Chapter 4

Unilateral dynamical systems

4.1 Introduction

The mathematical models of constrained mechanical systems in chapter 1 contain higher-order differential equations to model the dynamics. Additional inequalities are present that model the hard environmental or operational restrictions. In this chapter we will investigate a larger class of dynamical systems: namely systems that can be represented by differential or difference inequalities. To gain an understanding of modelling issues for systems that can be described by inequalities, we will study a special class of these systems in a behavioural setting. Up to now there are only two publications that we are aware of that deal with inequalities in a behavioural setting: a beginning of a theory has been presented in [41, 111]. In this chapter we will extend the results presented in 'A.A. ten Dam, Representations of dynamical systems described by behavioral inequalities, Proceedings European Control Conference ECC'93, Vol. 3, pages 1780-1783, Groningen, The Netherlands, June 28 - July 1, 1993'.

In the linear case important modelling issues are elimination of latent variables from a representation, establishing relations between different representations of the same system, finding efficient representations of a system, and deriving first-order representations (see chapter 2). We will therefore also concentrate on these issues. Emphasis will be on systems that can be represented by difference inequalities.

The remainder of this chapter is organized as follows. In section 4.2 we will introduce the notion of convex conical systems. In section 4.3 we formally define unilateral dynamical systems. It is investigated which properties allow dynamical systems to be described by a class of difference inequalities. In section 4.4 we discuss the elimination problem for difference inequalities. The notion of slack variables is introduced in a behavioural setting, and a particular representation is presented that will play an important role in the discussion on elimination of latent variables. In section 4.5 minimality issues in representations of
behavioural difference inequalities are treated. A number of open problems is identified. In section 4.6 dynamic inequalities are looked upon as a special case of interconnected systems. We will formally define the class of unilaterally constrained linear systems. We will briefly discuss first-order representations and the axiom of state. Directions for further research are given. Conclusions can be found in section 4.7.

4.2 **Convex conical behaviours**

Instead of linear dynamical systems we will discuss dynamical systems that have other properties. Some basic notions and properties are introduced, where we closely follow the line of thought presented in [125] for static inequalities.

**Definition 4.2.1** A dynamical system $\Sigma = (\mathcal{T}, \mathcal{W}, \mathfrak{B})$ is said to be *convex* if $\mathcal{W}$ is a real vector space over $\mathbb{R}$ and $\mathfrak{B}$ is a convex subset of $\mathcal{W}^\tau$, i.e. if $w_1, w_2 \in \mathfrak{B}$ then $\{(1 - \alpha)w_1 + \alpha w_2 | 0 \leq \alpha \leq 1\} \subseteq \mathfrak{B}$. It is said to be *conical* if $\mathfrak{B}$ is a cone, i.e. $\mathfrak{B}$ is closed under multiplication by a nonnegative scalar: $\{w \in \mathfrak{B}, \alpha \geq 0\} = \{\alpha w \in \mathfrak{B}\}$. If a system is both convex and conical it is called *convex conical*.

Linear systems are a special case of convex conical systems.

**Proposition 4.2.2** Let $\Sigma = (\mathcal{T}, \mathcal{W}, \mathfrak{B})$ be a dynamical system. Then $\Sigma$ is convex conical if and only if $\mathfrak{B}$ contains all nonnegative linear combinations of its elements, i.e. if $w_1, \ldots, w_n \in \mathfrak{B}$ and $\alpha_1, \ldots, \alpha_n \in [0, \infty)$ then $\alpha_1 w_1 + \ldots + \alpha_n w_n \in \mathfrak{B}$.

For a behaviour $\mathfrak{B}$ let $(-\mathfrak{B}) := \{w \in \mathcal{W}^\tau | -w \in \mathfrak{B}\}$.

**Proposition 4.2.3** Let $\Sigma = (\mathcal{T}, \mathcal{W}, \mathfrak{B})$ be a convex conical dynamical system. Then:

(i) The behaviour of the smallest linear system containing $\Sigma$, denoted by $\Sigma^L$, is given by $\mathfrak{B} - \mathfrak{B} = \{w \in \mathcal{W}^\tau | w = w_1 - w_2, w_1, w_2 \in \mathfrak{B}\}$.

(ii) The behaviour of the largest linear system contained in $\Sigma$, denoted by $\Sigma^L$, is given by $\mathfrak{B} \cap (-\mathfrak{B}) = \{w \in \mathcal{W}^\tau | w \in \mathfrak{B}$ and $w \in (-\mathfrak{B})\}$.

In $\mathbb{R}^n$ a subset is called convex polyhedral if it is the intersection of a finite collection of closed halfspaces (see chapter 3). The concept of polyhedrals will prove to be useful also in the present study.

**Definition 4.2.4** Let $\Sigma = (\mathcal{Z}, \mathcal{W}, \mathfrak{B})$ be a discrete-time dynamical system. Then $\Sigma$ is said to be a *finite-polyhedral (conical)* system if $\forall t_1, t_2 \in \mathcal{Z}, -\infty < t_1 \leq t_2 < \infty$, $\mathfrak{B}[t_1, t_2]$ is (a) polyhedral (cone) in $(\mathbb{R}^q)^{t_2 - t_1 + 1}$. In that case $\mathfrak{B}$ is said to be (a) finite-polyhedral (cone).

The use of the word conical is consistent with earlier use. If a convex conical system is also finite-polyhedral then it is a convex finite-polyhedral-conical system, and vice versa.
Proposition 4.2.5 Let $\Sigma = (Z, W, B)$ be a time-invariant discrete-time dynamical system. Let $\Sigma$ be finite-polyhedral. Then $\Sigma$ is complete if and only if $B$ is closed.

The question arises if the 'finite-polyhedral' condition on $\Sigma$ is a necessary condition in proposition 4.2.5. We return to this issue later.

So far, we have discussed inequality systems at a rather high level of abstraction. In the remainder of this chapter we focus on representation issues.

4.3 Behavioural inequalities

A behavioural difference inequality representation of a discrete-time dynamical system with the time axis $T = \mathbb{Z}$ and signal space $W$, as in the linear case, defined by two integers $L$ and $l$, and a map $f : \mathbb{W}^{L-l+1} \rightarrow \mathbb{R}^g$ (for some $g \in \mathbb{N}$). A difference inequality is given by:

$$f(w(t + L), ..., w(t + l)) \geq 0, \forall t \in \mathbb{Z}. \quad (4.1)$$

It is clear that the system $\Sigma = (Z, W, B)$, with

$$B = \{w : \mathbb{Z} \rightarrow \mathbb{W} \mid \text{equation (4.1) is satisfied}\} \quad (4.2)$$

defines a time-invariant dynamical system.

Definition 4.3.1 Let $\Sigma = (Z, W, B)$ be a discrete-time dynamical system. If there exists a map $f : \mathbb{W}^{L-l+1} \rightarrow \mathbb{R}^g$ such that $B$ allows a representation as in (4.2) then the system $\Sigma$ is said to be a unilateral dynamical system.

The notion of unilateral dynamical systems extends to the case $T = \mathbb{R}$ in the obvious way. It can be seen that the systems described in chapter 1 are all examples of unilateral dynamical systems: the map $f$ readily follows from the prescribing equations in those examples.

In the linear case systems that can be described by behavioural equalities of the form $R(\sigma, \sigma^{-1})w = 0$ are well studied. In this chapter we will focus on convex conical unilateral dynamical systems whose behaviour can be represented by

$$B = \{w \in (\mathbb{R}^g)^\mathbb{Z} \mid R(\sigma, \sigma^{-1})w \geq 0\}, \quad (4.3)$$

where $R \in \mathbb{R}^{g \times g}[s, s^{-1}]$. As in chapter 2 the number of columns is fixed and equals the number of manifest variables. The number of rows is equal to the number of equations used to describe the behaviour.

Equation (4.3) is the logical extension of the linear case to the convex conical case as far as the behavioural representation is concerned. In [143] it has been shown that a discrete-time dynamical system $\Sigma = (T, W, B)$ can be described by a difference equality if and only if
it is linear, time-invariant and complete. The question immediately arises what intrinsic properties of a dynamical system allow its behaviour to be represented as in (4.3).

It is clear that system $\Sigma = (\mathbb{Z}, \mathcal{W}, \mathfrak{B})$ with $\mathfrak{B}$ as in (4.3) is a convex conical system. Convex conical behaviours can also arise from certain nonlinear representations. For instance, the discrete-time system that is described by the nonlinear latent variable description

$$w(t) = \ell^2(t),$$

(4.4)
can also be described by the convex conical latent variable description

$$w(t) = r(t),$$

$$r(t) \geq 0,$$

(4.5)
and by the convex conical manifest description

$$w(t) \geq 0.$$  

(4.6)

This provides us with another motivation to study convex conical dynamical systems. (The first motivation being the mathematical models in the examples in chapter 1.)

In the remainder of this chapter the behaviour in (4.3) will be denoted by $\mathfrak{B}_I(R)$, to differentiate these behaviours from behavioural representations where only equalities are present, and from behavioural representations where both equalities and inequalities are present. Formally: let $R_1 \in \mathbb{R}^{q \times q}[s, s^{-1}]$ and $R_2 \in \mathbb{R}^{r \times q}[s, s^{-1}]$. Denote:

$$\mathfrak{B}_E(R_1) := \{w \in (\mathbb{R}^q)^\mathbb{Z} \mid R_1(\sigma, \sigma^{-1})w = 0\},$$

$$\mathfrak{B}_I(R_2) := \{w \in (\mathbb{R}^q)^\mathbb{Z} \mid R_2(\sigma, \sigma^{-1})w \geq 0\},$$

(4.7)

$$\mathfrak{B}_{EI}(R_1, R_2) := \{w \in (\mathbb{R}^q)^\mathbb{Z} \mid R_1(\sigma, \sigma^{-1})w = 0 \text{ and } R_2(\sigma, \sigma^{-1})w \geq 0\}.$$

The following definition is a generalization of the notion of lineality space for the static case (see chapter 3).

**Definition 4.3.2** The lineality behaviour $\mathfrak{B}_L$ of a nonempty convex conical system $\Sigma = (\mathbb{Z}, \mathcal{W}, \mathfrak{B})$ is defined as $\mathfrak{B}_L = \mathfrak{B} \cap (-\mathfrak{B})$. For system $\Sigma$, the system $\Sigma_L = (\mathbb{Z}, \mathcal{W}, \mathfrak{B}_L)$ is called the lineality system.

By proposition 4.2.3 the lineality system is the largest linear system in a given system.

**Lemma 4.3.3** Let $\Sigma = (\mathbb{Z}, \mathcal{W}, \mathfrak{B})$ be a discrete-time dynamical system with $\mathfrak{B} = \mathfrak{B}_I(R)$. Then $\mathfrak{B}_L = \mathfrak{B}_E(R)$.

Difference inequalities uniquely define the lineality behaviour. Since the reverse statement is not generally true one can not hope to find a difference inequality representation of a
behaviour from a characterization of its linearity behaviour. This implies that the results
that have been obtained in the behavioural approach to linear difference equalities are not
directly applicable to the inequality case. To obtain an inequality description a different
approach must be followed.

Definition 4.3.4 Let \( \alpha \) be a \( \mathbb{R}^q \)-valued sequence with compact support. Then the set
\( \mathcal{H} = \{ w \in (\mathbb{R}^q)^\mathbb{Z} \mid \sum_{t \in \mathbb{Z}} \alpha^T(t)w(t) \geq 0 \} \) is said to be a halfspace in \( (\mathbb{R}^q)^\mathbb{Z} \). If a set \( \mathcal{P} \) is the
intersection of a finite number of halfspaces \( \mathcal{H}_i \), i.e. \( \mathcal{P} = \cap_{i=1}^g \mathcal{H}_i \) for some \( g \in \mathbb{N} \), then the
set \( \mathcal{P} \) is said to be a polyhedral cone in \( (\mathbb{R}^q)^\mathbb{Z} \).

Note that if the sequence \( \alpha \) in definition 4.3.4 has compact support in \([t_0, t_1]\), then for \( w \in \mathcal{P} \) there is no requiremen
outside this interval \([t_0, t_1]\). For a polyhedral cone \( \mathcal{P} \) in \( (\mathbb{R}^q)^\mathbb{Z} \),
define \( \sigma^t \mathcal{P}, t \in \mathbb{Z} \), by \( \sigma^t \mathcal{P} := \{ \sigma^t w \in (\mathbb{R}^q)^\mathbb{Z} \mid w \in \mathcal{P} \} \). This leads to the following notion.

Definition 4.3.5 Let \( \Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{W}) \) be a discrete-time convex time-invariant complete
dynamical system. Then \( \Sigma \) is said to be shifted-polyhedral conical if there exists a polyhedral
cone \( \mathcal{P} \) in \( (\mathbb{R}^q)^\mathbb{Z} \) such that \( \mathcal{B} = \cap_{t \in \mathbb{Z}} \sigma^t \mathcal{P} \). In that case \( \mathcal{B} \) is said to be a shifted-polyhedral
cone.

The use of the word conical is consistent with earlier use. If a convex conical system is also
a shifted-polyhedral system then it is a convex shifted-polyhedral-conical system, and vice versa.

The importance of the notion of shifted polyhedral cones lies in the following result.

Theorem 4.3.6 Let \( \Sigma = (\mathbb{Z}, \mathcal{W}, \mathcal{B}) \) be a discrete-time convex finite-polyhedral time-
invariant complete dynamical system. Then: \( \Sigma \) is shifted-polyhedral conical if and only
if \( \exists R \in \mathbb{R}^{q \times q}[s, s^{-1}] \) such that
\[
\mathcal{B} = \{ w \in (\mathbb{R}^q)^\mathbb{Z} \mid R(\sigma, \sigma^{-1})w \geq 0 \}.
\]

It follows from definition 4.3.1 that if \( \Sigma = (\mathbb{Z}, (\mathbb{R}^q)^\mathbb{Z}, \mathcal{B}) \) satisfies the conditions in theorem
4.3.6 with \( \mathcal{B} \) a shifted-polyhedral cone then \( \Sigma \) is a unilateral dynamical system. The
question arises if the conditions \( \Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}) \) is complete, time-invariant and convex
finite-polyhedral conical, or equivalently (by proposition 4.2.5), \( \mathcal{B} \) is closed, shift-invariant
and a convex finite-polyhedral cone, are also sufficient for \( \Sigma \) to be a shifted polyhedral cone.
This conjecture, which was raised in [41], was disproven in [111] for the case \( T = \mathbb{Z}_+ \) by the
following illustrative and nontrivial counterexample.

Example 4.3.7 Define the following sequence of vectors in \( (\mathbb{R}^q)^\mathbb{Z}^+ \):

\[
\begin{align*}
k_1 & := (2, 0, 0, \ldots), \\
k_2 & := (1, 2, 0, 0, \ldots), \\
k_n & := (1, 1, \ldots, 1, 2, 0, 0, \ldots), \\
& \quad \text{for } n \geq 1. 
\end{align*}
\]
Suppose that this vector is in $\mathbb{R}^\mathbb{Z}_+$.

To disprove the conjecture it must be shown that there is no polynomial matrix $R(s)$ such that $w \in \mathcal{B} \Leftrightarrow R(\sigma)w \geq 0$. Let $\bar{K}_m$ be the projection of $K_m$ on the first $m$ coordinates of $\mathbb{Z}_+$, i.e.

$$K_m := \text{cone}((1,0,\ldots,0)^T, (1,2,0,\ldots,0)^T, \ldots, (1,1,\ldots,1,2)^T).$$

It is easy to see that from $(x_1,x_2,\ldots,x_{n+1})^T \in \bar{K}_{n+1}$ it follows that $(x_1,x_2,\ldots,x_n)^T \in \bar{K}_n$ and $(x_2,x_3,\ldots,x_{n+1})^T \in \bar{K}_n$. However, $(x_1,x_2,\ldots,x_n)^T, (x_2,x_3,\ldots,x_{n+1})^T \in \bar{K}_n$ does not imply that $(x_1,x_2,\ldots,x_{n+1})^T \in \bar{K}_{n+1}$. To see this, first observe that $(1,1,\ldots,1)^T \in \bar{K}_n$, and $(1-\frac{1}{2^{n-1}},1,\ldots,1)^T \in \bar{K}_n$. Now consider, in $\mathbb{R}^{n+1}$, the vector $(1-\frac{1}{2^{n-1}},1,\ldots,1)^T$. Suppose that this vector is in $\bar{K}_{n+1}$. Then from

$$(1,1,\ldots,1) = \frac{1}{2}(1,1,\ldots,1,2) + \frac{1}{4}(1,1,\ldots,1,2,0) + \ldots + \frac{1}{2^{n+1}}(2,0,\ldots,0),$$

it follows that $\lambda_1 < 0$. This contradicts the requirement that $\lambda_1 \geq 0$.

Now assume that there does exist a polyhedral cone $K \in \mathbb{R}^n$, for some $n \in \mathbb{N}$, such that

$$\{w \in \mathcal{B} \} \leftrightarrow \{ \forall t \in \mathbb{Z}_+, (w(t),w(t+1),\ldots,w(t+n-1))^T \in K \}.$$  

By construction one has that $\bar{K}_n \subseteq K$. But now from $\bar{K}_n \subseteq \bar{K}_{n+1}, \forall n \in \mathbb{N}$, and the fact that $(x_1,x_2,\ldots,x_n)^T \in \bar{K}_n$ and $(x_2,x_3,\ldots,x_{n+1})^T \in \bar{K}_n$ does not imply that $(x_1,x_2,\ldots,x_{n+1})^T \in \bar{K}_{n+1}$ it follows immediately that one can not conclude from $(w(t),w(t+1),\ldots,w(t+n-1))^T \in \bar{K}_n$. 

By construction one has that $\bar{K}_n \subseteq \bar{K}_n$. But now from $\bar{K}_n \subseteq \bar{K}_{n+1, \forall n \in \mathbb{N}$, and the fact that $(x_1,x_2,\ldots,x_n)^T \in \bar{K}_n$ and $(x_2,x_3,\ldots,x_{n+1})^T \in \bar{K}_n$ does not imply that $(x_1,x_2,\ldots,x_{n+1})^T \in \bar{K}_{n+1}$ it follows immediately that one can not conclude from $(w(t),w(t+1),\ldots,w(t+n-1))^T \in K$. 

Section 4.3: Behavioural inequalities
The above example can be used also to disprove the conjecture for the case $\mathbb{T} = \mathbb{Z}$, by redefining $k_n := (\ldots, 1, 1, 2, 0, 0, \ldots)$, where $k_n(n) = 2$.

If we look at the way proposition 2.2.6 is proven in the linear case, then the following emerges. In [143] the linear case is proven in two different ways. In the first proof, one of the essential observations is that a decreasing sequence of linear subspaces $L_t$ with $L_{t+1} \subseteq L_t$ attains a limit in a finite number of steps. This however need not be the case in convex polyhedral sets, not even in finite dimensions. Moreover, the convex cone that is obtained as the limit of this sequence need not to be polyhedral. Take the ice-cream cone $K$ that contains the closed convex cone $\mathbb{K}$ itself is not polyhedral. In the present case we have so far not been able to prove or disprove that similar statements hold for inequality behaviours. We will return to these difficulties in the section 4.5 in the context of minimal representations.

**Open Problem 4.3.8** Let $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathcal{B})$ be a discrete-time convex finite-polyhedral conical time-invariant dynamical system. Give necessary and sufficient conditions for $\mathcal{B}$ to be a shifted-polyhedral cone.

Note that a sufficient condition exists: if we require that $\mathcal{B} = \mathcal{B}_L$ then we are again in the linear case, but this is of course a highly unsatisfactory condition in a discussion on inequality behaviours. Note also that as long as we do not have necessary and sufficient conditions under which $\mathcal{B}$ is a shifted-polyhedral cone we can not even prove that the behaviour $\mathcal{B} := \{ w \in (\mathbb{R}^d)^{\mathbb{Z}} \mid (\sigma - 1)^k w \geq 0, k \in \mathbb{N}_+ \}$ can not be represented by a finite number of inequalities. We conclude that at least until problem 4.3.8 is solved, the finite-polyhedral condition on $\Sigma$ can not be omitted easily from proposition 4.2.5.

### 4.4 The elimination problem

In this section we focus on elimination of latent variables from a difference inequality representation.

Let $R_1 \in \mathbb{R}^{g \times q}[s, s^{-1}]$, $R_2 \in \mathbb{R}^{g \times d}[s, s^{-1}]$, $w : \mathbb{Z} \rightarrow \mathbb{R}^g$ and $\ell : \mathbb{Z} \rightarrow \mathbb{R}^d$. Consider the latent
variable difference inequality:

\[ R_1(\sigma, \sigma^{-1})w \geq R_2(\sigma, \sigma^{-1})\ell. \]  \hspace{1cm} (4.15)

The question arises whether or not the latent variable \( \ell \) can be eliminated from (4.15) to arrive at a representation \( R(\sigma, \sigma^{-1})w \geq 0 \) for some polynomial matrix \( R \).

Elimination of latent variables from difference inequality representations has been discussed also in [111], where it is argued that (4.15) implies that there is a \( l \in \mathbb{N} \), and there are vectors \( b_1, \ldots, b_n \in (\mathbb{R}^{a+d})^l \) such that for all \( t \in \mathbb{Z} \), there are \( \lambda_1(t), \ldots, \lambda_n(t) \), all in \( \mathbb{R}_+ \), such that

\[ \text{col}(w(t), \ell(t), \ldots, w(t+l-1), \ell(t+l-1)) = \lambda_1(t)b_1 + \ldots + \lambda_n(t)b_n. \]  \hspace{1cm} (4.16)

It remains to be shown that from (4.16), (4.15) follows. This is not discussed in [111].

Our approach to the elimination problem is based on the observation made in section 4.3 for representations (4.5) and (4.6). First however, we show by example that elimination of latent variables from a difference inequality is different from elimination of latent variables from a difference equality.

**Example 4.4.1** Consider the latent variable difference inequalities:

\[
\begin{align*}
R_1(\sigma, \sigma^{-1})w & \geq \ell, \\
R_2(\sigma, \sigma^{-1})w & \geq a\ell,
\end{align*}
\]  \hspace{1cm} (4.17)

with \( a \in \mathbb{R} \) and \( w \in \mathbb{R}^2 \). If \( a = 0 \) then the manifest behaviour can be described by \( R_2(\sigma, \sigma^{-1})w \geq 0 \) since the first inequality in (4.17) does not pose a restriction on \( w \). If \( a > 0 \) then the manifest behaviour can be described by \( 0 \cdot w \geq 0 \), as for a given time-series \( w \) it is always possible to find a time series \( \ell \) such that \( (w, \ell) \) satisfies (4.17). Finally, if \( a < 0 \) we can write the last inequality in (4.17) as \( \ell \geq \frac{1}{a}R_3(\sigma, \sigma^{-1})w \). Combining with \( R_1(\sigma, \sigma^{-1})w \geq \ell \) gives \( R_1(\sigma, \sigma^{-1})w - \frac{1}{a}R_2(\sigma, \sigma^{-1})w \geq 0 \), and this inequality describes the behaviour of \( w \) completely. For difference equalities, consider

\[
\begin{align*}
R_1(\sigma, \sigma^{-1})w & = \ell, \\
R_2(\sigma, \sigma^{-1})w & = a\ell.
\end{align*}
\]  \hspace{1cm} (4.18)

Substitution of the second equation in the first equation gives \( R_2(\sigma, \sigma^{-1})w = aR_1(\sigma, \sigma^{-1})w \) as the prescribing equation of the manifest behaviour, whatever the value of \( a \).

The difference inequality case discussed in example 4.4.1 can be generalized. The next result states that the number of equations necessary to describe the manifest behaviour depends on the values of reals present in the latent variable description. (The result can be seen as an extension of a result in [83], where static inequalities are discussed.)
Proposition 4.4.2 Consider the latent variable system $\Sigma_\ell = (\mathbb{Z}, (\mathbb{R}^q)\mathbb{Z}^\ell, (\mathbb{R}^q)\mathbb{Z}^\ell, \mathfrak{B}_\ell)$, with $\mathfrak{B}_\ell = \{(w, \ell) \in (\mathbb{R}^q)\mathbb{Z} \times (\mathbb{R}^q)\mathbb{Z} | R_1(\sigma, \sigma^{-1})w \geq a_1\ell, \ldots, R_g(\sigma, \sigma^{-1})w \geq a_g\ell\}$, $g \in \mathbb{N}$. Define $H_+ := \{i : a_i > 0\}$, $H_0 := \{i : a_i = 0\}$, $H_- := \{i : a_i < 0\}$, and $n_+ := \text{card}(H^+)$, $n_0 := \text{card}(H_0)$ and $n_- := \text{card}(H_-)$ (where $\text{card}$ denotes cardinality.) Then the latent variables can be eliminated by the $n_+ \cdot n_- + n_0$ inequalities

$$
R_i(\sigma, \sigma^{-1})w \geq 0, \ i \in H_0,
$$
$$
a_j R_k(\sigma, \sigma^{-1})w \geq a_k R_j(\sigma, \sigma^{-1})w, \ j \in H_+, k \in H_-.
$$

Consequently, if $n_0 = 0$ and either $n_+ = 0$ or $n_- = 0$ then there are no restrictions on the manifest variables $w$.

It is now easy to see that in case $R(\sigma, \sigma^{-1})w \geq \mathcal{A}\ell$, with $\mathcal{A} \in \mathbb{R}^q\mathbb{Z}$ and $\ell \in (\mathbb{R}^d)\mathbb{Z}$, the latent variables can also be eliminated by repeated use of the result in proposition 4.4.2: write $\ell = \text{col}(\ell_1, \ldots, \ell_d)$ and eliminate the $\ell_i$'s one after the other. The resulting set of inequalities contains a large number of equations, but is stated in terms of the manifest variable $w$ only. This also provides some clues for the elimination of the latent variables from $R(\sigma, \sigma^{-1})w \geq \mathcal{A}(\sigma, \sigma^{-1})\ell$. For instance if $\mathcal{A}(s, s^{-1})$ has positive or negative entries only, then the variables $w$ are not restricted.

The following representation in (4.19) will play an important role in our discussion of the elimination problem. Let $M \in \mathbb{R}^{q \times b}[s, s^{-1}]$, and $N \in \mathbb{R}^{q \times b}[s, s^{-1}]$, $b \in \mathbb{N}$.

$$
\begin{align*}
w &= M(\sigma, \sigma^{-1})\beta, \\
N(\sigma, \sigma^{-1})\beta &\geq 0.
\end{align*}
$$

\hspace{1cm} (4.19)

The variable $\beta$ appears in (4.19) as a unilaterally constrained latent variable.

Proposition 4.4.3 Let $\Sigma_\ell = (\mathbb{Z}, (\mathbb{R}^q)\mathbb{Z}^\ell, (\mathbb{R}^q)\mathbb{Z}^\ell, \mathfrak{B}_\ell)$ be a discrete-time time-invariant latent dynamical system represented by $R_1(\sigma, \sigma^{-1})w \geq R_2(\sigma, \sigma^{-1})\ell$, with $R_1 \in \mathbb{R}^{q \times q}[s, s^{-1}]$ and $R_2 \in \mathbb{R}^{q \times d}[s, s^{-1}]$. Then $\exists b \in \mathbb{N}$ and there are polynomial matrices $M \in \mathbb{R}^{q \times b}[s, s^{-1}]$ and $N \in \mathbb{R}^{q \times b}[s, s^{-1}]$ such that the manifest system $\Sigma = (\mathbb{Z}, (\mathbb{R}^q)\mathbb{Z}, \mathfrak{B})$ of $\Sigma_\ell$ can be described by $\mathfrak{B} = \{w \in (\mathbb{R}^q)\mathbb{Z} | \exists \beta \in (\mathbb{R}^b)\mathbb{Z} \text{ such that equation (4.19) holds}\}$. <

Since a behaviour $\mathfrak{B}_R(R)$ is a special case of a latent variable description (take $R_2 = 0$ in (4.15)), proposition 4.4.3 applies to systems $\Sigma = (\mathbb{Z}, (\mathbb{R}^q)\mathbb{Z}, \mathfrak{B}_R(R))$ as well. Note also that any representation (4.19) can be written as a latent variable description (4.15). Therefore, the remaining problem to be solved reads as follows. Can the latent variable $\beta$ be eliminated from (4.19) to arrive at an inequality representation $R(\sigma, \sigma^{-1})w \geq 0$ for some polynomial matrix $R$? The following proposition gives sufficient conditions. (The proof is straightforward from the results in chapter 2 and is omitted.)
**Section 4.4: The elimination problem**

**Proposition 4.4.4** Let \( M \in \mathbb{R}^{q \times b}[s, s^{-1}] \) and \( N \in \mathbb{R}^{g \times b}[s, s^{-1}] \). Let \( \Sigma_\beta = (\mathbb{Z}_n, (\mathbb{R}^q)^Z, (\mathbb{R}^b)^V, \mathfrak{B}_\beta) \) be a discrete-time time-invariant latent variable dynamical system. Suppose it induces the manifest behaviour with \( \mathfrak{B} = \{ w \in (\mathbb{R}^g)^Z \mid \exists \beta \in (\mathbb{R}^b)^V \text{ such that (4.19) holds} \} \). Then there exists a polynomial matrix \( R(s, s^{-1}) \) such that the manifest behaviour of \( \Sigma_\beta \) is given by \( \mathfrak{B} = \mathfrak{B}_I(R) \) if \( \beta \) is observable in \( w = M(\sigma, \sigma^{-1})\beta \), or \( N(s, s^{-1}) = 0 \).

The first condition in proposition 4.4.4 implies that there exists a polynomial matrix \( R'(s, s^{-1}) \) such that \( \beta = R'(\sigma, \sigma^{-1})w \). This gives \( \{(I - MR')(\sigma, \sigma^{-1})w = 0, (NR')(\sigma, \sigma^{-1})w \geq 0 \} \) as a model of the manifest behaviour. The second condition in proposition 4.4.4 implies that the prescribing equation is \( w = M(\sigma, \sigma^{-1})\beta \). Since we are now in the linear case, the elimination theorem 2.3.4 provides us with the manifest behaviour \( R''(\sigma, \sigma^{-1})w = 0 \) for some polynomial matrix \( R''(s, s^{-1}) \).

Up to now we have not been able to formulate necessary conditions in terms of the matrices \( M \) and \( N \). Another approach, which uses results from the linear case, is given next. We include it as a possible direction of future research.

Let \( \mathfrak{B} = \mathfrak{B}_I(R) \) with \( R \in \mathbb{R}^{q \times q}[s, s^{-1}] \). Define

\[
\mathfrak{B}_\alpha(R) := \{(w, \alpha) \in ((\mathbb{R}^q)^Z \times (\mathbb{R}^b)^V) \mid R(\sigma, \sigma^{-1}) - I \begin{bmatrix} w \\ \alpha \end{bmatrix} = 0, \alpha \geq 0 \}. \quad (4.20)
\]

**Definition 4.4.5** A **dynamical system with slack variables** is defined as \( \Sigma_\alpha = (\mathbb{Z}_n, (\mathbb{R}^q)^Z, (\mathbb{R}^b)^V, \mathfrak{B}_\alpha) \) with \( \mathbb{Z} \) the time-axis, \( (\mathbb{R}^q)^Z \) the space of manifest variables, \( (\mathbb{R}^b)^V \) the space of slack variables, and \( \mathfrak{B}_\alpha = \mathfrak{B}_\alpha(R) \) as in (4.20) for some polynomial matrix \( R \in \mathbb{R}^{q \times q}[s, s^{-1}] \).

Obviously, \( \Sigma_\alpha = (\mathbb{Z}_n, (\mathbb{R}^q)^Z, (\mathbb{R}^b)^V, \mathfrak{B}_\alpha) \) defines a slack variable representation of the manifest system \( \Sigma = (\mathbb{Z}_n, (\mathbb{R}^q)^Z, \mathfrak{B}) \), with manifest behaviour \( \mathfrak{B} = \mathfrak{B}_I(R) \).

Now consider an image representation (4.19). First introduce a slack variable \( \alpha \) in (4.19). This gives \( \{ w = M(\sigma, \sigma^{-1})\beta, \alpha = N(\sigma, \sigma^{-1})\beta, \alpha \geq 0 \} \) as a slack variable representation. The first two equations give an image representation of a system \( \Sigma \) with \( \beta \) as a latent variable. It follows that \( \Sigma \) is a controllable system [143]. From the elimination theorem for linear systems it follows that \( \Sigma \) can be represented by \( [R'_1 : R'_2] \cdot \text{col}(w, \alpha) = 0 \) for some polynomial matrices \( R'_1(s, s^{-1}) \) and \( R'_2(s, s^{-1}) \). Next, find a full row-rank representation \( [R_1 : -R_2] \cdot \text{col}(w, \alpha) = 0 \). Evidently, if \( R_2(s, s^{-1}) \) is a unimodular matrix then an equivalent representation is given by \( [R_2^{-1}R_1 : -I] \cdot \text{col}(w, \alpha) = 0 \). Finally, using that \( \alpha \geq 0 \), the slack variable \( \alpha \) can be deleted again to obtain the desired behaviour \( \mathfrak{B}_I(R) \), with \( R := R_2^{-1}R_1 \). However, the condition \( R_2(s, s^{-1}) \) is unimodular is not necessary. For instance if \( R_2(s, s^{-1}) = 0 \) then \( R_1(\sigma, \sigma^{-1})w = 0 \) is obtained. The use of slack variables representations in the elimination problem needs further research.

Many of the definitions and results in the present section have their counterpart for the case
\( \mathbb{T} = \mathbb{R} \). But in the continuous-time case there are additional modelling issues. Consider the following continuous-time example, which is a special case of the system in example 1.2.6.

**Example 4.4.6** Consider two carts running on the same track. Suppose that the motion of the left cart cannot be controlled. Let \( y_1 \) denote the position of the left cart, \( y_2 \) the position of the right cart, and \( u \) the control input for the right cart (see figure 4.1).

![Figure 4.1: Contact between two carts.](image)

Consider the following model

- **Left Cart:** \[ m_1 \frac{d^2 y_1}{dt^2} + d_1 \frac{dy_1}{dt} + k_1 y_1 = 0, \]  \hspace{1cm} (4.21)
- **Right Cart:** \[ m_2 \frac{d^2 y_2}{dt^2} + d_2 \frac{dy_2}{dt} + k_2 y_2 = u, \]  \hspace{1cm} (4.22)
- **Constraint:** \[ y_2 \geq y_1, \]  \hspace{1cm} (4.23)

where \( m_i, d_i, \) and \( k_i, i = \{1, 2\} \), are the systems parameters for the left cart, right cart, respectively. Let \( \Sigma_1 := (\mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathcal{B}_1) \), \( \Sigma_2 := (\mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathcal{B}_2) \), and \( \Sigma_3 := (\mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathcal{B}_3) \), with

\[
\begin{align*}
\mathcal{B}_1 &= \{ (y_1, y_2) \in \mathcal{L}^{loc}_1 \times \mathcal{L}^{loc}_1 \mid \text{equation (4.21) is satisfied} \}, \\
\mathcal{B}_2 &= \{ (y_1, y_2) \in \mathcal{L}^{loc}_1 \times \mathcal{L}^{loc}_1 \mid \text{equation (4.22) is satisfied} \}, \\
\mathcal{B}_3 &= \{ (y_1, y_2) \in \mathcal{L}^{loc}_1 \times \mathcal{L}^{loc}_1 \mid \text{equation (4.23) is satisfied} \}.
\end{align*}
\]

The interconnection of \( \Sigma_1, \Sigma_2 \) and \( \Sigma_3 \) is given by \( \Sigma_1 \land \Sigma_2 \land \Sigma_3 = (\mathbb{R}, \mathbb{R}^2, \mathcal{B}) \) with \( \mathcal{B} := \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 \). This gives

\[ \mathcal{B} = \{ (y_1, y_2) \in \mathcal{L}^{loc}_1 \times \mathcal{L}^{loc}_1 \mid \text{equations (4.21) - (4.23) are satisfied} \}. \]  \hspace{1cm} (4.25)

In contrast to representation (2.8) we have not obtained a single prescribing differential equation for the system \( \Sigma \).

Suppose that we are interested specifically in the behaviour of the left cart. An interesting
question is the following. If we interpret the variables that model the right cart as latent variables, can we eliminate these variables? In other words, does there exist a polynomial matrix $R(s)$ such that the behaviour of the first cart in the above example can be explained as time-trajectories of the differential inequality $R(\sigma)y_1 \geq 0$ (where $\sigma$ denotes differentiation with respect to time)? The answer depends on the mathematical framework one works in. Physics tells us that when the carts make contact, there may be an impulse that makes the left cart move. The key problem is that we do not know a priori when and how the right cart comes in contact with the left cart. Moreover, if the carts remain in contact, then the right cart may not exert a force on the left cart since the interaction force is not modelled in (4.25)! (See also example 2.3.2.) These observations imply of course that the control $u$ must be chosen appropriately. (This is related to a reachability question studied in [4] where also transitions are studied that do not involve discrete jumps.) On the other hand, these observations also tell us that (4.25) is not a mathematical model of the system in figure (4.1). If we are interested only in the set of possible trajectories of the left cart, then the latent variables, i.e. the pair $y_2, u$, can be eliminated if one works in a convex conical setting. They can not be eliminated in a framework that includes impulse modelling. We will leave this modelling issue for the moment, but return to it in chapter 5.

In this section we have discussed the elimination problem. It remains to be shown that there exists an elimination theorem for difference inequalities.

### 4.5 On minimal representations

In this section we focus on efficient representations of behaviours that can be represented by

$$\mathcal{B}_I(R) = \{w \in \mathbb{R}^q_{\mathbb{Z}} \mid R(\sigma, \sigma^{-1})w \geq 0\},$$

with $R \in \mathbb{R}^{q \times q}[s, s^{-1}]$. We would like to associate the notion of minimality to a representation $\mathcal{B}_I(R)$. As in the linear case, minimality will always refer to keeping the number of equations as small as possible.

**Definition 4.5.1** Let $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathbb{B})$. Let $R \in \mathbb{R}^{q \times q}[s, s^{-1}]$ and $R' \in \mathbb{R}^{d \times q}[s, s^{-1}]$. The systems of difference inequalities $R(\sigma, \sigma^{-1})w \geq 0$ and $R'(\sigma, \sigma^{-1})w \geq 0$ are said to be equivalent if $\mathcal{B}_I(R) = \mathcal{B}_I(R')$. The system of difference inequalities $R(\sigma, \sigma^{-1})w \geq 0$ is said to be a minimal inequality (or, for short, minimal) if: $\{\mathcal{B}_I(R) = \mathcal{B}_I(R')\} \Rightarrow \{g \leq g'\}$. <

In for instance [143], it is shown that a kernel representation $R(\sigma, \sigma^{-1})w = 0$ is minimal if and only if $R(s, s^{-1})$ has full row-rank. It is easy to see that an inequality system $\{R(\sigma, \sigma^{-1})w \geq 0\}$ with $R \in \mathbb{R}^{1 \times q}[s, s^{-1}]$ is minimal, and also that $\mathcal{B}_I(R) = \mathcal{B}_E(R)$ if $R(s, s^{-1}) \neq 0$ and $R(s, s^{-1})$ has full row-rank.
The following result will be useful in proving proposition 4.5.3.

**Lemma 4.5.2** Let \( R \in \mathbb{R}^{g \times q}[s, s^{-1}] \), \( g > 1 \), be such that \( \mathcal{B}_I(R) = \mathcal{B}_E(R) \). Then \( R(\sigma, \sigma^{-1})w \geq 0 \) is a minimal inequality if and only if there exists a \( F \in \mathbb{R}^{(g-1) \times q}[s, s^{-1}] \) such that \( \mathcal{B}_I(R) = \mathcal{B}_E(F) \) and \( F(\sigma, \sigma^{-1})w = 0 \) is a minimal equality.

**Proposition 4.5.3** The following holds:

(i) Not every inequality behaviour \( \mathcal{B}_I(R) \) has a full row-rank inequality representation.

(ii) For every \( q \in \mathbb{N} \), there exists an inequality system in \( q \) variables such that the minimum number of rows in the minimal inequality representation exceeds \( q \).

Let \( \mathcal{B} = \mathcal{B}_I(R) \) with \( R \in \mathbb{R}^{g \times q}[s, s^{-1}] \). Denote the \( k \)th row of matrix \( R \) by \( R_k \). Define for \( k \in \mathbb{N} \),

\[
\mathcal{B}^k := \{ w \in (\mathbb{R}^q)\mathbb{Z} \mid R_k(\sigma, \sigma^{-1})w \geq 0, i \neq k \}. \tag{4.27}
\]

\[
\mathcal{B}_k := \{ w \in (\mathbb{R}^q)\mathbb{Z} \mid R_k(\sigma, \sigma^{-1})w \geq 0 \}. \tag{4.28}
\]

Clearly \( \mathcal{B} \subseteq \mathcal{B}^k \) for all \( k \in \mathbb{N} \).

**Definition 4.5.4** Let \( \Sigma = (\mathbb{Z}, \mathbb{W}, \mathcal{B}) \) with \( \mathcal{B} = \mathcal{B}_I(R) \). The difference inequality \( R_k(\sigma, \sigma^{-1})w \geq 0 \) is said to be a redundant inequality in representation \( \mathcal{B}_I(R) \) if \( \mathcal{B}^k = \mathcal{B} \).

Recall from chapter 2 that a representation \( R(\sigma, \sigma^{-1})w = 0 \) contains a redundant equation if and only if \( R(s, s^{-1}) \) does not have full row-rank. In the remainder, equalities as well as inequalities will be referred to as equations. And for instance the phrase ’redundant equation’ will refer to either a redundant equality or a redundant inequality (in a representation).

Apart from redundancy there is another way to reduce the number of inequality equations, at the ’cost’ of introducing equalities. This idea has been used already in lemma 4.5.2. Recall from lemma 4.3.3 that the linearity behaviour of \( \mathcal{B}_I(R) \) is given by \( \mathcal{B}_E(R) \). Associate with \( R_k(s, s^{-1}) \), \( k \in \mathbb{N} \), the linearity behaviour \( \mathcal{B}_{L,k} := \mathcal{B}_E(R_k) \). Note that \( \mathcal{B}_L = \bigcap_{k=1}^g \mathcal{B}_{L,k} \).

**Definition 4.5.5** Let \( \Sigma = (\mathbb{Z}, \mathbb{W}, \mathcal{B}) \) with \( \mathcal{B} = \mathcal{B}_I(R) \). The difference inequality \( R_k(\sigma, \sigma^{-1})w \geq 0 \) is called an implicit equality in representation \( \mathcal{B}_I(R) \) if \( \mathcal{B} \subseteq \mathcal{B}_{L,k} \).

Implicit equalities may be replaced by equalities without increasing the number of equations in a representation. It is easy to see that if all inequalities in a representation are implicit equalities then \( \mathcal{B} = \mathcal{B}_L \), or equivalently \( \mathcal{B}_I(R) = \mathcal{B}_E(R) \) (from lemma 4.3.3). In that case \( \mathcal{B}_I(R) \) will be called an implicit linear behaviour.

As in the static case, a natural approach to study minimality issues in inequality behaviours is to allow equalities in the representation.
**Definition 4.5.6** Let $\Sigma = (\mathbb{Z}, \mathcal{W}, \mathcal{B})$. Let $R_1 \in \mathbb{R}^{g \times q}[s, s^{-1}]$, $R_2 \in \mathbb{R}^{g \times q}[s, s^{-1}]$, $R'_1 \in \mathbb{R}^{g' \times q}[s, s^{-1}]$, and $R'_2 \in \mathbb{R}^{g' \times q}[s, s^{-1}]$. The systems of difference equations and inequalities

$$R_1(\sigma, \sigma^{-1})w = 0 \land R_2(\sigma, \sigma^{-1})w \geq 0$$

and

$$R'_1(\sigma, \sigma^{-1})w = 0 \land R'_2(\sigma, \sigma^{-1})w \geq 0$$

are said to be equivalent if $\mathcal{B}_E(R_1, R_2) = \mathcal{B}_E(R'_1, R'_2)$. The system of difference equations and inequalities \(R_1(\sigma, \sigma^{-1})w = 0 \land R_2(\sigma, \sigma^{-1})w \geq 0\) is said to be a minimal equality/inequality (or, for short, minimal) if: \(\mathcal{B}_E(R_1, R_2) = \mathcal{B}_E(R'_1, R'_2)\) $\Rightarrow \{g + r \leq g' + r'\}$. \(<$

It is immediate that if $R_1(\sigma, \sigma^{-1})w = 0 \land R_2(\sigma, \sigma^{-1})w \geq 0$ is a minimal system of equations then $R_1(s, s^{-1})$ has full row-rank.

We are interested in the relation between two polynomial matrices $R(s, s^{-1})$ and $R'(s, s^{-1})$ when they satisfy

$$R(\sigma, \sigma^{-1})w \geq 0 \Rightarrow R'(\sigma, \sigma^{-1})w \geq 0.$$

(4.29)

Based on the static case, one may expect that such a relation should be the extension of Farkas' theorem to the behavioural case. The following result is useful. The proof is straightforward and hence omitted.

**Lemma 4.5.7** Let $R \in \mathbb{R}^{g \times q}[s, s^{-1}]$ and $R' \in \mathbb{R}^{g' \times q}[s, s^{-1}]$. If $\{R(\sigma, \sigma^{-1})w \geq 0 \Rightarrow R'(\sigma, \sigma^{-1})w \geq 0\}$ then also $\{R(\sigma, \sigma^{-1})w = 0 \Rightarrow R'(\sigma, \sigma^{-1})w = 0\}$. \(<$

We introduce the notion of positive polynomials.

**Definition 4.5.8** A polynomial matrix $H \in \mathbb{R}^{n \times m}[s, s^{-1}]$ is said to be a positive polynomial matrix if $H(s, s^{-1}) = H_s s^L + \ldots + H_t s^t$ for constant matrices $H_k \in \mathbb{R}^{n \times m}$, $l \leq k \leq L$. The set of all positive polynomial matrices is denoted by $\mathbb{R}_+^{n \times m}[s, s^{-1}]$. \(<$

The following result is not difficult to prove, we will need it in the remainder.

**Lemma 4.5.9** Let $H \in \mathbb{R}^{g \times q}[s, s^{-1}]$. Then $H$ is a positive polynomial matrix if and only if $w \in (\mathbb{R}^g)^Z$, $w(t) \geq 0$, $\forall t \in \mathbb{Z}$, implies $H(\sigma, \sigma^{-1})w \geq 0$. \(<$

**Definition 4.5.10** Let $H \in \mathbb{R}_+^{g \times q}[s, s^{-1}]$. Then $H(s, s^{-1})$ is said to be a posimodular matrix if $H(s, s^{-1})$ is unimodular and $H^{-1} \in \mathbb{R}_+^{g \times q}[s, s^{-1}]$. \(<$

Posimodular matrices are important because for any matrix $R \in \mathbb{R}^{g \times q}[s, s^{-1}]$ one has that $\mathcal{B}_I(R) = \mathcal{B}_I(HR)$ whenever $H(s, s^{-1})$ is a posimodular matrix. The following result is the extension of lemma 3.4.4 to the polynomial case.

**Proposition 4.5.11** Let $H \in \mathbb{R}_+^{g \times q}[s, s^{-1}]$. Then $H(s, s^{-1})$ is posimodular if and only if there exist a permutation matrix $\Pi \in \mathbb{R}^{g \times g}$ and a polynomial matrix $\Delta \in \mathbb{R}_+^{g \times q}[s, s^{-1}]$ with \[\Delta(s, s^{-1}) = \text{diag}(a_i s^{d_i}), \ a_i > 0, \ d_i \in \mathbb{Z}, \ i \in q, \ \text{such that} \ H(s, s^{-1}) = \Delta(s, s^{-1}) \Pi.\] \(<$
It is easy to see that for the case holds. Posimodular matrices in \( \mathbb{R}^{d \times g}[s] \) are the nonnegative constant matrices, independent of the indeterminate \( s \), whose inverse is again nonnegative.

We are now in a position to give an extension of Farkas’ theorem from the static case to the behavioural case for full row-rank polynomial matrices.

**Proposition 4.5.12** Let \( R \in \mathbb{R}^{d \times q}[s, s^{-1}] \) be a full row-rank polynomial matrix. Let \( R' \in \mathbb{R}^{d \times q}[s, s^{-1}] \). Then: \( \{ R(\sigma, \sigma^{-1})w \geq 0 \Rightarrow R'(\sigma, \sigma^{-1})w \geq 0 \} \) if and only if there exists a unique polynomial matrix \( H \in \mathbb{R}^{d'}_{+} \times g[s, s^{-1}] \) such that \( R'(s, s^{-1}) = H(s, s^{-1})R(s, s^{-1}) \). <

In order to extend proposition 4.5.12 to the general case, i.e. without the assumption that the polynomial matrix \( R(s, s^{-1}) \) has full row-rank, there remain some difficulties. One could try to extend the original proof given by Farkas in [59]. However, this proof explicitly uses the fact that every scalar that is unequal to zero is invertible. Such a general statement does not hold for elements of \( \mathbb{R}^{d \times q}[s, s^{-1}] \). The most promising approach for the dynamic case seems to be the usage of mathematical tools such as the separation theorem of Hahn-Banach (see for instance [127]). The basic mathematical preliminaries read as follows. Denote \( \mathbb{E} := (\mathbb{R}^{d})_{\mathbb{Z}} \) with the topology of point-wise convergence. The dual of \( \mathbb{E} \), denoted by \( \mathbb{E}' \), consists of all \( \mathbb{R}^{d} \)-valued sequences that have compact support. Let \( R \in \mathbb{R}^{d \times q}[s, s^{-1}] \). Let \( \mathcal{B} = \mathcal{B}_{1}(R) \). The polar cone of \( \mathcal{B} \), denoted by \( \mathcal{B}_{\mathbb{Z}} \), is given by \( \{ w^{*} \in \mathbb{E}' : \forall w \in \mathcal{B} : \sum_{t \in \mathbb{Z}} w^{*}(t)w(t) \geq 0 \} \). We would like to establish that \( \mathcal{B}_{\mathbb{Z}} = \{ w^{*} \in \mathbb{E}' : \exists \alpha \in \mathbb{E}', \alpha \geq 0 \text{ such that } w^{*} = R^{T}(\sigma^{-1}, \sigma)\alpha \} \), but we have so far not been able to prove or disprove these statements. The statements, together with the fact that \( \{ \mathcal{B}_{1} \subseteq \mathcal{B}_{2} \} \) implies \( \{ \mathcal{B}_{2} \subseteq \mathcal{B}_{1} \} \) are believed to be useful in a proof of the following conjecture.

**Conjecture 4.5.13** Let \( R \in \mathbb{R}^{d \times q}[s, s^{-1}] \) and \( R' \in \mathbb{R}^{d \times q}[s, s^{-1}] \). Then: \( \{ R(\sigma, \sigma^{-1})w \geq 0 \Rightarrow R'(\sigma, \sigma^{-1})w \geq 0 \} \) if and only if there exists a polynomial matrix \( H \in \mathbb{R}^{d'}_{+} \times g[s, s^{-1}] \) such that \( R'(s, s^{-1}) = H(s, s^{-1})R(s, s^{-1}) \). <

It is easy to show that if there are no implicit equalities in a system \( \{ R_{1}(\sigma, \sigma^{-1})w = 0 \wedge R_{2}(\sigma, \sigma^{-1})w \geq 0 \} \) then \( \exists w' \in \mathcal{B}_{EH}(R_{1}, R_{2}) \) and \( \exists w' \in \mathcal{B} \) such that \( R_{2}(\sigma, \sigma^{-1})w'(t') > 0 \). If we follow the line of proof in proposition 3.3.12 then we arrive at \( (H(\sigma, \sigma^{-1})R_{2}(\sigma, \sigma^{-1})w')(t') = 0 \). If \( H \in \mathbb{R}^{d'}_{+} \times g[s, s^{-1}] \) it is concluded that \( H(s, s^{-1}) = 0 \). However, if it is not known that \( H \in \mathbb{R}^{d'}_{+} \times g[s, s^{-1}] \) such a conclusion is not possible. This discussion leads to the following conjecture.

**Conjecture 4.5.14** Let \( R_{1} \in \mathbb{R}^{d \times q}[s, s^{-1}] \) and \( R_{2} \in \mathbb{R}^{r \times q}[s, s^{-1}] \). Then the system of difference equations \( \{ R_{1}(\sigma, \sigma^{-1})w = 0 \wedge R_{2}(\sigma, \sigma^{-1})w \geq 0 \} \) is minimal if and only if there are no redundant equations, nor implicit equalities. Moreover, in a minimal representation both the number of equalities and the number of inequalities are unique. <
Of interest are operations that preserve minimality. It is easy to show that addition or subtraction of rows of a matrix \(R(s, s^{-1})\) does not necessarily leave the behaviour \(\mathcal{B}_f(R)\) invariant. To see this consider \(R = \text{col}(s^2 + s + 1, s + 1)\) and \(R' = \text{col}(s^2, s + 1)\). The matrix \(R(s, s^{-1})\) can be obtained from \(R'(s, s^{-1})\) by adding the second row to the first row. The signal \(w = \{\ldots, 1, -\frac{1}{2}, 1, -\frac{1}{2}, 1, \ldots\}\) belongs to \(\mathcal{B}_f(R)\) but not to \(\mathcal{B}_f(R')\). A similar example can be given for continuous time signals: the differential operator does not preserve sign. Clearly the positynomial matrices (see proposition 4.5.11) are only a subset of the class of unimodular matrices. It is remarked that representations that are not minimal may be related by unimodular matrices that are not positynomial. Consider \(R := \text{col}(s + 1, -s - 1)\). Then \(\mathcal{B}_f(R) = \mathcal{B}_f(R')\) with \(R' := \text{col}(-s - 1, s + 1)\), and \(R'(s, s^{-1}) = H(s, s^{-1})R(s, s^{-1})\) where \(H = \text{col}([-1 \, 0], [0 \, -1])\). Matrix \(H(s, s^{-1})\) is not positynomial. Note that also \(R'(s, s^{-1}) = H'(s, s^{-1})R(s, s^{-1})\) with \(H' = \text{col}([0 \, 1], [1 \, 0])\). Matrix \(H'(s, s^{-1})\) is positynomial, thus supporting conjecture 4.5.12.

**Proposition 4.5.15** Let \(R_1 \in \mathbb{R}^{q \times q}[s, s^{-1}], R_2 \in \mathbb{R}^{r \times q}[s, s^{-1}]\) and \(\mathcal{B} = \mathcal{B}_{EI}(R_1, R_2)\). Assume that the system of difference equations \(\{R_1(\sigma, \sigma^{-1})w = 0 \wedge R_2(\sigma, \sigma^{-1})w \geq 0\}\) is minimal. If

\[
\begin{bmatrix}
R_1'(s, s^{-1}) \\
R_2'(s, s^{-1})
\end{bmatrix}
= \begin{bmatrix}
U(s, s^{-1}) & 0 \\
S(s, s^{-1}) & H(s, s^{-1})
\end{bmatrix}
\begin{bmatrix}
R_1(s, s^{-1}) \\
R_2(s, s^{-1})
\end{bmatrix},
\]

with \(U \in \mathbb{R}^{q \times q}[s, s^{-1}]\) unimodular, \(H \in \mathbb{R}^{r \times r}[s, s^{-1}]\) positynomial and \(S \in \mathbb{R}^{r \times q}[s, s^{-1}]\), then \(\mathcal{B} = \mathcal{B}_{EI}(R_1', R_2')\) and the system of difference equations \(\{R_1'(\sigma, \sigma^{-1})w = 0 \wedge R_2'(\sigma, \sigma^{-1})w \geq 0\}\) is also minimal. \(<\)

In the linear case, the Smith form is an important mathematical tool, amongst other, in a discussion on minimality issues. Let the Smith form of \(R(s, s^{-1})\) be given by \(R' = \text{diag}(\Delta, 0)\), and let \(R(s, s^{-1}) = U(s, s^{-1})R'(s, s^{-1})V(s, s^{-1})\), with \(U(s, s^{-1}), V(s, s^{-1})\) unimodular matrices of appropriate dimensions. Then we would like to conclude that \(R(\sigma, \sigma^{-1})w \geq 0\) if and only if \((R'V)(\sigma, \sigma^{-1})w \geq 0\). This holds if \(U(s, s^{-1})\) is a positynomial matrix, which is not true in general. Although one can not reduce representations issues to the single equation case by direct application of the Smith form, it may still provide good inroads to representation issues of difference inequalities. This is left as part of future research.

We end this section with two examples. The first example shows that minimal representations of difference inequalities differ from minimal representations of differential inequalities.

**Example 4.5.16** Consider the system with behavioural equations:

\[
\begin{align*}
\sigma w_1(t) &= w_2(t), \\
bw_1(t) &\geq 0, \\
w_2(t) &\geq 0.
\end{align*}
\]
Chapter 4: Unilateral dynamical systems

If $T = \mathbb{R}$, $\sigma$ denotes the differential operator, and if $T = \mathbb{Z}$, $\sigma$ denotes the shift operator. Obviously the equality can not be deleted from the representation. Now for $T = \mathbb{R}$ the differential operator does not preserve sign. It follows that both inequalities are also necessary in the representation. However, for $T = \mathbb{Z}$ a minimal representation of the system is given by:

$$
\begin{align*}
\sigma w_1(t) &= w_2(t), \\
w_2(t) &\geq 0,
\end{align*}
$$

(4.32)

since $w_1(t) \geq 0$ is already implied by the other equations in (4.31). It is not difficult to see that in this example the case $T = \mathbb{Z}_+$ is similar to the case $T = \mathbb{R}$, at least with respect to minimality issues.

From example 4.5.16 it can be seen that continuous-time dynamical systems have characteristics that are different from the characteristics of discrete-time dynamical systems over $\mathbb{Z}$.

The following example illustrates that the construction of trajectories of a system of difference inequalities poses the same problems as in the linear case (see lemma 2.4.5).

**Example 4.5.17** Consider the system $\mathcal{B} = \mathcal{B}_I(R)$ with $R = \text{col}(s - 1, -s + \frac{1}{2})$. Let $w(t) = -1$ for $t \leq -1$ and $w(0) = \frac{1}{2}$. Then for $t \leq -1$ we have $w(t + 1) - w(t) = 0$ and $-w(t + 1) - \frac{1}{2}w(t) = 1 + \frac{1}{2} \geq 0$. Furthermore: $w(0) - w(-1) = \frac{1}{2} + 1 \geq 0$ and $-w(0) - \frac{1}{2}w(-1) = -\frac{1}{2} + \frac{1}{2} = 0$. It follows that for $t \leq -1$, $(R(\sigma, \sigma^{-1})w)(t) \geq 0$. However, for $t = 0$ the following equations need to be satisfied: $w(1) \geq w(0) = \frac{1}{2}$ and $-w(1) \geq \frac{1}{2}w(0) = \frac{1}{4}$. Clearly such $w(1)$ does not exist.

At present it is not known whether or not for behavioural inequalities there holds a result similar to the one in lemma 2.4.5.

### 4.6 Constrained linear systems

A system $\Sigma = (\mathcal{T}, \mathcal{W}, \mathcal{B})$ with $\mathcal{B} = \mathcal{B}_{EI}(R_1, R_2)$ is a special case of an interconnection of two systems. Define $\Sigma_1 = (\mathcal{T}, \mathcal{W}, \mathcal{B}_1)$ with $\mathcal{B}_1 = \mathcal{B}_{EI}(R_1)$ and define $\Sigma_2 = (\mathcal{T}, \mathcal{W}, \mathcal{B}_2)$ with $\mathcal{B}_2 = \mathcal{B}_{I}(R_2)$. Then $\Sigma = \Sigma_1 \wedge \Sigma_2$, and $\mathcal{B} = \mathcal{B}_1 \cap \mathcal{B}_2$. Figure 4.1 in example 4.4.6 depicts such an interconnected system. In this example, if $y_2 > y_1$ the carts do not have an influence on each other. In mechanical systems this is referred to as the ’free motion phase’, where free is to be understood with respect to the constraint. If, on the other hand, $y_2 = y_1$, the carts have made contact. This is commonly referred to as the ’constrained motion phase’, of which example 2.3.2 gives an example. One of the first questions that springs to mind is whether or not control can be used to make all transitions between motion phases in a smooth manner.
The discussion in this section serves as a motivation for our approach to inequality systems in the remainder of this thesis. In particular, it serves as an introduction to the problems we will discuss in more detail in chapter 5 for first-order representations. To gain a better understanding of interconnections of systems described by dynamic inequalities we will be concerned with behaviours that can be represented by dynamic equalities, and where additional (in)equalities model restrictions on the behaviour of the system (see the examples in chapter 1). We will call such systems \textit{unilaterally constrained linear dynamical systems}. We will now formally define this class of systems.

\textbf{Definition 4.6.1} Let \( \Sigma = (\mathbb{Z}, \mathbb{W}, \mathcal{B}) \) with \( \mathcal{B} = \mathcal{B}_E(R) \cap \mathcal{B}_f(R') \), \( R \in \mathbb{R}^{q \times q}[s, s^{-1}] \) and \( R' \in \mathbb{R}^{r \times r}[s, s^{-1}] \). Then \( \Sigma \) is said to be a \textit{unilaterally constrained linear dynamical system}, or for short, a \textit{constrained linear system}, if for all \( i \in \mathcal{Q} \) for which there exist a \( j \in \mathcal{Q} \) such that \( R_{ij}(s, s^{-1}) \neq 0 \) there also exists a \( k \in \mathcal{Q} \) such that \( R_{ki}(s, s^{-1}) \neq 0 \). If in addition \( \mathcal{B} = \mathcal{B}_E(R) \cap \mathcal{B}_E(R') \), then \( \Sigma \) is said to be a \textit{bilaterally constrained linear dynamical system}.

When restricting the attention to constrained linear systems, systems that can be represented by, for instance \( \{ \sigma w_1 + w_1 = 0, w_1 + w_2 \geq 0 \} \) are excluded from the analysis. It is easy to see that in this particular example \( w_1 \) is not restricted by the inequality. If we interpret \( w_2 \) as a latent variable then proposition 4.4.2 leads to the same conclusion.

Constrained linear systems are an important class of systems: in applications, restrictions are often imposed on (models of) unconstrained systems, leading directly to the class of systems in definition 4.6.1. For constrained linear systems there arises a practical way to deal with input/output inequality representations and with first-order representations. For linear systems there always exists a decomposition of the manifest variables in input and output variables ([143], see also (2.13)). This leads to the following result.

\textbf{Theorem 4.6.2} Let \( \Sigma = (\mathbb{Z}, \mathbb{W}, \mathcal{B}) \) be a constrained linear system with \( \mathcal{B} = \mathcal{B}_E(R_1) \cap \mathcal{B}_f(R_2) \), \( R_1 \in \mathbb{R}^{q \times q}[s, s^{-1}] \) and \( R_2 \in \mathbb{R}^{r \times r}[s, s^{-1}] \). Then there exist a partition of \( \mathbb{R}^q \cong \mathbb{R}^p \times \mathbb{R}^m \) and matrices \( P \in \mathbb{R}^{p \times p}[s, s^{-1}] \) with \( \det(P) \neq 0 \), \( Q \in \mathbb{R}^{p \times m}[s, s^{-1}] \), where \( P^{-1}Q \) is proper, and matrices \( T \in \mathbb{R}^{r \times p}[s, s^{-1}] \) and \( S \in \mathbb{R}^{r \times m}[s, s^{-1}] \) such that \( \Sigma = (\mathbb{Z}, \mathbb{V} \times \mathbb{U}, \mathcal{B}_{i/o}) \) with

\[ \mathcal{B}_{i/o} = \{(y, u) \in (\mathbb{R}^p)^{\mathbb{Z}} \times (\mathbb{R}^m)^{\mathbb{Z}} \mid P(\sigma, \sigma^{-1})y = Q(\sigma, \sigma^{-1})u \text{ and } T(\sigma, \sigma^{-1})y + S(\sigma, \sigma^{-1})u \geq 0 \}, \]

where \( \Sigma = (\mathbb{Z}, \mathbb{V} \times \mathbb{U}, \mathcal{B}) \) is an input/output dynamical system, with \( \mathcal{B} = \{(y, u) \in (\mathbb{R}^p)^{\mathbb{Z}} \times (\mathbb{R}^m)^{\mathbb{Z}} \mid P(\sigma, \sigma^{-1})y = Q(\sigma, \sigma^{-1})u \} \). Moreover, if \( P \) is unimodular then there exists a matrix \( S' \in \mathbb{R}^{r \times m}[s, s^{-1}] \) such that

\[ \mathcal{B}_{i/o} = \{(y, u) \in (\mathbb{R}^p)^{\mathbb{Z}} \times (\mathbb{R}^m)^{\mathbb{Z}} \mid P^{-1}(\sigma, \sigma^{-1})y = S'(\sigma, \sigma^{-1})u \text{ and } S'(\sigma, \sigma^{-1})u \geq 0 \}. \]
For the linear subsystem \( P(\sigma, \sigma^{-1})y = Q(\sigma, \sigma^{-1})u \), the decomposition of \( w \) is made such that \( u \) can be interpreted as an input that is available for control purposes. This immediately raises the question whether or not we can make a further decomposition of the input \( u \) into a part that is free, and a part that is not free for a unilaterally constrained system. For bilaterally constrained linear systems this question has been investigated in [131] in a first-order setting. We postpone a further discussion on this subject till section 7.6 and chapter 8.

From a representation \( R(\sigma)w \geq 0 \) one can always obtain a first-order representation that still contains the manifest variables \( w \). Let \( R(s) = \sum_{k=0}^{L} R_k s^k \). One possible choice is based on the linear case in (2.14).

\[
E\sigma x + F x + Gw = 0, \quad Qx \geq 0, \quad (4.33)
\]

with \( x := \text{col}[w, \sigma w, \ldots, \sigma^L w] \), and where matrices \( E, F, G \) and \( Q \) satisfy

\[
E = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
I & 0 & \ldots & 0 & 0 \\
0 & I & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & I & 0
\end{bmatrix},
F = - \begin{bmatrix}
I & 0 & \ldots & 0 \\
0 & I & \ldots & 0 \\
0 & 0 & I & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & I & 0
\end{bmatrix},
G = \begin{bmatrix}
I \\
0 \\
0 \\
0 \\
\vdots
\end{bmatrix},
Q = [R_0 \ R_1 \ \ldots \ R_L].
\]

A class of systems that is often taken as a starting point for controller design and simulation is the classical input/state/output representation. When restrictions are added, one arrives at an input/state/output representation with constraints on the state and input:

\[
\begin{align*}
\sigma x &= Ax + Bu, \\
y &= Cx + Du, \\
0 &\leq Sx + Tu.
\end{align*} \quad (4.35)
\]

There are several important issues that still need research. For instance, an important question is whether or not every constrained driving variable representation can be represented also as an input/state constrained input/state/output representation, and vice versa. An other representation that may be useful in resolving modelling issues is a so called driving variable representation [89]. Adding constraints leads to

\[
\begin{align*}
\sigma x &= A'x + B'v, \\
w &= C'x + D'v, \\
0 &\leq Qx.
\end{align*} \quad (4.36)
\]

[131]
The latter could then be called a constrained driving variable representation. Establishing relations between all these representations is left as part of future research.

In the linear case, an input/state/output representation of a system in (2.16) is a representation that is first-order in the state-space variable, denoted by $x$, zero-th order in the input variable $u$ and the output variable $y$, and that satisfies the axiom of state of definition 2.4.6. From a general input/output description of a unilaterally constrained dynamical system one would now like to proceed directly to a constrained input/state/output representation, i.e. a representation where the equalities are first-order in the state-space variable, denoted by $x$, and zero-th order in the input variable $u$ and the output variable $y$. However, direct application of the results from the linear case can lead to representations where the control variable does not appear in zero-th order form, or where the state does not appear in a first-order form.

**Example 4.6.3** Consider the discrete-time scalar system

\[
\begin{align*}
y(t) &= u(t-1), \\
y(t) &\geq u(t-4).
\end{align*}
\]

Equivalent representations are given by

\[
\begin{align*}
x(t+1) &= u(t), \\
y(t) &= x(t), \\
0 &\geq u(t-4) - u(t-1),
\end{align*}
\]

and

\[
\begin{align*}
x(t+1) &= u(t), \\
y(t) &= x(t), \\
0 &\geq x(t-3) - x(t).
\end{align*}
\]

Direct application of realization theories from the linear case can also lead to systems that do not satisfy the axiom of state.

**Example 4.6.4** Consider the discrete-time system $\Sigma$ represented by

\[
\begin{align*}
y(t+1) &= u(t), \\
y(t) + y(t-3) &\geq 0.
\end{align*}
\]
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An input/state/output representation of the linear equation \( y(t + 1) = u(t) \) is given by

\[
x(t + 1) = u(t), \quad y(t) = x(t).
\]

This leads to the full behaviour \( B : x(t + 1) = u(t), y(t) = x(t), x(t) + x(t - 3) \geq 0 \). (The latent variable \( x \) can be eliminated again to obtain the manifest behaviour.) Now consider two trajectories of the full behaviour, denoted by \((u_1, x_1, y_1)\) and \((u_2, x_2, y_2)\). Choose

\[
u_1(t) = \begin{cases} 0 & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}
\]

This gives \( x_1(t) = 0 \) if \( t \neq 1 \) and \( x_1(1) = 1 \). Obviously this satisfies \( x_1(t) + x_1(t - 3) \geq 0 \), for all \( t \in \mathbb{Z} \). Next, choose

\[
u_2(t) = \begin{cases} -1 & \text{if } t \text{ odd}, \\ 1 & \text{if } t \text{ even or } t = 0. \end{cases}
\]

This gives \( x_2(t) = -1 \) if \( t \) is even (or \( t = 0 \)), and \( x_2(t) = 1 \) if \( t \) is odd. It follows that \( x_2(t) + x_2(t - 3) \geq 0 \), for all \( t \in \mathbb{Z} \). Moreover, \( x_2(1) = 1 = x_1(1) \). Now consider the trajectory \((u', x', y')\) with

\[
(u'(t), x'(t), y'(t)) = \begin{cases} (u_1(t), x_1(t), y_1(t)) & \text{if } t < 1, \\ (u_2(t), x_2(t), y_2(t)) & \text{if } t \geq 1. \end{cases}
\]

Then \( x'(2) + x'(-1) = x_2(2) + x_1(-1) = -1 + 0 < 0 \). It follows that the axiom of state is not satisfied.

\[<\]

It is tempting to call \( x \) in example 4.6.4 a state. But in that case we have a state that does not contain sufficient information about the past in order to determine the future behaviour! Further research on this subject is necessary. Representations (4.34) and (4.36) are good starting points for such a study.

In case of static constraints on the inputs and outputs the situation becomes much clearer. We focus our attention on subclasses of systems that are still large enough to cover the systems in the examples in chapter 1 (in a linear setting).

Consider again the system

\[
P(\sigma, \sigma^{-1})y = Q(\sigma, \sigma^{-1})u, \\
0 \leq Ty + Su,
\]

with \( P \in \mathbb{R}^{p \times p}[s,s^{-1}] \), \( \det(P) \neq 0 \), \( Q \in \mathbb{R}^{p \times m}[s,s^{-1}] \), and \( P^{-1}Q \) is proper. Standard
realization theory for linear systems can be used to obtain

\[ \sigma x = Ax + Bu \]
\[ y = Cx + Du, \]
\[ 0 \leq Ty + Su. \]  \hspace{2cm} (4.38)

It can be seen that in this case the axiom of state does hold. Moreover, using the output equation, the inequality can be put on the state and input variables. This will be our starting point in the next chapter.

4.7 Conclusions

In this chapter the beginning of a theory on linear dynamical systems described by behavioural difference inequalities has been presented. The notions of convexity and conicity have been introduced in a behavioural setting. It has been shown that so called shifted-polyhedral conical systems can be described by a difference inequality. It is still an open problem which properties of a system make it shifted-polyhedral conical. We have also discussed the elimination problem for representations that involve latent variables. We have derive sufficient conditions under which latent variables can be eliminated from a representation. It remains to be shown that there exists an elimination theorem for behavioural inequalities.

A key problem in efficient representation of inequality systems is to establish necessary and sufficient conditions for a system of equalities and inequalities to contain the minimal number of equations. The extension of Farkas’ theorem, which was particularly useful in the static case, could only be proven for the special class of full row-rank matrices. For this, the class of posimodular matrices was introduced. Posimodular matrices turned out to be useful also in obtaining equivalent representations of inequality systems that are not necessarily minimal.

We formally defined the class of unilaterally constrained dynamical systems as a special class of inequality systems. Some problems with respect to finding constrained input/state/output systems have been identified. In particular it was shown that direct application of realization theory for linear systems to inequality systems may result in representations that do not satisfy the axiom of state. More work needs to be done on this aspect of modelling inequality systems, especially on use of results and ideas from the field of linear systems theory.

Throughout this chapter we have shown that inequality systems have characteristics that are very different from characteristics of linear systems. Clearly, there remains a lot of work to be done on modelling unilateral systems, and a number of open research problems have been identified. In the remainder of this thesis we will be concerned with constrained in-
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put/state/output representations to gain insight in some of the system theoretical problems identified in this chapter.

Appendix 4.A: Proofs

Proof of proposition 4.2.2:
The proof resembles the proof of the static case in [125]. (⇒): Let \( w_1, w_2 \in \mathcal{B} \). From \( \Sigma \) is convex it follows that \( w := \frac{1}{2}(w_1 + w_2) \in \mathcal{B} \). As \( \mathcal{B} \) is a cone it now follows that \( w_1 + w_2 = 2w \in \mathcal{B} \). Hence \( \mathcal{B} \) is closed under addition. (⇐): By assumption one has that if \( w_1, w_2 \in \mathcal{B}, \ 0 < \alpha < 1 \), then \((1 - \alpha)w_1 + \alpha w_2 \in \mathcal{B}\). From closed under addition it follows that \((1 - \alpha)w_1 + \alpha w_2 \in \mathcal{B}\). Hence \( \Sigma \) is convex. \(<\)

Proof of proposition 4.2.3:
First it is proven that \( \mathcal{B} \cap (-\mathcal{B}) \) and \( \mathcal{B} - \mathcal{B} \) are linear behaviours. Let \( \Sigma \) be convex conical. Then \( \mathcal{B} \) is a convex cone. By proposition 4.2.2 \( \mathcal{B} \) is closed under addition and nonnegative scalar multiplication. It follows that \( \mathcal{B} - \mathcal{B} \) and \( \mathcal{B} \cap (-\mathcal{B}) \) are closed under addition and nonnegative scalar multiplication. Hence \( \mathcal{B} - \mathcal{B} \) and \( \mathcal{B} \cap (-\mathcal{B}) \) will be linear behaviours if they also contain \( w = 0 \) and are closed under multiplication by \(-1\). It is easily verified that this holds. Next it is proven that \( \mathcal{B} - \mathcal{B} \) is the smallest linear behaviour containing \( \mathcal{B} \). Let \( \mathcal{L} \) be a subspace such that \( \mathcal{B} \subseteq \mathcal{L} \subseteq (\mathcal{B} - \mathcal{B}) \), and let \( w \in \mathcal{B} - \mathcal{B} \). Then there are \( w_1, w_2 \in \mathcal{B} \) such that \( w = w_1 - w_2 \). Since \( \mathcal{B} \subseteq \mathcal{L} \) it follows that \( w_1, w_2 \in \mathcal{L} \) and by linearity \( w_1 - w_2 \in \mathcal{L} \). Hence \( w \in \mathcal{L} \). So \( \mathcal{B} - \mathcal{B} \subseteq \mathcal{L} \). We have obtained that \( \mathcal{B} - \mathcal{B} = \mathcal{L} \). Finally, it is shown that \( \mathcal{B} \cap (-\mathcal{B}) \) is the largest linear behaviour contained in \( \mathcal{B} \). Let \( \mathcal{L} \) be a linear subspace such that \( (\mathcal{B} \cap (-\mathcal{B})) \subseteq \mathcal{L} \subseteq \mathcal{B} \). Let \( w \in \mathcal{L} \). Since \( \mathcal{L} \) is a linear behaviour \(-w \in \mathcal{L} \). From \( \mathcal{L} \subseteq \mathcal{B} \) we have \( w \in \mathcal{B} \) and \(-w \in \mathcal{B} \). Hence \( w \) is a linear behaviour contained in \( \mathcal{B} \). It follows that \( \mathcal{L} = \mathcal{B} \cap (-\mathcal{B}) \). \(<\)

Proof of proposition 4.2.5:
The proof is adapted (with a small change in the 'if' proof) from the linear case [143]. (If): Let \( \Sigma \) be complete. It must be shown that \( \mathcal{B} \) is closed. Assume \( w_n \in \mathcal{B}, \ n \in \mathbb{N}, \) and \( \lim_{n \to \infty} w_n = w \) (with convergence in the topology of point-wise convergence). Now \( \{w_n \}_{n=1}^{\infty} \Rightarrow \{w_n\}_{[0,t]}^{\infty} \Rightarrow \{w\}_{[0,t]} \). Observe that \( \mathcal{B}_{[0,t]} \) is polyhedral, hence closed and finite-dimensional. Consequently, since \( w_n\}_{[0,t]}^{\infty} \in \mathcal{B}_{[0,t]} \) we obtain \( w\}_{[0,t]}^{\infty} \in \mathcal{B}_{[0,t]} \) for all \(-\infty < t_0 \leq t_1 < \infty\). By completeness, this implies \( w \in \mathcal{B} \), as desired. (Only if): Assume that \( \mathcal{B} \) is closed. Let \( w : \mathbb{Z} \rightarrow \mathbb{R}_{+}^{d} \) be such that \( w\}_{[0,t]}^{\infty} \in \mathcal{B}_{[0,t]} \) for all \(-\infty < t_0 \leq t_1 < \infty\). It will be shown that \( w \in \mathcal{B} \). By assumption one has \( w\}_{[-n,n]}^{\infty} \in \mathcal{B}_{[-n,n]} \). There exists \( w_n \in \mathcal{B}, \ n \in \mathbb{N} \) such that \( w_n\}_{[-n,n]}^{\infty} \rightarrow w\}_{[-n,n]} \). Obviously \( \{w_n \}_{n=1}^{\infty} \Rightarrow w \). Since \( \mathcal{B} \) is closed, this implies \( w \in \mathcal{B} \), as desired. Hence \( \Sigma \) is complete. \(<\)

Proof of lemma 4.3.3:
The statement follows immediately from \( w, -w \in \mathcal{B}_{\Sigma} \). \(<\)
Proof of theorem 4.3.6:
To show sufficiency let \( \mathfrak{B} \) be a shifted polyhedral cone. Then there exists a polyhedral cone in \((\mathbb{R}^q)^\mathbb{Z}\) such that \( \mathfrak{B} = \cap_{i \in \mathbb{Z}^d} \mathfrak{P}_i \) with \( \mathfrak{P}_i = \cap_{i \in \mathbb{Z}^d} \mathfrak{G}_i \). Let \( \mathfrak{G}_i = \{ w \in (\mathbb{R}^q)^\mathbb{Z} \mid \sum_{t \in \mathbb{Z}^d} \alpha_i(t) w(t) \geq 0 \} \), with \( \alpha_i \) compact support in \([t_i, t_i']\). Define \( t_i := \min t_i \) and \( t_i' = \max t_i', i \in q \). Define \( \Delta := t_i' - t_i \). Define \( \alpha_i^T := \alpha_i |_{[t_i, t_i']} \) From shift-invariance of \( \mathfrak{B} \), and \( \alpha_i \) fixed it follows that \( \{ w \in \mathfrak{B} \} \leftrightarrow \{ \alpha_i^T \text{col}(w(t), \ldots, w(t + \Delta)) \geq 0, \forall i \in q, \forall t \in \mathbb{Z} \} \). Take \( R \in \mathbb{R}^{q \times q} [s, s^{-1}] \) such that \( (R(\sigma, \sigma^{-1})w)(t) = \alpha_i \text{col}(w(t), \ldots, w(t + \Delta)), \forall t \in \mathbb{Z} \). This yields: \( w \in \mathfrak{B} \Leftrightarrow R(\sigma, \sigma^{-1})w \geq 0 \). Reversing the steps in the proof above gives necessity.

Proof of proposition 4.4.2:
First observe that if \( a_i = 0 \) the equation \( R_i(\sigma, \sigma^{-1})w \geq 0 \) is obtained without a condition on \( \ell \). If in the remaining equations either all \( a_i > 0 \) or all \( a_i < 0 \), then we are done. So, consider the case where the \( a_i \)'s take both strictly positive and strictly negative values. For fixed \( k \in H_+ \) write the \( n_+ \) equations \( a_k R_k(\sigma, \sigma^{-1})w \leq a_k a_k \ell \). This gives \( n_+ \) equations \( a_j R_k(\sigma, \sigma^{-1})w \geq a_k R_k(\sigma, \sigma^{-1})w, j \in H_+ \). Repeating this for each \( k \in H_- \) gives the \( n_-n_+ \) equations in the statement. To show the reverse, take \( w \) such that \( R(\sigma, \sigma^{-1})w \geq 0 \) and \( a_k R_k(\sigma, \sigma^{-1})w \geq a_k R_k(\sigma, \sigma^{-1})w \) for all \( i \in H_0, j \in H_+ \) and \( k \in H_- \). Define \( \ell(t) \) such that: \( \max_{k \in H_+} \frac{1}{a_k}(R_k(\sigma, \sigma^{-1})w)(t) \leq \ell(t) \leq \min_{j \in H_+} \frac{1}{a_j}(R_j(\sigma, \sigma^{-1})w)(t) \). This gives that for all \( k \in H_-, (R_k(\sigma, \sigma^{-1})w)(t) \geq a_k \ell(t) \) and for all \( j \in H_+, (R_j(\sigma, \sigma^{-1})w)(t) \geq a_j \ell(t) \). Combined with the difference inequalities with \( R_i(\sigma, \sigma^{-1})w \geq 0 \) for \( i \in H_0 \) gives the latent variable description in \( \mathfrak{B}_\ell \).

Proof of proposition 4.4.3:
Let \( R_i(\sigma, \sigma^{-1})w \geq R_j(\sigma, \sigma^{-1})\ell \). Define \( \beta := \text{row}(w, \ell), M(s, s^{-1}) := [I : 0] \) and \( N(s, s^{-1}) := [R_i(s, s^{-1}) : -R_j(s, s^{-1})] \). Then \( w = M(\sigma, \sigma^{-1}) \) and \( N(\sigma, \sigma^{-1})\beta \geq 0 \).

Proof of lemma 4.5.2:
First observe that any equality behaviour \( \mathfrak{B}_E(R) \) can be written equivalently as an inequality behaviour \( \mathfrak{B}_I(\text{col}(R, -R)) \). From \( \mathfrak{B}_I(R) = \mathfrak{B}_E(R) \) follows that \( R(s, s^{-1}) \) does not have full row-rank. (If \( R(s, s^{-1}) \) has full row-rank, it is surjective [143].) In that case \( \exists w \) such that \( R(\sigma, \sigma^{-1})w > 0 \). This contradicts \( \mathfrak{B}_I(R) = \mathfrak{B}_E(R) \). Next observe that for arbitrary \( F \in \mathbb{R}^{q \times q} [s, s^{-1}] \) there holds \( \mathfrak{B}_I(F) = \mathfrak{B}_I(\text{col}(F, -F)) = \mathfrak{B}_I(\text{col}(F - \sum_k F_k)) \). It is now straightforward to prove that \( \mathfrak{B}_I(\text{col}(F - \sum_k F_k)) \) is a minimal inequality behaviour if and only if \( \mathfrak{B}_E(F) \) is a minimal equality behaviour. This implies that \( F(s, s^{-1}) \) has full row-rank. It follows that \( \mathfrak{B}_I(R) \) is minimal if and only if \( p = q - 1 \).

Proof of proposition 4.5.3:
The proof is by counterexamples using equality representations. Other examples can easily be coined, already in the static case, using the theory presented in chapter 3.

(i): Let \( R := \text{col}(s + 1, -s - 1) \). Obviously, \( R(s, s^{-1}) \) does not have full row-rank. Since \( \mathfrak{B}_I(R) = \mathfrak{B}_E(R) \), the conditions of lemma 4.5.2 are satisfied with \( g = 2 \). It follows that \( \mathfrak{B}_I(R) \) is minimal as an inequality representation.

(ii): Take arbitrary \( q \in \mathbb{N} \). Take a square matrix \( R \in \mathbb{R}^{q \times q} [s, s^{-1}] \) with \( \det(R) \neq 0 \). By lemma 4.5.2 the inequality behaviour \( \mathfrak{B}_I(\text{col}(R, -\sum_k R_k)) \) is minimal, but the number of
rows equals $q + 1$, which exceeds $q$.

**Proof of lemma 4.5.9:**
(Only if): This is immediate. (If): We have $H = H_k s^k + \ldots + H_0 s^0$ for some constant matrices $H^k \in \mathbb{R}^d \times g$, $\ell \leq k \leq L$. Take $w = (\ldots, 0, 0, w_1, 0, 0, \ldots)$ with $w_1 \geq 0$ but otherwise arbitrary. Then for $\ell \leq k \leq L$ there holds: $0 \leq H(\sigma, \sigma^{-1})(\sigma^k w) = H_k w_1$. Since $w_1$ is arbitrary it follows that necessarily $H_k \geq 0$. It follows that $H \in \mathbb{R}^d \times g[s, s^{-1}]$. <

**Proof of proposition 4.5.11:**
The proof follows the proof of lemma 3.4.4 for the static case. The 'if' part is immediate. To prove the 'only if' part let $H \in \mathbb{R}^g \times g[s, s^{-1}]$ be posimodular. First consider the case $g = 1$. It is easy to see that in that case $H(s, s^{-1}) = \alpha_d s$ for some scalar $\alpha > 0$ and $d \in \mathbb{Z}$. Now consider the general case. Denote $F(s, s^{-1}) := H^{-1}(s, s^{-1})$. Let $I_k$ denote the $k$th row of the identity polynomial in $\mathbb{R}^g \times g$. Take an arbitrary $i \in q$. Let $(HF)_{ij}(s, s^{-1})$ denote the $ij$th element of the matrix product $(HF)(s, s^{-1})$. Then $(HF)_{ij}(s, s^{-1}) = \sum_{k=1}^g H_{ik}(s, s^{-1}) F_{kj}(s, s^{-1})$. From $(HF)_{ii}(s, s^{-1}) = 1$ (and $H(s, s^{-1})$ and $F(s, s^{-1})$ posimodular) now follows that $\exists \ell \in g$ such that $H_{i\ell}(s, s^{-1}) \neq 0$ and $F_{i\ell}(s, s^{-1}) \neq 0$. Furthermore, let $\forall j \neq i$ there holds: $0 = (HF)_{ij}(s, s^{-1}) = \sum_{k=1}^g H_{ik}(s, s^{-1}) F_{kj}(s, s^{-1})$. This implies that $H_{i\ell}(s, s^{-1}) F_{j\ell}(s, s^{-1}) = 0$ for all $\ell \neq i$ and for all $j \neq i$. We have: $H_{i\ell}(s, s^{-1}) F_{j\ell}(s, s^{-1}) = 0$ for all $j \neq i$. Since $H_{i\ell}(s, s^{-1}) \neq 0$ then it now follows that $F_{j\ell}(s, s^{-1}) = 0$ for all $j \neq i$. This gives that for the $\ell$th row of $(F(s, s^{-1})$ there holds: $F_{\ell}(s, s^{-1}) = \ell(s, s^{-1}) I_\ell$ with $\ell \in \mathbb{R}_+ [s, s^{-1}]$. Now suppose that $\exists \ell' \neq \ell$ such that $(H)_{i\ell'}(s, s^{-1}) \neq 0$. Similar reasoning as above then gives that $F_{\ell'}(s, s^{-1}) = \ell'(s, s^{-1}) I_\ell$ with $\ell' \in \mathbb{R}_+ [s, s^{-1}]$. The latter now contradicts that $H^{-1}(s, s^{-1})$ is unimodular. It follows that $H_{i\ell}(s, s^{-1}) = \mu_{i\ell}(s, s^{-1}) I_\ell$, with $\mu_{i\ell} \in \mathbb{R}_+ [s, s^{-1}]$. We have obtained that $\forall i \in g$, $\exists \ell(i) \in g$ such that $H_{i\ell}(s, s^{-1}) = \mu_{i\ell(i)}(s, s^{-1}) I_{\ell(i)}$. Again using unimodularity of $H(s, s^{-1})$ now gives that for $i' \neq i$ one has $\ell(i') \neq \ell(i)$. It follows that $((\ell(1), \ldots, \ell(g))$ is a permutation of $(1, 2, \ldots, g)$. Define $\delta_i(s, s^{-1}) := \mu_{i\ell(i)}(s, s^{-1})$, $\Delta(s, s^{-1}) := \text{diag}(\delta_i(s, s^{-1}))$ and $\Pi := \text{col}(I_{\ell(1)}, \ldots, I_{\ell(g)})$. We obtain $H(s, s^{-1}) = \Delta(s, s^{-1}) \Pi$. Now from $H(s, s^{-1})$ is posimodular it follows that $\delta_i(s, s^{-1}) = \alpha_i s^d_i$, with $\alpha_i > 0$ and $d_i \in \mathbb{Z}$.

**Proof of proposition 4.5.12:**
We only need to prove the 'only if' implication. From the assumption and lemma 4.5.7 it follows that: $R(\sigma, \sigma^{-1}) w = 0 \Rightarrow R'(\sigma, \sigma^{-1}) w = 0$. It now follows that there exists a unique matrix $H \in \mathbb{R}^d \times g[s, s^{-1}]$ such that $R'(s, s^{-1}) = H(s, s^{-1}) R(s, s^{-1})$ [143]. It remains to be shown that $H \in \mathbb{R}^d \times g[s, s^{-1}]$. Since $R(s, s^{-1})$ has full row-rank it follows that for every nonnegative signal $f \in (\mathbb{R}^g)^\mathbb{Z}$ there exist a time-series $w \in (\mathbb{R}^g)^\mathbb{Z}$ such that $(R(\sigma, \sigma^{-1}) w)(t) = f(t) \geq 0$. Using the assumption it now follows that $0 \leq R'(\sigma, \sigma^{-1}) w = H(\sigma, \sigma^{-1}) R(\sigma, \sigma^{-1}) w = H(\sigma, \sigma^{-1}) f$. Since $f$ was chosen arbitrary it follows from lemma 4.5.9 that $H \in \mathbb{R}^d \times g[s, s^{-1}]$.

**Proof of proposition 4.5.15:**
First observe that by relation (4.30): $\mathcal{B}_{EI}(R_1, R_2) \subseteq \mathcal{B}_{EI}(R'_1, R'_2)$. From (4.30) and the
Appendix 4.A: Proofs

assumptions on the entries in the \((g+r) \times (g+r)\) block matrix in (4.30) follows that

\[
\begin{bmatrix}
R_1 \\
R_2
\end{bmatrix} = \begin{bmatrix}
U^{-1} & 0 \\
H^{-1}SU^{-1} & H^{-1}
\end{bmatrix}
\begin{bmatrix}
R'_1 \\
R'_2
\end{bmatrix},
\]

where we omitted the arguments. Observe that the entries in the \((g+r) \times (g+r)\) block matrix again satisfy the assumptions. It follows that \(\mathcal{B}_{EI}(R'_1, R'_2) \subseteq \mathcal{B}_{EI}(R_1, R_2)\). This gives \(\mathcal{B}_{EI}(R'_1, R'_2) = \mathcal{B}_{EI}(R_1, R_2)\). Since we did not change the number of equations, \(\{R'_1(\sigma, \sigma^{-1})w = 0 \land R'_2(\sigma, \sigma^{-1})w \geq 0\}\) is also minimal.

\[\triangleright\]

Proof of theorem 4.6.2:

From the linear case [144, theorem VIII.7] it follows that there exist matrices \(P \in \mathbb{R}^{p \times p}[s, s^{-1}]\) and \(Q \in \mathbb{R}^{p \times m}[s, s^{-1}]\) with the desired properties. It follows that there exist matrices \(T \in \mathbb{R}^{r \times p}[s, s^{-1}]\) and \(S \in \mathbb{R}^{r \times m}[s, s^{-1}]\) such that \(R_2(\sigma, \sigma^{-1})w = [T(\sigma, \sigma^{-1}) : S(\sigma, \sigma^{-1})] \cdot \text{col}(y, u)\). The second part follows from \(y = (P^{-1}Q)(\sigma, \sigma^{-1})u\) and the definition (omitting the arguments) \(S' := TP^{-1}Q + S\). This proves the statements in the theorem. \(\triangleright\)