhuis, A linear programming algorithm for invariant polyhedral sets of discrete-time linear systems, Systems & Control Letters, Vol. 25, pages 337-341, 1995'.

The basic research question we will investigate in this chapter is the following. Take an arbitrary initial condition such that the polyhedral inequality constraints in (5.2) are satisfied. Under which conditions do trajectories of the system remain in the polyhedral set defined by these inequalities? Research in this area started in the mid-eighties on autonomous state-space systems subject to state restrictions [15, 16, 136]. The number of papers on invariance properties of polyhedral restrictions is by now quite large and many analysis
and design issues have been solved. In the literature on this subject attention is usually focussed on autonomous systems, and control design issues by static state feedback. Overviews of the theory have appeared in [18, 73]. Our contributions are a discussion of both open-loop and closed-loop control issues, a number of new results on invariance properties of polyhedral restrictions, and application of the results to constrained mechanical systems.

The remainder of this chapter is as follows. We will discuss the continuous-time case separately from the discrete-time case. At first emphasis will be on discrete-time systems with $\mathbb{T} = \mathbb{Z}_+$. When a trajectory of a linear autonomous system remains in a given subset $\Omega$ of the state-space for any initial condition that is in $\Omega$, this is usually referred to as positive invariance of the set $\Omega$ [15]. In section 5.2 we will give some basic definitions and results on positively invariant sets from the literature on this subject. In section 5.3 we will establish conditions under which open-loop and closed-loop control can be used to guarantee that trajectories remain within a given polyhedral cone. In section 5.4 we will discuss the continuous-time case with $\mathbb{T} = \mathbb{R}_+$. We will briefly discuss controller design issues in section 5.5, where we also discuss some extensions of the theory. In section 5.6 we will investigate unilaterally constrained mechanical systems. Our findings are then discussed to provide a motivation for the research in the remainder of this thesis. In section 5.7 we will briefly review some results that have appeared in the literature on what is commonly referred to as the nonnegative realization problem. Conclusions are given in section 5.8.

## 5.2 Positively invariant sets

In this section we will recapitulate some of the results that appeared in the literature. We focus on autonomous discrete-time systems subject to inequality constraints. Let the system equations be given by

$$x(t + 1) = Ax(t),$$

(5.3)

for all $t \in \mathbb{Z}_+$, with $x \in (\mathbb{R}^n)^{\mathbb{Z}_+}$ and $A \in \mathbb{R}^{n \times n}$. Let the constraint be given by

$$Cx(t) \geq d,$$

(5.4)

for all $t \in \mathbb{Z}_+$, with $C \in \mathbb{R}^{p \times n}$ and $d \in \mathbb{R}^p$. Our basic assumption will be that $\{x \in \mathbb{R}^n \mid Cx \geq d\} \neq \emptyset$. As remarked earlier, state constraints may originate from inequality constraints on the output, or from knowledge of the physical properties that are modelled by the state variables.

**Definition 5.2.1** ([14, 15]) A subset $\Omega \subseteq \mathbb{R}^n$ is called a *positively invariant* set of system (5.3) if for any initial state $x(0) \in \Omega$, the complete trajectory of the state vector $x(t)$ of system (5.3) remains in $\Omega$. 

\[
\text{positive invariance set}
\]
Chapter 5: On invariant properties

In this chapter \(\Omega\) is taken to be a polyhedral set. In case of linear subspaces, a set \(\Omega\) is said to be invariant for a system (5.3) if \(A(\Omega) \subseteq \Omega\) [150]. In the literature on polyhedral restrictions this kind of invariance is generally referred to as positive invariance. We will do likewise.

The following result was initially established in [15, 16] under the restrictive assumption that the zero vector must be in the interior of the polyhedral set. The result is derived using a Lyapunov approach. Use of Haar’s Lemma eliminates this restriction [71] (see also [43, 70]). Recall from chapter 3 the notation \(\mathfrak{P}_I(C,d) = \{x \in \mathbb{R}^n \mid Cx \geq d\}\).

**Proposition 5.2.2** ([71]) Let \(C \in \mathbb{R}^{p \times n}\) and \(d \in \mathbb{R}^p\) be such that \(\mathfrak{P}_I(C,d) \neq \emptyset\). Then \(\mathfrak{P}_I(C,d)\) is a positively invariant set for system (5.3) if and only if \(\exists H \in \mathbb{R}^{p \times p}_+\) such that \(CA = HC\) and \(Hd \geq d\).

Next we give a series of results for some special types of polyhedral sets. Some of these results can be found in [15] for the case where all entries of the vector \(d\) are strictly negative. The extension to the general case is given in [43]. Recall from chapter 3, equation (3.10) the following notation for a matrix \(A = (a_{ij})\).

\[
A^+ = (a^+_{ij}) \text{ with } a^+_{ij} := \max(0,a_{ij}), \quad \text{and} \\
A^- = (a^-_{ij}) \text{ with } a^-_{ij} := \min(0,a_{ij})
\]

(5.5) (5.6)

(and analogous for a vector). It follows that \(A = A^+ + A^-\) and \((A^+ - A^-) \geq 0\).

The following lemma will be useful.

**Lemma 5.2.3** ([43]) Let \(d_1,d_2 \in \mathbb{R}^p_+\). Let \(H \in \mathbb{R}^{p \times p}\) be such that \((H^+ - I) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \leq 0\), with \(H^+ = \begin{bmatrix} H^+ & -H^- \\ -H^- & H^+ \end{bmatrix}\). Then \(-d_1 \leq x \leq d_2\) implies \(-d_1 \leq Hx \leq d_2\).

Under the conditions of lemma 5.2.3 the polyhedral set \(\{z \in \mathbb{R}^n \mid -d_1 \leq z \leq d_2\}\) is a positively invariant set of the system \(z(t+1) = Hz(t)\).

**Definition 5.2.4** Let \(S\) be a nonempty set in \(\mathbb{R}^n\). Then \(S\) is said to be symmetrical if \(x \in S \Rightarrow -x \in S\).

It is immediate that a nonempty symmetrical polyhedral set can be represented by \(\{x \in \mathbb{R}^n \mid -d \leq Cx \leq d\}\) for some matrix \(C\) and vector \(d \geq 0\). If \(A = (a_{ij})\) is a real matrix then define \(|A| := (|a_{ij}|)\). It can be seen that \(|A| = A^+ - A^-\).

**Proposition 5.2.5** ([15, 43]) Let \(C \in \mathbb{R}^{p \times n}\) and \(d \in \mathbb{R}^p_+\). Then \(\{x \in \mathbb{R}^n \mid -d \leq Cx \leq d\}\) is a positively invariant set for system (5.3) if and only if \(\exists H \in \mathbb{R}^{p \times p}_+\) such that \(CA = HC\).
and $|H|d \leq d.$

The well-known geometric approach by Wonham [150] also includes a treatment of invariance properties of linear subspaces.

**Proposition 5.2.6** ([43, 150]) Let $C \in \mathbb{R}^{p \times n}$. Then $\{x \in \mathbb{R}^n \mid Cx = 0\}$ is a positively invariant set for system (5.3) if and only if $\exists H \in \mathbb{R}^{p \times p}$ such that $CA = HC$.  

Next we generalize the result in proposition 5.2.5 to nonsymmetrical polyhedral sets, where we make an additional assumption on the representation of the polyhedral set.

**Proposition 5.2.7** ([15, 43]) Let $C \in \mathbb{R}^{p \times n}$ with $\text{rank}(C) = p$, and let $d_1, d_2 \in \mathbb{R}_+^p$. Then $\{x \in \mathbb{R}^n \mid -d_1 \leq Cx \leq d_2\}$ is a positively invariant set for system (5.3) if and only if $\exists H \in \mathbb{R}^{p \times p}$ such that $CA = HC$ and $(H^+ - I) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \leq 0$, with $H^+ := \begin{bmatrix} H^+ & -H^- \\ -H^- & H^+ \end{bmatrix}$.  

The rank assumption in the above proposition is not necessary to prove invariance of the polyhedral set $\{x \in \mathbb{R}^n \mid -d_1 \leq Cx \leq d_2\}$. The rank assumption is made to reduce the size of the unknown matrix $H(\in \mathbb{R}^{p \times 2p})$ in the proof of this proposition to the size of another matrix $H(\in \mathbb{R}^{p \times p})$.

The results above show that in order to determine whether or not a given polyhedral set is a positively invariant set the existence of a certain matrix $H$ must be established. Since all polyhedral restrictions on the state can be written in the form $\{x \in \mathbb{R}^n \mid Cx \geq d\}$ for some matrix $C$ and vector $d$ we will only discuss such representations in the remainder.

In order to verify whether a matrix $H$ exists that satisfies the conditions in proposition 5.2.2, the Linear Program defined by (3.9) and (3.12) can be adapted as follows. Assume that the matrices $A$ and $C$, and the vector $d$ are known. Define matrix $Q := (q_{ij})$ as $Q := CA$. Define the feasible set $\mathcal{F}$ by:

$$\mathcal{F} = \{(H, N, \zeta, \eta) \in \mathbb{R}_+^{p \times p} \times \mathbb{R}^{p \times n} \times \mathbb{R}^p \times \mathbb{R}^p \mid HC + N = Q, Hd + \eta = d + \zeta, \hspace{1cm} (5.7)\}$$

$$N = (n_{ij}) \text{ with } \begin{cases} n_{ij} \geq 0 \text{ when } q_{ij} \geq 0 \\ n_{ij} \leq 0 \text{ when } q_{ij} < 0 \end{cases}$$

$$\eta = \text{col}(\eta_1, \ldots, \eta_p) \text{ with } \begin{cases} \eta_i \geq 0 \text{ when } d_i \geq 0 \\ \eta_i \leq 0 \text{ when } d_i < 0 \end{cases}$$

Note that $\mathcal{F} \neq \emptyset$ since $(0, Q, 0, d) \in \mathcal{F}$. The entries of the matrix $N$ and vector $\eta$ are artificial variables, and the vector $\zeta$ serves as a vector of slack variables. Now using, as in (3.12), that $N = N^+ + N^-$ and $(N^+ - N^-) \geq 0$, and analogously for the vector $\eta$, we define the
object function:

\[
\emptyset = \min \left\{ \sum_{1 \leq i \leq p, 1 \leq j \leq n} \left( n_{ij}^+ - n_{ij}^- + \eta_{ij}^+ - \eta_{ij}^- \right) \mid (H, N, \zeta, \eta) \in \mathcal{F} \right\}.
\]

(5.8)

The Linear Program defined by (5.8) and (5.7) has an optimal solution with

\[
0 \leq \emptyset \leq \sum_{1 \leq i \leq p, 1 \leq j \leq n} ((CA)_{ij}^+ - (CA)_{ij}^- + d_i^+ - d_i^-).
\]

(5.9)

The following result can be derived analogously to proposition 3.3.7.

**Corollary 5.2.8** ([43]) Let \( C \in \mathbb{R}^{p \times n} \) and \( d \in \mathbb{R}^p \) be such that \( \mathcal{P}_f(C, d) \neq \emptyset \). Then \( \mathcal{P}_f(C, d) \) is a positively invariant set for system (5.3) if and only if \( \emptyset = 0 \), with \( \emptyset \) defined as in (5.8).

The Linear Program defined by (5.8) and the feasible set (5.7) can be rewritten in \( p \) independent 'smaller' linear programs. The following algorithm is based on rewriting the equations \( HC + N = Q \) and \( Hd + \eta = d + \zeta \) as a system of \( p \) equations.

**Algorithm 5.2.9** Let \( C \in \mathbb{R}^{p \times n} \), \( A \in \mathbb{R}^{n \times n} \) and \( d \in \mathbb{R}^p \).

(a) Set \( i := 0 \), \( Q := CA \) and \( \emptyset := \sum_{1 \leq i \leq p, 1 \leq j \leq n} (q_{ij}^+ - q_{ij}^- + d_i^+ - d_i^-) \).

(b) Set \( i := i + 1 \). Set \( H_i := Q_i \) and \( \eta := d_i \). Define the feasible set

\[
\mathcal{F}_i := \{ (H_i, N_i, \zeta, \eta) \in \mathbb{R}^{1 \times p} \times \mathbb{R}^{1 \times n} \times \mathbb{R} \times \mathbb{R} \mid H_i C + N_i = Q_i, H_i d_i + \eta = d_i + \zeta, N_i = (n_{ij}) \text{ with } n_{ij} \geq 0 \text{ when } q_{ij} \geq 0, n_{ij} \leq 0 \text{ when } q_{ij} < 0, j \in n_i, \eta \geq 0 \text{ when } d_i \geq 0 \text{ and } \eta \leq 0 \text{ if } d_i < 0 \}.
\]

(c) Solve the Linear Program

\[
\emptyset_i = \min \left\{ \sum_{1 \leq j \leq n} (n_{ij}^+ - n_{ij}^- + \eta^+ - \eta^-) \mid (H_i, N_i, \zeta, \eta) \in \mathcal{F}_i \right\}.
\]

(d) If \( i < p \) go to step (b).

(e) Set \( \emptyset := \max_{i \in \mathbb{R}} \emptyset_i \) and \( H := \text{col}(H_1, \ldots, H_p) \).

Algorithm 5.2.9 together with corollary 5.2.8 leads to the following result. The proof is omitted.

**Proposition 5.2.10** Let \( C \in \mathbb{R}^{p \times n} \) and \( d \in \mathbb{R}^p \) be such that \( \mathcal{P}_f(C, d) \neq \emptyset \). Then \( \mathcal{P}_f(C, d) \) is a positively invariant set for system (5.3) if and only if algorithm 5.2.9 gives \( \emptyset = 0 \).

It is remarked that algorithm 5.2.9 can be terminated as soon as for some index \( i \in p \) one has \( \emptyset_i > 0 \). If \( \emptyset_i = 0 \) for all \( i \in p \) then proposition 5.2.10 gives that the matrix \( H := \text{col}(H_1, \ldots, H_p) \), from the algorithm, is the one we are looking for.

In the literature on positively invariant sets often a Linear Program is used to determine whether or not a polyhedral set is a positively invariant set (see for instance [28, 70, 71, 73, ...]}
137]). For autonomous systems, the Linear Program reads

\[
\begin{align*}
\text{minimize } & \epsilon \\
\text{subject to } & \begin{cases}
CA = HC, \\
Hd \geq \epsilon d, \\
H \geq 0, \\
\epsilon \geq 0,
\end{cases}
\end{align*}
\]

(5.10)

where \(H\) and \(\epsilon\) are the variables. A set \(\mathcal{P}(C, d)\), with \(d \leq 0\), is a positively invariant set when (5.10) admits a solution \(\epsilon \leq 1\) [71]. The condition \(d \leq 0\) is a neccessary condition. To see this, consider the system \(\{x(t + 1) = \frac{1}{2}x(t) \land -x(t) \geq 1\}\). The polyhedral set \(\{x \in \mathbb{R}^n \mid x \leq -1\}\) is not a positively invariant set for system \(x(t + 1) = \frac{1}{2}x(t)\), whereas \(H = \frac{1}{2}\) and \(\epsilon = 0\) satisfy the set of equations in (5.10).

Compared to the formulation in (5.8) there are a number of differences. We made an effort to construct a Linear Program (5.7) and (5.8), for which a simple so called start/basis solution exists. Such a start/basis solution can reduce the computational effort considerably. A start/basis solution can not be given easily for the Linear Program (5.10), since there is no a priori guarantee that the set of equations in (5.10) allows a solution. Moreover, formulation (5.8) is applicable to arbitrary nonempty polyhedral sets, whereas formulation (5.10) is applicable only to polyhedral sets that contain the zero vector.

### 5.3 Controlled holdable sets

In the previous section we have reviewed some basic results for autonomous systems where the state is required to remain in a nonempty convex polyhedral set. In this section we will focus on unilateral constraints that determine a polyhedral cone. We will first derive another set of necessary and sufficient conditions for a polyhedral cone to be a positively invariant set. After this analysis we will discuss open-loop and closed-loop control of unilaterally constrained linear systems.

To shorten notation we denote in this section \(\mathcal{P}_{I}(C) := \{x \in \mathbb{R}^n \mid Cx \geq 0\}\).

First we consider again the autonomous system in (5.3) subject to the restrictions on the state as in (5.4) but now with \(d = 0\), i.e. we consider the system

\[
\begin{align*}
x(t + 1) &= Ax(t), \\
Cx(t) &\geq 0,
\end{align*}
\]

(5.11)

for all \(t \in \mathbb{Z}_+\), with \(A \in \mathbb{R}^{n \times n}\) and \(C \in \mathbb{R}^{p \times n}\). From proposition 5.25 it follows that the condition \(CA = HC\) (with \(H \in \mathbb{R}_{+}^{p \times p}\)) is a necessary and sufficient condition for the nonempty polyhedral cone \(\mathcal{P}_{I}(C)\) to be a positively invariant set.
Let $\mathfrak{P}$ be a nonempty convex polyhedral cone. We define a number of sets. First, let $\mathcal{A}(\mathfrak{P})$ denote the set of matrices $A$ that leave a convex polyhedral cone $\mathfrak{P}$ invariant. Formally,

\[ \mathcal{A}(\mathfrak{P}) := \{ A \in \mathbb{R}^{n \times n} \mid A(\mathfrak{P}) \subseteq \mathfrak{P} \}. \]

(5.12)

Define the set $\mathcal{A}_1(C)$ as the set of all system matrices $A$ such that the polyhedral cone \( \{ x \in \mathbb{R}^n \mid Cx \geq 0 \} \), for a fixed matrix $C \in \mathbb{R}^{p \times n}$, is a positively invariant set for the system $x(t + 1) = Ax(t)$, with $T = \mathbb{Z}_+$. Formally,

\[ \mathcal{A}_1(C) := \{ A \in \mathbb{R}^{n \times n} \mid \exists H \in \mathbb{R}^{p \times p} \text{ such that } CA = HC \}. \]

(5.13)

Recall from chapter 3 that a polyhedral cone is finitely generated. In the remainder of this section we assume that an image representation of a polyhedral cone \( \{ x \in \mathbb{R}^n \mid Cx \geq 0 \} \) is given by $\mathfrak{F}(N) = \{ x \in \mathbb{R}^n \mid \exists \ell \in \mathbb{R}_+^l \text{ such that } x = N\ell \}$. We immediately obtain: $\mathfrak{P}_1(C) = \mathfrak{F}(N)$ implies $(CN \geq 0)$. A representation $\mathfrak{F}(N)$ can be found from a representation $\mathfrak{F}_1(C)$ by applying the algorithms in [98, 132]. Now define the following set, analogous to the one in (5.13).

\[ \mathcal{A}_2(N) := \{ A \in \mathbb{R}^{n \times n} \mid \exists H \in \mathbb{R}^{p \times p}_+ \text{ such that } AN = NH \}. \]

(5.14)

It is remarked that the existence of a nonnegative matrix $H$ such that $AN = NH$ can be verified by a Linear Program that can be set up completely analogously to the one in (5.8).

**Proposition 5.3.1** Let $C \in \mathbb{R}^{p \times n}$ and $N \in \mathbb{R}^{n \times l}$ be such that $\mathfrak{P} = \{ x \in \mathbb{R}^n \mid Cx \geq 0 \} = \{ x \in \mathbb{R}^n \mid x = N\ell, \ell \geq 0 \}$. Then the following holds.

(i) $\mathcal{A}(\mathfrak{P}) = \mathcal{A}_1(C) = \mathcal{A}_2(N)$.

(ii) $A \in \mathcal{A}(\mathfrak{P})$ if and only if $CAN \geq 0$.

Proposition 5.3.1 provides us with necessary and sufficient conditions for a polyhedral cone to be a positively invariant set of an autonomous system. The condition $CAN \geq 0$ is especially interesting because it can be verified directly once the matrix $N$ is known. There exist algorithms to determine a matrix $N$ (see for instance [98, 132]). Since the conditions in proposition 5.3.1 do not depend on a specific representation of the polyhedral cone, it suffices to check any of these conditions for only one choice of the matrices $C$ and $N$. The characterization $CA = HC$ is usually taken as a starting point in the theory on positively invariant sets [18, 73], whereas the characterization $AN = NH$ is used for instance in the theory on positive systems ([1, 56], see also section 5.7).

In the remainder of this section we will discuss the control system

\[ x(t + 1) = Ax(t) + Bu(t), \]

(5.15)

with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, subject to inequality constraints on the state.
We start with the concept of strongly controlled holdable sets.

**Definition 5.3.2** A set $\Omega \subseteq \mathbb{R}^n$ is said to be a *strongly controlled holdable set* for the system (5.15) if for any initial state $x(0) \in \Omega$, and control $u \in (\mathbb{R}^m)^{\mathbb{Z}^+}$ there holds $x_u(t) \in \Omega$, with $x_u(t+1) := Ax(t) + Bu(t)$.

If $B = 0$ then a strongly controlled holdable set is a positively invariant set.

**Proposition 5.3.3** Let $C \in \mathbb{R}^{p \times n}$. Then the set $\Psi_I(C)$ is a strongly controlled holdable set for system (5.15) if and only if $CB = 0$ and $\exists H \in \mathbb{R}^{p \times p}$ such that $CA = HC$.

We will return to the conditions in the above proposition in section 5.6 where we will discuss unilaterally constrained mechanical systems. It is clear that the conditions whether a set is a strongly controlled holdable set are restrictive ones. We introduce a weaker notion, which can also be found in for instance [14] (where it is called 'holdable with respect to a system'). We will make a distinction between open-loop and closed-loop control in the remainder.

**Definition 5.3.4** A subset $\Omega \subseteq \mathbb{R}^n$ is said to be an *open-loop controlled holdable set*, if for any initial state $x(0) \in \Omega$ there exist a control $u \in (\mathbb{R}^m)^{\mathbb{Z}^+}$, such that $x_u(t) \in \Omega$, with $x_u(t+1) := Ax(t) + Bu(t)$.

It is clear that for open-loop controlled holdability there must hold $\{Cx \geq 0\} \Rightarrow \{\exists u \text{ such that } CAx + CBu \geq 0\}$. In [83] a projection algorithm is presented that computes a matrix $Q$ and a vector $q$ such that $\{x \in \mathbb{R}^n \mid Qx \geq q\} = \{x \in \mathbb{R}^n \mid \exists u \text{ such that } CAx + CBu \geq q'\}$, where $q'$ is a known vector.

For polyhedral cones we can now give another characterization of open-loop controlled holdability.

**Proposition 5.3.5** Let $C \in \mathbb{R}^{p \times n}$ and $N \in \mathbb{R}^{n \times t}$ be such that $\Psi_I(C) = \mathcal{N}_I(N) = \Psi$. Then the set $\Psi$ is an open-loop controlled holdable set for the system (5.15) if and only if there exists a matrix $U \in \mathbb{R}^{n \times t}$ such that $CAN + CBU \geq 0$.

The condition in proposition 5.3.5 is a necessary condition also for a nonempty convex polyhedral set to be an open-loop controlled holdable set.

We now turn to closed-loop control issues.

**Definition 5.3.6** A subset $\Omega \subseteq \mathbb{R}^n$ is said to be a *closed-loop controlled holdable set by static state feedback* for the system (5.15) if there exists a gain matrix $F \in \mathbb{R}^{m \times n}$ such that $x \in \Omega \Rightarrow (A + BF)x \in \Omega$.

The following result is immediate from proposition 5.3.1.
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Proposition 5.3.7 Let $C \in \mathbb{R}^{p \times n}$ and $N \in \mathbb{R}^{n \times f}$ be such that $\Psi_f(C) = \mathcal{N}_f(N) (= \Psi)$. Then the polyhedral cone $\Psi$ is a closed-loop controlled holdable set by static state feedback for system (5.15) if and only if $\exists F \in \mathbb{R}^{m \times n}$ such that $CAN + CBFN \geq 0$.

Suppose that the condition $CAN + CBU \geq 0$ in proposition 5.3.5 holds for some matrix $U$. In order to conclude from open-loop controlled holdability that a set is also closed-loop controlled holdable by static state feedback, one must establish that there exists a matrix $F$ such that the condition $CAN + CBFN \geq 0$ in proposition 5.3.7 is satisfied. However, it may not be possible to solve the inequality $CBFN \geq CBU$ for $F$, as shown in the following example.

Example 5.3.8 Consider the scalar system $x(t + 1) = u(t)$, $t \in \mathbb{Z}_+$. Let $\Psi = \{x \in \mathbb{R} \mid x \geq 0\}$. Take $C := 1$ and $N := [1 : 0]$. Then also $\Psi = \{x \in \mathbb{R} \mid x = Nt, \ t \geq 0\}$. With $U := [1 : 1]$ there holds $CAN + CBU = [1 : 1] \geq 0$. Let $F \in \mathbb{R}$. Then $CAN + CBFN = F[1 : 0]$. In order to conclude closed-loop controlled holdability by static state feedback from open-loop controlled holdability we must have $F[1 : 0] \geq [1 : 1]$. Clearly such an $F$ does not exist.

A sufficient condition to solve $F$ from $CBFN \geq CBU$ is that $FN = U$ can be solved for $F$, which is certainly true when $N$ has full column-rank. The following result is straightforward.

Corollary 5.3.9 Let $C \in \mathbb{R}^{p \times n}$. Let $\Psi = \Psi_f(C)$. If $\Psi$ is a closed-loop controlled holdable set by static state feedback for the system (5.15) then it is also an open-loop controlled holdable set (for the same system). If in addition there exists a representation $\mathcal{N}_f(N)$ of the polyhedral cone $\Psi$ with $N$ full column-rank then the reverse statement also holds.

Note that the condition $N$ full column-rank is not necessary. If we take $N := [1 : 1]$ in example 5.3.8 then $F = 1$ is a solution to $CBFN \geq CBU$. For equality constraints it is well known that open-loop invariance and closed loop invariance (by static state feedback) are equivalent notions [150]. A linear subspace can always be written as $Cx = 0$ and $x = Nt$, for a full row-rank matrix $C$, and full column-rank matrix $N$. The equality $CBFN = CBU$ (derived analogously to equation $CBFN \geq CBU$ in the inequality case) can now be solved for $F$. The equivalence of open-loop and closed-loop controlled holdability for systems subject to polyhedral restrictions is still an open issue.

Combining proposition 5.3.1 with the notion of closed-loop controlled holdable sets allows us to state the following result.

Theorem 5.3.10 Let $C \in \mathbb{R}^{p \times n}$ and $N \in \mathbb{R}^{n \times f}$ be such that $\Psi_f(C) = \mathcal{N}_f(N) = \Psi$. Then the following conditions are equivalent.

(i) $\Psi$ is a closed-loop controlled holdable set by static state feedback for system (5.15).
(ii) $\exists F \in \mathbb{R}^{m \times n}$ such that $(A + BF) \in A_1(C)$.
(iii) $\exists F \in \mathbb{R}^{m \times n}$ such that $(A + BF) \in A_2(N)$.
(iv) $\exists F \in \mathbb{R}^{m \times n}$ such that $C(A + BF)N \geq 0$. 


It is easy to see that the condition $C(A + BF)N \geq 0$ is not sufficient for an arbitrary nonempty convex polyhedral set $\mathcal{P}_f(C, d)$ to be a closed-loop controlled holdable set. The condition does guarantee the existence of a nonnegative matrix $H$ such that $C(A + BF) = HC$. There is no guarantee however that this matrix $H$ also satisfies $Hd \geq d$, which must also hold, according to the result in proposition 5.2.2. (See also the remark after proposition 5.3.5.)

Next, consider system (5.15) with the control law

$$u = Fx + v.$$  \hfill (5.16)

Controls as in (5.16) have been studied in [61], where bounds on the control $u$ lead to a combined constraint on the state $x$ and the new control $v$. If we apply the control in (5.16) to a system described by $x(t+1) = Ax(t) + Bu(t)$ the equations $x(t+1) = (A+BF)x(t) + Bv(t)$ are obtained. The resulting system contains a closed loop, and the control $v$ can be used for feedforward control. The research in this section leads to the following result. (The proof is omitted.)

**Proposition 5.3.11** Let $C \in \mathbb{R}^{p \times n}$ and $N \in \mathbb{R}^{n \times t}$ be such that $\mathcal{P}_f(C) = \mathcal{N}_f(N) \equiv \mathcal{P}$. Let $F \in \mathbb{R}^{m \times n}$ and let $v \in (\mathbb{R}^m)^{2+}$. Then the following conditions are equivalent:

1. $\mathcal{P}$ is an open-loop controlled holdable set for the system $x(t+1) = (A+BF)x(t) + Bv(t)$.
2. $\exists V \in \mathbb{R}^{m \times t}$ such that $C(A + BF)N + CBV \geq 0$. \hfill $\triangle$

Further work is necessary to see if proposition 5.3.11 and the ideas in for instance [61] can be combined to design controllers as in (5.16) that make the controlled system satisfy additional stability requirements.

### 5.4 Continuous-time systems

In this section we will discuss unilaterally constrained continuous-time linear systems. Consider the system

$$\frac{dx}{dt}(t) = Ax(t), \forall t \in \mathbb{R}_+,$$  \hfill (5.17)

with $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. Let the inequality constraint be given by

$$Cx(t) \geq d,$$  \hfill (5.18)

for all $t \in \mathbb{R}_+$, with $C \in \mathbb{R}^{p \times n}$ and $d \in \mathbb{R}^p$. Again, our basic assumption will be that the set $\mathcal{P} = \{x \in \mathbb{R}^n \mid Cx \geq d\} \neq \emptyset$.

It is clear that all the definitions made in the previous section on positively invariant sets and controlled holdable sets carry over from the discrete-time case to the continuous-time
case. It is immediate that the convex polyhedral set defined by the constraint (5.18) is a positively invariant set for system (5.17) if and only if

\[ \{Cx \geq d\} \Rightarrow \{Ce^hx \geq d\}. \]  

(5.19)

Before stating the continuous-time analogue of proposition 5.2.2 we need the following notion (see also [29]).

**Definition 5.4.1** Let \( H \in \mathbb{R}^{p \times p} \). Then \( H = (h_{ij}) \) is said to be an **essentially nonnegative** matrix if \( h_{ij} \geq 0 \), for all \( j \neq i \), \((i,j) \in p\). \( \triangleright \)

We introduce the notation \( H^e \geq 0 \) to indicate that a matrix \( H \) is essentially nonnegative. The following proposition was stated in [137] under some restrictive conditions. The general result was given in [29], where there is still the mild restriction that \( Cx \geq d \) does not contain redundant inequalities.

**Proposition 5.4.2** ([29, 137]) Let \( C \in \mathbb{R}^{p \times n} \) and \( d \in \mathbb{R}^p \) be such that \( \mathcal{V}_I(C,d) \neq \emptyset \). Then \( \mathcal{V}_I(C,d) \) is a positively invariant set for system (5.17) if and only if \( \exists H \in \mathbb{R}^{p \times p}, H^e \geq 0 \), such that \( CA = HC \) and \( Hd \geq 0 \). \( \triangleright \)

Given the result for the discrete-time case in proposition 5.3.1, it might be expected that for polyhedral cones a necessary and sufficient condition to be a positively invariant set in the continuous-time case is that \( CAN^e \geq 0 \) holds true. However, this need not be so, as the next example shows.

**Example 5.4.3** Consider in \( \mathbb{R}^2 \) the stable system \( \frac{dx}{dt} = Ax = \begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix} x \), subject to the constraint \( \{x_1 \geq 0 \land x_1 + x_2 \geq 0\} \). We obtain that the constraint set \( \mathcal{V} \) can be given as \( \mathcal{V} = \{x \in \mathbb{R}^2 | Cx \geq 0\} \) with \( C := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \), or equivalently \( \mathcal{V} = \{x \in \mathbb{R}^2 | x = Nl, l \geq 0\} \) with \( N := \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \). It can be verified that \( CN \geq 0 \). It can be also verified that the essentially nonnegative matrix \( H = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \) satisfies the equation \( CA = HC \). By proposition 5.4.2, \( \mathcal{V} \) is a positively invariant set for system \( \frac{dx}{dt} = Ax \). However, \( CAN = \begin{bmatrix} -1 & -1 \\ -2 & 0 \end{bmatrix} \) is not an essentially nonnegative matrix. \( \triangleright \)

The example shows that post-multiplication of an essentially nonnegative matrix by a non-negative matrix does not preserve the essentially nonnegative characteristic.
In order to verify whether a matrix $H \geq 0$ exists that satisfies the conditions in proposition 5.4.2 the Linear Program defined by (3.9) and (3.12) can be adapted as follows. Assume that the matrices $A$ and $C$, and the vector $d$ are known. Define matrix $Q := CA$. Define the feasible set $\mathcal{F}$ by:

$$\mathcal{F} = \{(H,N,\zeta) \in \mathbb{R}^{p \times p} \times \mathbb{R}^{p \times n} \times \mathbb{R}^p | HC + N = Q, Hd = \zeta, H \geq 0, \}.$$  

(5.20)

Note that $\mathcal{F} \neq \emptyset$ since $(0,Q,0) \in \mathcal{F}$. The entries of the matrix $N$ serve as artificial variables, and the vector $\zeta$ serves as a vector of slack variables. Now using, as in (3.12), that $N = N^+ + N^−$ and $(N^+ - N^-) \geq 0$ we define the Linear Programming problem:

$$0 = \min \left\{ \sum_{1 \leq i \leq p, 1 \leq j \leq n} (n^+_{ij} - n^-_{ij}) | (H,N,\zeta) \in \mathcal{F} \right\}.$$  

(5.21)

The Linear Program defined by (5.20) and (5.21) has an optimal solution with

$$0 \leq 0 \leq \sum_{1 \leq i \leq p, 1 \leq j \leq n} ((CA)^+_{ij} - (CA)^-_{ij}).$$  

(5.22)

The following result can be derived analogously to proposition 3.3.7.

**Corollary 5.4.4** Let $C \in \mathbb{R}^{p \times n}$ and $d \in \mathbb{R}^p$ be such that $\mathcal{H}_f(C,d) \neq \emptyset$. Then $\mathcal{H}_f(C,d)$ is a positively invariant set for system (5.17) if and only if $0 = 0$, with $0$ defined as in (5.21). <}

Now consider the control system

$$\dot{x}(t) = Ax(t) + Bu(t).$$  

(5.23)

The following result is immediate from proposition 5.4.2. (The proof is omitted.)

**Proposition 5.4.5** Let $C \in \mathbb{R}^{p \times n}$ and $d \in \mathbb{R}^p$. Then the nonempty polyhedral set $\mathcal{H} = \{x \in \mathbb{R}^n | Cx \geq d \}$ is a closed-loop controlled holdable set for system (5.23) if and only if there exist a gain matrix $F$ and a matrix $H \geq 0$ such that $C(A+BF) = HC$ and $Hd \geq 0$. <

From proposition 5.4.5 the following result is immediate, and can also be found in [14, Theorem 7.2.1].

**Corollary 5.4.6** Let $\mathcal{H} = \{x \in \mathbb{R}^n | x \geq 0 \}$. Then $\mathcal{H}$ is a controlled holdable set by static state feedback for system (5.23) if and only if $\exists F \in \mathbb{R}^{m \times n}$ such that $A+BF \geq 0$. <

Additional requirements on the spectrum of matrix $(A+BF)$ can be used to ensure that the state of the closed-loop system tends to zero if $t \to \infty$ while the trajectory remains in
the nonnegative orthant (see [14, 109]).

5.5 On controller design and extensions

In this section we first briefly discuss some controller design issues. Next, we show that several research questions that have been posed and solved for unconstrained linear systems ([80, 150]) can be posed (as well as partially solved) for constrained linear systems also.

In the literature on positively invariant sets, attention is almost exclusively focussed on what we have called closed-loop controlled holdability by static state feedback. Two approaches have been developed for verifying the conditions on positively invariant sets: a Linear Programming approach and an approach which uses eigenstructure assignment. In the latter approach conditions on the eigenvalues of certain matrices are used to construct feedback laws. In [73] the Linear Programming approach is said to be the one of greater generality and suitable also for its ease of integrating other constraints, such as performance and robustness requirements. The eigenstructure approach gives more insight into the structural and spectral properties of the system. Both approaches have been applied successfully in the literature [73].

In case a feedback control is to be designed such that the constrained system becomes asymptotically stable as well, there have emerged several approaches. For instance in the discrete-time case, we obtained in this chapter that the conditions $C(A + BF) = HC$ and $Hd \geq d$ for some matrix $H \geq 0$ become important. These conditions are the basic starting point in controller design issues.

Let the system be represented by

$$x(t + 1) = Ax(t) + Bu(t),$$

(5.24)

$t \in \mathbb{Z}_+$. Assume symmetrical constraints on the state $x$:

$$-d \leq Cx \leq d,$$

(5.25)

where $C \in \mathbb{R}^{p \times n}$ and $d \in \mathbb{R}^p$. The state constrained control problem consists of finding a feedback matrix $F \in \mathbb{R}^{m \times n}$ such that, with $u =Fx$ there holds

(i) $-d \leq Cx \leq d$, and

(ii) system $x(t + 1) = (A + BF)x(t)$ is asymptotically stable.

Common assumptions are rank($C$) = $p$ and $d > 0$ [73]. For a fixed feedback matrix $F$ the notion of positively invariant sets can be used to obtain the following conditions from proposition 5.2.5.

$$CA + CBF = HC,$$

(5.26)
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\[ |H|d \leq d, \]  
\[ A + BF \text{ is stable.} \]  
\[(5.27)\]  
\[ (5.28)\]

(For the case \( p \geq n \) when \( C \) has full-column rank, it is always possible, for any stabilizing \( F \), to satisfy (5.26). Select for instance \( H = C(A + BF)(C^TC)^{-1}C^T \) [73].)

Next assume symmetrical constraints on the control \( u \):

\[ -g \leq u \leq g, \]  
\[ (5.29)\]

where \( g \in \mathbb{R}^m \).

The regulator problem for system (5.24) subject to controller constraints (5.29) consists of finding a feedback matrix \( F \in \mathbb{R}^{m \times n} \) such that, with \( u = Fx \) there holds

(i) \(-g \leq Fx \leq g,\)

(ii) system \( x(t + 1) = (A + BF)x(t) \) is asymptotically stable.

The regulator problem (sometimes with additional state constraints) is discussed in for instance [13, 18, 19, 72, 136]. For a fixed feedback matrix \( F \) the notion of positively invariant sets can be used to obtain the following conditions from proposition 5.2.5.

\[ FA + FBF = HF, \]  
\[ (5.30)\]

\[ |H|g \leq g, \]  
\[ (5.31)\]

\[ A + BF \text{ is stable.} \]  
\[ (5.32)\]

For the regulator problem, the matrices \( F \) and \( H \) are the unknowns. If \( F \) is required to have full row-rank, then (5.30) has a solution \( H \) if and only if \( \text{rank} (\text{col} ([FA + FBF], [F])) = n \) (a result obtained by Porter in 1977). Solutions of the regulator problem where \( F \) is required to have full row-rank are discussed in [13] via a trial and error method. First a stabilizing feedback matrix \( F \) is chosen. Then a solution \( H \) of (5.30) is obtained, and then it is checked whether \( H \) satisfies (5.31). If this inequality is not satisfied, then another choice of \( F \) is made. The results in [72] are interesting also because it treats restrictions \(-\rho \leq Cx \leq \rho, \rho \in \mathbb{R}^p, \) where \( \rho > 0 \) is considered to be a design parameter, and \( C \) is fixed. In depth overviews of controller design issues can be found in [18, 73].

In the remainder we briefly discuss other control problems.

First consider the discrete-time system (5.15) but now interconnected with the dynamic controller [150]

\[ w(t + 1) = Ww(t) + Kx(t), \]
\[ u(t) = Fx(t) + Gw(t). \]  
\[ (5.33)\]

**Definition 5.5.1** A subset \( \Omega \subseteq \mathbb{R}^n \) is said to be a closed-loop controlled holdable set by dynamic feedback if there exist matrices \( W, K, F \) and \( G \) such that \( \Omega \) is a controlled holdable
set for the system (5.15) with the dynamic controller (5.33).

From (5.15) and (5.33) it follows that the closed-loop system is given by:

\[
\begin{bmatrix}
  x(t+1) \\
  w(t+1)
\end{bmatrix} =
\begin{bmatrix}
  A + BF & BG \\
  K & W
\end{bmatrix}
\begin{bmatrix}
  x(t) \\
  w(t)
\end{bmatrix}.
\] (5.34)

Since the closed-loop system is autonomous, the set \( \mathcal{P} = \{x \in \mathbb{R}^n \mid Cx \geq 0\} \) is a controlled holdable set by dynamic feedback for system (5.15) if and only if \( \mathcal{P} \) is a positively invariant set for system (5.34).

**Proposition 5.5.2** Let \( C \in \mathbb{R}^{p \times n} \). Then the polyhedral cone \( \mathcal{P} = \mathcal{P}_I(C,0) \) is a closed-loop controlled holdable set by dynamic feedback for system (5.15) if and only if there exists a matrix \( H \geq 0 \) such that \( C(A + BF) = HC \) and \( CBG = 0 \).

This result can be generalized to arbitrary nonempty polyhedral sets; we omit the details. Note that the conditions in proposition 5.5.2 are independent of the matrices \( W \) and \( K \). It is easy to see that the result in proposition 5.5.2 also holds for the continuous-time case if we replace the condition \( H \geq 0 \) by \( H \geq 0 \).

From theorem 5.3.10 it is now easy to see that closed-loop holdability by static feedback is a necessary condition for closed-loop holdability by dynamic feedback. Since there is a condition on matrix \( G \) it follows that the reverse need not be true.

Next, suppose that there is a disturbance input \( \delta \in \mathbb{R}^d \) that enters a discrete-time system. Consider the following system equation.

\[
x(t + 1) = Ax(t) + Bu(t) + E\delta(t).
\] (5.35)

Assume that \( Q\delta \geq q \), with \( q \leq 0 \). The question arises under which conditions does there hold that for all disturbances \( \delta \) with \( Q\delta \geq q \), one can keep the state in a given polyhedral set \( Cx \geq d \) by suitable choice of the control \( u = Fx \). It is clear that there must then hold

\[
\{Cx \geq d \wedge Q\delta \geq q\} \Rightarrow \{\exists F \text{ such that } C(A + BF)x + CE\delta \geq d\}.
\] (5.36)

Since equation (5.36) must hold for any \( x \) and \( \delta \) that satisfy the polyhedral restrictions, use of Haar’s lemma, with the obvious redefinition of the notion of closed-loop controlled holdability to cover systems as in (5.35), gives the following result.

**Proposition 5.5.3** The nonempty set \( \mathcal{P}_I(C,d) \) is a closed-loop controlled holdable set for system (5.35) where \( Q\delta \geq q \), with \( q \leq 0 \), if and only if there exist a gain matrix \( F \) and \( \exists H \geq 0 \) with \( H = [H_1 : H_2] \), all of appropriate dimensions, such that \( C(A + BF) = H_1 C \), \( CE = H_2 Q \) and \( H_1 d + H_2 q \geq d \).
Note that if $Q = 0$ then the conditions in proposition 5.5.3 reduce to: $\exists H_1 \geq 0$ such that $C(A + BF) = H_1 C$, $CE = 0$ and $H_1 d + H_2 q \geq d$ must then hold for all $q \leq 0$. These conditions can be stated alternatively as: $\text{im}(E) \subseteq \ker(C)$ and the set \{ $x \in \mathbb{R}^n \mid Cx \geq d$ \} is a closed-loop controlled holdable set by static state feedback for the system $x(t + 1) = Ax(t) + Bu(t)$. Note that this result can not be extended to the continuous-time case, for instance using proposition 5.4.2, since matrix $H$ in proposition 5.5.3 is not square. In [11, 12] other approaches are presented to deal with disturbances. Further research on this subject is left as part of future research.

Finally, we consider observers in a continuous-time setting. Assume that the system is given by (5.23) with an output equation $z = Cx$. Consider the full-order observer
\[
\dot{x} = A\hat{x} + Bu + J(z - C\hat{x}),
\]
where $J$ is an arbitrary matrix, which is referred to as the gain matrix of the observer [90]. Assume the restriction $Qz \geq d$. This leads to: $Qz = QCx \geq d$. It is natural to consider the same constraint on the output $\hat{z} = C\hat{x}$ of the observer. This gives $QC\hat{x} \geq d$. If we now set $u = -K\hat{x}$ then we obtain the closed loop system
\[
\begin{bmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{bmatrix} =
\begin{bmatrix}
A & -BK \\
JC & A - BK - JC
\end{bmatrix}
\begin{bmatrix}
x \\
\hat{x}
\end{bmatrix}.
\]

**Proposition 5.5.4** Consider the system (5.38). Then the constraints $QCx \geq d$ and $QC\hat{x} \geq d$ are satisfied for trajectories of this system that start in these sets if and only if there exist nonnegative matrices $H_2, H_3$ and essentially nonnegative matrices $H_1, H_4$ such that $QCA = H_1 QC$, $-QCBK = H_2 QC$, $QCJC = H_3 QC$, and $QC(A - BK - JC) = H_4 QC$, and $(H_1 + H_2)d \geq d$ and $(H_3 + H_4)d \geq d$.

Further research is necessary to see how the open-loop and closed-loop conditions established in this section can be used in controller design of unilaterally constrained linear systems.

### 5.6 Mechanical systems

In this section we will first investigate what the theory of positively invariant sets and controlled holdable sets brings us for unilaterally constrained mechanical systems. Next we will briefly recapitulate our results so far, as a motivation for the research in the remainder of this thesis.

The study of control/structure interaction in large spacecraft or complex robotic systems is facilitated by assuming an ideal, linear mathematical model for the dynamics (e.g. [6]). Such a model can for instance be obtained by (feedback) linearization of the non-linear
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Let the unconstrained system equation be given by

\[ M\ddot{y} + D\dot{y} + Ky = Lu, \]  

(5.39)

where \( T = \mathbb{R}_+ \) or \( T = \mathbb{Z}_+ \). Vector \( y \in (\mathbb{R}^d)^T \) is a generalized system coordinate vector, \( u \in (\mathbb{R}^m)^T \) the generalized force vector, \( M \in \mathbb{R}^{d \times d} \) the generalized positive definite inertia matrix, \( D \in \mathbb{R}^{d \times d} \) the generalized structural damping matrix, \( K \in \mathbb{R}^{d \times d} \) the generalized structural stiffness matrix, and \( L \in \mathbb{R}^{d \times m} \) the actuator force distribution matrix.

Assume the following restriction

\[ C_1 y + C_2 \sigma y \geq d, \]  

(5.40)

with \( C \in \mathbb{R}^{p \times d}, d \in \mathbb{R}^p \).

We introduce some terminology. Restriction (5.40) is said to be a holonomic inequality if \( C_2 = 0 \). Restriction (5.40) is said to be a nonholonomic inequality if \( C_2 \neq 0 \).

Let us put the linear models in a standard first-order form by choosing as state \( x := [y^T, \sigma y^T]^T \). Then (5.39) and (5.40) can be written equivalently as:

\[ \sigma x = Ax + Bu := \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1}L \end{bmatrix} u, \]  

(5.41)

and

\[ Cx \geq d, \]  

(5.42)

where \( C := [C_1 \ C_2] \).

Let us first consider holonomic constraints, i.e., assume that matrix \( C_2 = 0 \). We first give a result for the discrete-time case. Observe that the set \( \{x \in \mathbb{R}^n | [C_1 \ 0] x \geq 0 \} \) can also be represented as \( \{x \in \mathbb{R}^n | x_1 = N\ell_1, x_2 = [I - I]\ell_2, \ \ell_1, \ell_2 \geq 0 \} \) for some matrix \( N \).

Proposition 5.3.5 leads to the following result.

Corollary 5.6.1 The nonempty set \( \mathcal{P} = \{x \in \mathbb{R}^n | [C_1 \ 0] x \geq d \} \), with \( C_1 \neq 0 \), is not an open-loop controlled holdable set for the discrete-time system (5.41).

In the remainder we prefer to discuss the continuous-time case only, mainly for ease of reference in subsequent chapters. Similar results hold for the discrete-time case.

Corollary 5.6.2 Consider the continuous-time system (5.41). Then the nonempty set \( \mathcal{P} = \{x \in \mathbb{R}^n | [C_1 \ 0] x \geq d \} \), with \( C_1 \neq 0 \), is

(i) not a positively invariant set (with \( L = 0 \),
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(ii) not a strongly controlled holdable set,
(iii) not a closed-loop controlled holdable set by static state feedback,
(iv) not a closed-loop controlled holdable set by dynamic state feedback.

In fact, it can be seen that the only holonomic constraint set that is positively invariant is the set represented by \( \{ y \in \mathbb{R}^{d} | 0y \geq 0 \} \), which implies that the position is not constrained at all.

In the remainder we will discuss nonholonomical constraints. It is easy to show that it follows from proposition 5.3.3 that when matrix \( L \) in (5.41) is nonsingular then \( \Psi = \{ x \in \mathbb{R}^{n} | [0 C_{2}] x \geq d \} \) is not a strongly controlled holdable set for system (5.41).

Corollary 5.6.3 The nonempty set \( \Psi = \{ x \in \mathbb{R}^{n} | [C_{1} C_{2}] x \geq d \} \), with \( C_{2} \neq 0 \), is

(i) a positively invariant set for system (5.41) with \( B = 0 \) if and only if \( \exists H \in \mathbb{R}^{d \times d}, H \geq 0 \), such that \( -C_{2}M^{-1}K = HC_{1}, C_{1} - C_{2}M^{-1}D = HC_{2}, \) and \( HD \geq 0 \).

(ii) a closed-loop controlled holdable set by static state feedback for system (5.41) with \( u = \text{row}(F_{1}, F_{2}) x \), if and only if \( \exists H \in \mathbb{R}^{d \times d}, H \geq 0 \), such that \( C_{2}M^{-1}(LF_{1} - K) = HC_{1}, C_{1} + C_{2}M^{-1}(LF_{2} - D) = HC_{2}, \) and \( HD \geq 0 \).

Let us investigate the result in corollary 5.6.3 for the scalar case \( y, u \in \mathbb{R} \). It is immediate that when \( C_{1} = 0 \) and \( B = 0 \) then the condition in (i) becomes \( K = 0 \), and when \( C_{1} = 0 \) and \( B \neq 0 \) then the condition in (ii) reduces to \( F_{1} = K/L, F_{2} = (MH + D)/L \) with \( HD \geq 0 \).

Proposition 5.6.4 Consider system (5.41) with \( x \in \mathbb{R}^{2} \). Then the nonempty set \( \Psi = \{ x \in \mathbb{R}^{2} | [c_{1} c_{2}] x \geq 0 \} \), with \( c_{1}, c_{2} \neq 0 \), is a positively invariant set for system (5.41) with \( B = 0 \) if and only if the system matrix \( A \) has real eigenvalues only.

The proposition states that a nonholonomically constrained scalar mechanical system can not possess oscillations: the feedback gain should be chosen such that the resulting closed-loop system matrix has real eigenvalues only.

In chapter 4, example 4.4.6, we showed that direct application of the theory of unilateral dynamical systems to a holonomically constrained mechanical system is not possible. In that chapter the question was raised whether a suitable choice of the control would solve this problem. In this section we have shown that for any (nonempty) inequality constraint \( |C_{1} : 0| x \geq d \), with \( C_{1} \neq 0 \), there exist initial conditions such that whatever the applied control, the resulting trajectory does not remain in the polyhedral set under consideration. Example 4.4.6 deals with a special case of a constrained mechanical system. It follows from the physics of the problem in that example that certain points on the boundary set, i.e., points \( x \) with \( x_{1} = 0 \) and \( x_{2} < 0 \), pose problems. But physics also tell us that as time proceeds, the position and the velocity of the cart in the real world are well-defined. We arrive again at the same conclusion as in chapter 4: the set of equations in example 4.4.6 does not constitute a model of a constrained mechanical system.
In general, models for real-world systems are complex ones. Interactions between subsystems are common, but it may not be possible to identify directly those states of the system where problems may occur as easily as in example 4.4.6. Clearly, there is a need for a theory to recognize and deal with such difficulties during simulation of unilaterally constrained dynamical systems. We will propose a framework that can handle, amongst others, a large class of unilaterally constrained mechanical systems in a system theoretical setting in the remainder of this thesis, from chapter 6 onwards.

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5.7 On nonnegative realizations

In this section we briefly review some results from the literature on so-called nonnegative realizations of discrete-time dynamical systems. We include this section for two reasons. Firstly, nonnegative realizations deal with a special class of constrained linear systems. Secondly, positively invariant polyhedral sets play a role in the theory on nonnegative realizations. The present overview is not intended to be complete and we refer to [1, 76, 97, 112] and the references therein for more details. (For proofs of the results in this section we also refer to the cited literature.)

We will start our discussion on nonnegative realizations with an input/state/output representation

\[
\begin{align*}
x(t + 1) &= Ax(t) + Bu(t), \\
y(t) &=Cx(t) + Du(t),
\end{align*}
\]

for all \( t \in \mathbb{Z}_+ \). If we add the polyhedral restrictions \( u \geq 0, x \geq 0 \) and \( y \geq 0 \) then a constrained linear system is obtained. Such constraints on input, state, and output arise frequently in for instance economics and biomathematics, where the systems are called linear compartmental systems (see for instance [76]).

**Definition 5.7.1** A nonnegative discrete-time input/state/output system (with \( T = \mathbb{Z}_+ \)) is a system (5.43) in which \( A \in \mathbb{R}^{n \times n}_+, B \in \mathbb{R}^{n \times m}_+, C \in \mathbb{R}^{p \times n}_+, \) and \( D \in \mathbb{R}^{p \times m}_+ \).

In the remainder of this section we restrict ourselves to the single-input/single-output case.

A problem that has received considerable attention is the following (see for instance [1, 56, 57, 75, 76, 97, 108, 112]). Let \( P(s) \in \mathbb{R}^{p \times p}[s] \) and \( Q(s) \in \mathbb{R}^{p \times m}[s] \) be such that \( (P^{-1}Q)(s) = \sum_{i=0}^{\infty} M_i s^i \), with \( M_i \geq 0 \), for all \( i \in \mathbb{Z}_+, \) i.e. let \( (P(s), Q(s)) \) be such that the impulse response is nonnegative.

**Problem 5.7.2** Find necessary and sufficient conditions such that a system \( P(\sigma)y = Q(\sigma)u \) with nonnegative impulse response \( \sum_{i=0}^{\infty} M_i s^i \) with \( M_i \geq 0 \), for all \( i \in \mathbb{Z}_+ \) allows for a nonnegative input/state/output representation.
A state-of-the-art reference on this subject is [1], from which the following result is taken. Recall from chapter 3 that for a polyhedral cone $\mathfrak{P}$, its dual is denoted by $\mathfrak{P}^\#$.

**Theorem 5.7.3** ([1, 97]) Let $H(s)$ be a rational transfer function, $H(s) = h^T (Is - F)^{-1} g$, with $[g : Fg : \ldots : F^{n-1} g]$ and $[h^T : (hF)^T : \ldots : (hF^{n-1})^T]^T$ both have rank $n$. Let $\mathcal{R}$ denote the cone spanned by the infinite sequence of column vectors $g, Fg, F^2g, \ldots$. Then the following holds.

(i) If $H(s)$ has a nonnegative input/state/output representation (5.43), there exist a finite number of column vectors $p_i$, $i \in \mathbb{Z}_+$ ($r < \infty$), such that with $\mathfrak{P} := cone\{p_1, \ldots, p_r\}$ there holds $\mathcal{R} \subseteq \mathfrak{P}$, $F(\mathfrak{P}) \subseteq \mathfrak{P}$, and $h \subseteq \mathfrak{P}^\#$.

(ii) Suppose that there exist a finite number of column vectors $p_i$, $i \in \mathbb{Z}_+$ ($r < \infty$), such that with $\mathfrak{P} := cone\{p_1, \ldots, p_r\}$ there holds $\mathcal{R} \subseteq \mathfrak{P}$, $F(\mathfrak{P}) \subseteq \mathfrak{P}$, and $h \subseteq \mathfrak{P}^\#$. Then there exist a nonnegative input/state/output representation.  

The condition $F(\mathfrak{P}) \subseteq \mathfrak{P}$ in the above theorem was also encountered in our discussion on positively invariant sets. In [1] the result above is taken as a starting point to discuss the nonnegative realization problem in much more detail. One of the main difficulties is a check on the finitely generated conditions of the cone $\mathfrak{P}$.

Another interesting question is how the above fits in the behavioural framework, i.e. when do we call $\Sigma = (\mathbb{Z}_+, \mathbb{R}, \mathfrak{B})$ a nonnegative system? A beginning of a theory on nonnegative systems in a behavioural context is given in [112]. A motivation for the definitions below is given in [112].

Let MPUM stand for most powerful unfalsified model. The notion of MPUM is important in case a model is sought that explains a given collection of time-series (see [144], section XIV for details).

**Definition 5.7.4** ([112]) Let $\{w_1, \ldots, w_r\} \subseteq (\mathbb{R}^q)^+ \mathbb{Z}_+$ denote a collection of time-series. Define $\text{MPUM}\{w_1, \ldots, w_r\} := \cap \{\mathfrak{B} \subseteq (\mathbb{R}^q)^+ | \text{\mathfrak{B} is behaviour}, \forall i \in \mathbb{Z}, w_i \in \mathfrak{B}\}$.  

**Definition 5.7.5** ([112]) A system $\Sigma = (\mathbb{Z}_+, \mathbb{R}, \mathfrak{B})$ is said to be nonnegative if, for some $r \in \mathbb{N}$, $\mathfrak{B} = \text{MPUM}\{w_1, \ldots, w_r\}$ with all $w_i$ nonnegative.  

If $\Sigma = (\mathbb{Z}_+, \mathbb{R}, \mathfrak{B})$ is a nonnegative system then we will also call $\mathfrak{B}$ nonnegative. At first sight the definition above seems a long way from the notion of nonnegative input/state/output representations. In [112] the notion of a nonnegative system is related to finitely generated cones. Consider the autonomous system with $\mathfrak{B} := \{y \in (\mathbb{R}^q)^+ | p(\sigma)y = 0\}$, where $p(s) = s^n + p_{n-1}s^{n-1} + \ldots + p_0 \in \mathbb{R}[s]$. Associate with $p(s)$ the linear map $(y_0, y_1, \ldots, y_{n-1}) \mapsto (y_1, y_2, \ldots, y_n)$, defined by $y_n := -p_{n-1}y_{n-1} \ldots - p_0 y_0$.

**Proposition 5.7.6** ([112]) Let $0 \neq p(s) \in \mathbb{R}[s]$ be given. Consider the system $\Sigma = (\mathbb{Z}_+, \mathbb{R}, \mathfrak{B})$ with $\mathfrak{B} := \{w \in (\mathbb{R}^q)^+ | p(\sigma)w = 0\}$, and associated linear map $P : \mathbb{R}^n \to \mathbb{R}^n$, where $n := \deg p(s)$. Then $\mathfrak{B}$ is nonnegative if and only if there exists a closed convex cone
\( B \subseteq \mathbb{R}_+^n \) such that \( \text{int}(B) \neq \emptyset \) and such that \( P(B) \subseteq B \).  

Nonnegative input/output systems are also discussed in [112].

**Definition 5.7.7** ([112]) The input/output system \( \Sigma = (\mathbb{Z}_+, \mathbb{R}^2, B) \) with \( B = \{ (y, u) \mid p(\sigma)y = q(\sigma)u \} \) is called input/output nonnegative if for all \( u \in (\mathbb{R}_+)^2 \) there exists a \( y \in (\mathbb{R}_+)^2 \) with \( (y, u) \in B \).

**Proposition 5.7.8** ([112]) The input/output system \( \Sigma = (\mathbb{Z}_+, \mathbb{R}^2, B) \) with \( B = \{ (y, u) \mid p(\sigma)y = q(\sigma)u \} \) is input/output nonnegative if and only if the impulse response of \( B \) is nonnegative.

Much more is said on nonnegative behaviours in [112], but this is outside the scope of the present thesis.

## 5.8 Conclusions

In this chapter we have studied linear dynamical systems subject to polyhedral restrictions. We have discussed under which conditions a polyhedral set is an invariant set for linear dynamical systems in the sense that if one starts in such a set, one can remain in that set. In the literature emphasis is usually on static state feedback issues. We have discussed open-loop and closed-loop controlled holdability notions and presented necessary and sufficient conditions for invariance of a polyhedral set for these notions. Relations between several invariance notions have been established. For unilaterally constrained mechanical systems it has been shown that the theory of invariant polyhedral sets does not provide a solution to the difficulties that are encountered in modelling such systems. The need for a framework has been identified to handle unilaterally constrained dynamical systems that do not possess an invariance property of the kind discussed in this chapter.

In this chapter emphasis has been on controller analysis issues. Further research is necessary to see whether or not the open-loop and closed-loop conditions established in this chapter are useful in controller design of unilaterally constrained linear systems.

## Appendix 5.A: Proofs

**Proof of proposition 5.2.2:**

The set \( \mathcal{P}_f(C, d) \) is a positively invariant set if and only if \( \forall x \in \mathcal{P}_f(C, d) \) there holds: \( \{ Cx \geq d \Rightarrow CAx \geq d \} \). The result now follows from Haar's Lemma.
Proof of lemma 5.2.3:

Observe that \( \{ x \in \mathbb{R}^n \mid -d_1 \leq x \leq d_2 \} = \{ x \in \mathbb{R}^n \mid \begin{bmatrix} I \\ -I \end{bmatrix} x \geq \begin{bmatrix} -d_1 \\ -d_2 \end{bmatrix} \} \). Note that \( H^\pm \in \mathbb{R}^{2p \times 2p}_+ \). From \( H^\pm \begin{bmatrix} I \\ -I \end{bmatrix} = \begin{bmatrix} H \\ -H \end{bmatrix} \) and the assumptions it follows that \( H^\pm \) satisfies the conditions in Haar’s Lemma. It follows that \(-d_1 \leq x \leq d_2 \) implies \(-d_1 \leq Hx \leq d_2 \).

Proof of proposition 5.2.5:

First observe that \( \{ x \in \mathbb{R}^n \mid -d \leq Cx \leq d \} = \{ x \in \mathbb{R}^n \mid \begin{bmatrix} C \\ -C \end{bmatrix} x \geq \begin{bmatrix} -d \\ -d \end{bmatrix} \} \).

\((\Rightarrow)\) : Since \( 0 \in \Psi_f(C,d) \) we can apply Haar’s Lemma to obtain that \( \exists H \in \mathbb{R}^{2p \times 2p}_+ \) such that \( \begin{bmatrix} C \\ -C \end{bmatrix} A = H \begin{bmatrix} C \\ -C \end{bmatrix} \), and \((H - I) \begin{bmatrix} -d \\ -d \end{bmatrix} \geq 0\). Denote \( H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \). We obtain that \( CA = (H_{11} - H_{12})C \). Set \( H := H_{11} - H_{12} \). Then \( |H| \leq H_{11} + H_{12} \). It is now easy to show that \( CA = HC \) and \( |H|d \leq d \).

\((\Leftarrow)\) : Define \( H^\pm := \begin{bmatrix} H^+ & -H^- \\ -H^- & H^+ \end{bmatrix} \). From \( |H| \leq H^+ - H^- \) and the assumptions it follows that \( H^\pm \begin{bmatrix} d \\ d \end{bmatrix} \leq \begin{bmatrix} d \\ d \end{bmatrix} \). The result now follows from lemma 5.2.3.

Proof of proposition 5.2.6:

(Only if): Observe that \( \{ x \in \mathbb{R}^n \mid Cx = 0 \} = \{ x \in \mathbb{R}^n \mid 0 \leq Cx \leq 0 \} \). Application of proposition 5.2.5 with \( d = 0 \) gives that \( \exists H \in \mathbb{R}^{p \times p}_+ \) such that \( CA = HC \). (The condition \( |H|d \leq d \) is trivially satisfied.) (If): Let \( x \in \ker(C) \). Then \( CAx = HCx = 0 \). Hence: \( A(\ker(C)) \subseteq \ker(C) \). This can be written also as: \( \begin{bmatrix} C \\ -C \end{bmatrix} x \geq 0 \Rightarrow \begin{bmatrix} C \\ -C \end{bmatrix} Ax \geq 0 \).

Application of Haar’s Lemma and proposition 5.2.2 now gives that \( \{ x \in \mathbb{R}^n \mid Cx = 0 \} \) is a positively invariant set for system (5.3).

Proof of proposition 5.2.7:

Note that \( \{ x \in \mathbb{R}^n \mid -d_1 \leq Cx \leq d_2 \} = \{ x \in \mathbb{R}^n \mid \begin{bmatrix} C \\ -C \end{bmatrix} x \geq \begin{bmatrix} -d_1 \\ -d_2 \end{bmatrix} \} \).

\((\Rightarrow)\) : From the assumptions it follows that \( 0 \in \{ x \in \mathbb{R}^n \mid -d_1 \leq Cx \leq d_2 \} \). Proposition 5.2.2 gives that \( \exists H \in \mathbb{R}^{2p \times 2p}_+ \) such that \( \begin{bmatrix} C \\ -C \end{bmatrix} A = H \begin{bmatrix} C \\ -C \end{bmatrix} \), and \((H - I) \begin{bmatrix} -d_1 \\ -d_2 \end{bmatrix} \geq 0\). Denote \( H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \). It follows that \( CA = (H_{11} - H_{12})C \) and \( CA = (H_{22} - H_{21})C \).

Since \( C \) has full row-rank we obtain \( H_{11} - H_{12} = H_{22} - H_{21} \). Define \( H := H_{11} - H_{12} \). Then \( CA = HC \). From \( d_1, d_2 \geq 0 \) it now follows that \( H^+d_1 \leq H_{11}d_1, H^+d_2 \leq H_{22}d_2, -H^-d_1 \leq
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\[ H_2 d_1, \text{and } -H^{-1} d_2 \leq H_1 d_2. \] It follows that \[ H^{\pm} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \leq \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \leq \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}. \]

(\Leftarrow): Straightforward from lemma 5.2.3.

**Proof of proposition 5.3.1:**

(i): Let \( A \in \mathcal{A}(\mathcal{P}) \). Then for all \( x \in \mathcal{P} \) one has that \( Ax \in \mathcal{P} \). Consequently, \( Cx \geq 0 \Rightarrow CAx \geq 0 \). Haar’s Lemma now gives \( A \in \mathcal{A}_1(C) \). To see that \( A \in \mathcal{A}_2(N) \), observe that \( A \in \mathcal{A}(\mathcal{P}) \) implies that there exist \( \ell_1, \ell_2 \geq 0 \) such that for \( x \in \mathcal{P} \) there holds \( x = N\ell_1 \Rightarrow Ax = N\ell_2 \). This gives \( AN\ell_1 = N\ell_2 \). In particular, take \( \ell_i(t) = e_i \), with \( e_i \) the \( i \)th unit vector in \( \mathbb{R}^l \). We obtain that there exist vectors \( \ell'_1, \ldots, \ell'_l \in \mathbb{R}^l_+ \), such that \( ANe_i = N\ell'_i \) \( (i \in \ell) \). Define \( H := \text{row}(\ell'_1, \ldots, \ell'_l) \). This gives \( A \in \mathcal{A}_2(N) \). We have obtained: \( \mathcal{A}(\mathcal{P}) \subset (\mathcal{A}_1(C) \cap \mathcal{A}_2(N)) \). On the other hand, if \( A \in \mathcal{A}_1(C) \) or \( A \in \mathcal{A}_2(N) \) it is immediate that \( A \in \mathcal{A}(\mathcal{P}) \). This gives the statement in (i).

(ii): Assume that \( CAN \geq 0 \). Take arbitrary \( x \in \mathcal{P}_I(N) \). There exists \( \ell \geq 0 \) such that \( x = N\ell \). Consequently: \( CAN \geq 0 \Rightarrow CAN\ell \geq 0 \Rightarrow CAx \geq 0 \). Thus for all \( x \in \mathcal{P}_I(C) \), \( CAx \geq 0 \). This implies \( A \in \mathcal{A}(\mathcal{P}) \). Now suppose \( A \in \mathcal{A}(\mathcal{P}) \). From the statement in (i) we obtain that \( \exists H_1 \geq 0 \) such that \( CA = H_1 C \) and \( \exists H_2 \geq 0 \) such that \( AN = NH_2 \). This gives \( CAN = H_1 CN \) (and also \( CAN = CNH_2 \)). From \( H_1, H_2 \geq 0 \) and \( CN \geq 0 \) (c.f. corollary 3.6.7) it follows that \( CAN \geq 0 \).

**Proof of proposition 5.3.3:**

The set \( \mathcal{P}_I(C) \) is a strongly controlled holdable set for system \( x(t + 1) = Ax(t) + Bu(t) \) if and only if \( CAx(t) \geq 0 \) implies \( CAx(t + 1) \geq 0 \), or equivalently if and only if \( \forall u(t) \) there holds \([C \ 0] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \Rightarrow [CA \ CB] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \geq 0 \). Proposition 5.2.2 now gives that this holds if and only if \( CB = 0 \) and \( \exists H \in \mathbb{R}^{n \times p} \) such that \( CA = HC \).

**Proof of proposition 5.3.5:**

Suppose that \( \mathcal{P}_I(C) \) is an open-loop controlled holdable set. Then for all \( x \in \mathcal{P}_I(C) \) there exist a control \( u \) such that \( CAx(t) \geq 0 \) implies \( CAx(t) + CBu(t) \geq 0 \). In particular, take \( x_i(t) = Ne_i \), with \( e_i \) the \( i \)th unit vector in \( \mathbb{R}^l \). We obtain that there exist vectors \( u_1, \ldots, u_l \in \mathbb{R}^n \), such that \( CANe_i + CBu \geq 0 \) \( (i \in \ell) \). Define \( U := \text{row}(u_1, \ldots, u_l) \). This gives the desired result. Now take \( x = N\ell, \ell \geq 0 \), and assume that \( C(A + BU) \geq 0 \). Then \( C(A + BU)\ell \geq 0 \). Define \( u := U\ell \). Then we have \( CAx + Bu \geq 0 \), and we are done.

**Proof of theorem 5.3.10:**

The result follows from definition 5.3.6 and proposition 5.3.1.

**Proof of proposition 5.4.2:**

The proof is split into two parts. In the first part we will assume that \( \{Cx \geq d\} \) contains no redundant inequalities. In that case the proof is taken from [29], adapted to our notation. In the second part we will prove the case where \( \{Cx \geq d\} \) does contain redundant inequalities.

**Proof of part 1:**

(Necessity): Relation (5.19) should be valid for any \( x \in \mathcal{P}_I(C, d) \). In particular it must
hold for points \( x \) that satisfy

\[ C_i x = d_i, \quad i \in \mathbb{P} \]
\[ C_j x \geq d_j, \quad j \in \mathbb{P}, \quad j \neq i. \]

First take \( i \in \mathbb{P} \) fixed. Consider an infinitesimal move from any point \( x \) that satisfy the above equations. In order for this move to be feasible we must have:

\[
\begin{cases}
C_i x = d_i, \\
C_j x \geq d_j, \quad j \neq i
\end{cases} \Rightarrow C_i \dot{x} = C_i Ax \geq 0.
\]

Application of Haar’s Lemma now gives that the above implication is true if and only if

\[ \exists h_{ij} \in \mathbb{R}, \quad h_{ij} \geq 0 \quad \text{for} \quad j \neq i, \quad \text{such that} \quad \sum_{j=1}^{p} h_{ij} C_i = C_i A \quad \text{and} \quad \sum_{j=1}^{p} h_{ij} d_j \geq 0. \]

Denote \( h_i := \text{row}(h_{i1}, \ldots, h_{ip}) \). Repeating the above for all \( i \in \mathbb{P} \) gives the essentially nonnegative matrix \( H := \text{col}(h_1, \ldots, h_p) \) and the relations \( CA = HC \) and \( Hd \geq 0 \).

(Sufficiency): Let \( CA = HC \) and \( Hd \geq 0 \) for an essentially nonnegative matrix \( H \). Consider the power series representation \( e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \). From \( CA = HC \) we obtain that \( CA^k = H^k C, \forall k \in \mathbb{Z}_+ \). It follows that \( Ce^{At} = (\sum_{n=0}^{\infty} \frac{H^n t^n}{n!}) C = e^{Ht} C \). A classic result gives \([130]\) \( \{H \geq 0\} \Leftrightarrow \{e^{Ht} \geq 0 \ \forall t \geq 0\} \). From \( Hd \geq 0 \) it follows that \( H \dot{d} = d' \) with \( d' \in \mathbb{R}^p, \ d' \geq 0 \).

For any nonnegative value of \( t, \ (e^{Ht} - I)d \) can be expanded as follows:

\[ (e^{Ht} - I)d = (t + H \frac{t^2}{2} + \ldots + \frac{H^n}{n!} \frac{t^{n+1}}{n+1} + \ldots)Hd = \int_0^t (e^{H\tau} d')d\tau. \]

Since \( d' \) is nonnegative, and matrix \( e^{Ht} \) is nonnegative for any value of \( t \) such that \( 0 \leq t \leq t \), we obtain that \( \forall t \geq 0: (e^{Ht} - I)d \geq 0 \). We have obtained that \( Ce^{At} = e^{Ht} C \) and \( e^{Ht} d \geq d \).

Since \( e^{Ht} \geq 0 \) it follows that \( Cx \geq d \) implies \( Ce^{At} x \geq d \). This completes the first part of the proof.

Proof of part 2: Next suppose that \( \{Cx \geq d\} \) contains redundant inequalities. Inspection of the proof in [29] shows that we only need to prove the ‘only if’ part. Without loss of generality assume that matrices \( C \) and \( d \) are partitioned such that the redundant equations are arranged to be the last \( k \) inequalities in \( \{Cx \geq d\} \). (This can always be done using the results in section 3.3 and permuting the rows of \( C \) and \( d \).) Let

\[
\begin{bmatrix}
C_n & d_n \\
C_r & d_r
\end{bmatrix}, \quad \text{with} \quad C_n \in \mathbb{R}^{(p-k)\times n}, \ d_n \in \mathbb{R}^{(p-k)}, \ C_r \in \mathbb{R}^{k \times n}, \ \text{and} \ d_r \in \mathbb{R}^k
\]

such that \( C_n x \geq d_n \) does not contain any redundant equations and \( \{C_n x \geq d_n\} \Rightarrow \{C_r x \geq d_r\} \). Since \( C_n x \geq d_n \) does not contain redundant inequalities there exists an essentially nonnegative matrix \( M \in \mathbb{R}^{(p-k)\times(p-k)} \) such that \( C_n A = MC_n \) and \( Md_n \geq 0 \ ([29]) \). From Haar’s Lemma follows that \( \exists F \in \mathbb{R}_+^{k \times (p-k)} \) such that \( C_r = FC_n \) and \( Fd_n \geq d_r \). Define the diagonal matrix \( \Delta_n \in \mathbb{R}^{(p-k)\times(p-k)} \) with \( \Delta_n = \text{diag}(\delta, \ldots, \delta) \) such
that for matrix $M_n := M + \Delta_n$, we have $M_n \geq 0$. Next define the diagonal matrix
$\Delta_r \in \mathbb{R}^{k \times k}$ with $\Delta_r = \text{diag}(\delta, \ldots, \delta)$. Observe that $F \Delta_n = \Delta_r F$. Now define the essentially nonnegative matrix $H$ as $H := \begin{bmatrix} M & 0 \\ FM_n & -\Delta_r \end{bmatrix}$. It follows that $HC = H \begin{bmatrix} C_n & \\ C_r \end{bmatrix}$
$$\begin{bmatrix} MC_n \\ FM_n C_n - \Delta_r C_r \end{bmatrix} = \begin{bmatrix} MC_n \\ F(M + \Delta_n) C_n - \Delta_r C_r \end{bmatrix} = \begin{bmatrix} MC_n \\ FMC_n + F\Delta_n C_n - \Delta_r C_r \end{bmatrix} = \begin{bmatrix} C_n A \\ FC_n A + \Delta_r FC_n - \Delta_r C_r \end{bmatrix} = \begin{bmatrix} C_n A \\ C_r A + \Delta_r C_r - \Delta_r C_r \end{bmatrix} = \begin{bmatrix} C_n \\ C_r \end{bmatrix} A = CA. \text{ And } Hd =
H \begin{bmatrix} d_n \\ d_r \end{bmatrix} = \begin{bmatrix} Md_n \\ FM_n d_n - \Delta_r d_r \end{bmatrix} = \begin{bmatrix} Md_n \\ FMd_n + F\Delta_n d_n - \Delta_r d_r \end{bmatrix} \geq \begin{bmatrix} 0 \\ \Delta_rFd_n - \Delta_r d_r \end{bmatrix}.$$ 

Proof proposition 5.5.2:

Define the matrix $C' = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}$ to obtain: $\{Cx \geq 0\}$ if and only if $x$ satisfies $\{C' \begin{bmatrix} x \\ w \end{bmatrix} \geq 0\}$. Application of proposition 5.2.2 now gives that $\exists H \geq 0, H = col([H_1 \; H_2], [H_3 \; H_4])$, all of appropriate dimensions, such that $C(A + BF) = H_1 C, 0 = H_3 C, 0 = H_4 0, CBG = H_2 0,$
and $H_0 \geq 0$. Take $H = H_1, H_2 = 0, H_3 = 0, \text{ and } H_4 = 0$. This gives the statement in the proposition.

Proof of proposition 5.5.3:

Define the matrices $C' = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}$, $C_F = [C(A + BF) \; CE]$ and the vector $d' = \begin{bmatrix} d \\ q \end{bmatrix}$.

Rewrite (5.36) as $C' \begin{bmatrix} x \\ \delta \end{bmatrix} \geq d' \Rightarrow C_F' \begin{bmatrix} x \\ \delta \end{bmatrix} \geq d$. Application of proposition 5.2.2 now gives that $\exists H \geq 0, H = [H_1 : H_2]$, all of appropriate dimensions, such that $C(A + BF) = H_1 C, CE = H_2 Q$, and $H_1 d + H_2 q \geq d$. This gives the statement in the proposition.

Proof of proposition 5.5.4:

Define the matrices $C' = \begin{bmatrix} QC & 0 \\ 0 & QC \end{bmatrix}$, $A_F = \begin{bmatrix} A & -BK \\ JC & A - BK - JC \end{bmatrix}$ and the vector $d' = \begin{bmatrix} d \\ d \end{bmatrix}$. Application of proposition 5.2.2 now gives that $\exists H \geq 0$, such that $C'A_F = HC'$. Write $H = col([H_1 \; H_2], [H_3 \; H_4])$, all of appropriate dimensions. We obtain $QCA = H_1 QC,$ 
$-QC BK = H_2 QC, QC JC = H_3 QC,$ and $QC(A - BK - JC) = H_4 QC,$ and $(H_1 + H_2)d \geq d$ and $(H_3 + H_4)d \geq d$. From $H_2, H_3 \text{ nonnegative, and } H_1, H_4 \text{ essentially nonnegative the statement in the proposition follows}$.
Proof of proposition 5.6.4:
From corollary 5.6.3 it follows that we must have that there $\exists h \in \mathbb{R}$ such that $-c_2 k/m = hc_1$, $c_1 - c_2 d/m = hc_2$. Solving both equations for $h$, and equating the resulting solutions gives $c_2^2 k/m - d/m c_1 c_2 + c_1^2 = 0$. Since $c_1$ and $c_2$ are real, we obtain that necessarily: $d^2 \geq 4km$. Now consider matrix $A$. The characteristic equation is $s^2 + d/m s + k/m = 0$. The condition $d^2 \geq 4km$ now gives that $A$ has only real eigenvalues. Reversing the 'only if' proof shows that the condition is also sufficient. $\triangleright$