Chapter 2

Linear dynamical systems

2.1 Introduction

In this chapter we will present a brief overview of the behavioural approach to modelling linear dynamical systems. All results in this chapter are taken from the literature, notably [143, 144, 145].

Mathematical models for control systems often involve differential or difference equations. Usually the models contain higher-order derivatives (in case of continuous-time models) or expressions to be evaluated at different time instants (in case of discrete-time models) of the variables of interest. Classical systems theory often focuses on systems that are already in first-order form. Moreover, in many cases an explicit distinction is made between input and output variables. Such a distinction however, may not be clear a priori. Examples that illustrate this already in a linear context can be found in for instance [145, 146]. Another example is given by the constrained robotic manipulator in example 1.2.4. Interaction of the manipulator with its environment, for instance grapple of an object, will inevitable mean restrictions on (possibly all) the positions, velocities and forces that are used to model the behaviour of the manipulator.

In the behavioural theory, which encompasses a foundation for the theory of deterministic dynamical systems, all variables are initially treated on an equal footing. Another important feature of the behavioural framework is that it offers a mathematical theory to discuss interconnected systems. This is important for us since an aerospace system usually consists of a collection of interacting (sub)systems.

Behavioural theory is well developed, especially for linear dynamical systems. We will focus on those results and ideas from the behavioural theory that will be used in the sequel in the context of dynamic inequality systems. An introduction to the behavioural approach to systems and control can be found in [146].
2.2 Behavioural equalities

In the behavioural theory a basic distinction is made between a system and its representation(s). The idea that a system is a set of possible trajectories is formalized in the following definition.

**Definition 2.2.1** A dynamical system $\Sigma$ is defined as a triple $\Sigma := (T, W, B)$, where $T \subseteq \mathbb{R}$ is the time set, $W$ is the space of variables, and $B$ is a subset of $W^T$. $B$ called the *behaviour* of the system.

In this thesis we concentrate on the case $W = \mathbb{R}^q$. In the present chapter we will usually focus on the discrete-time case with $T = \mathbb{Z}$ and the continuous-time case with $T = \mathbb{R}$. (For the case $T = \mathbb{Z}^+$ or $\mathbb{R}^+$ we refer to the literature.)

Let $\sigma^t := W^T \to W^T$ denote the backward t-shift: $(\sigma^t f)(t') = f(t' + t), \forall t, t' \in T$.

**Definition 2.2.2** A dynamical system $\Sigma := (T, W, B)$ with $T = \mathbb{Z}$ or $\mathbb{R}$ is said to be *linear* if $W$ is a linear vector space and $B$ is a linear subspace of $W^T$.

**Definition 2.2.3** A dynamical system $\Sigma := (T, W, B)$ with $T = \mathbb{Z}$ or $\mathbb{R}$ is said to be *time-invariant* if $\sigma^t B = B$ for all $t \in T$ (shift-invariance of $B$). If $T = \mathbb{Z}$ then this condition is equivalent to $\sigma B = B$.

The analogon of this definition when the time-axis is $\mathbb{Z}^+$ or $\mathbb{R}^+$ requires $\sigma^t B \subseteq B$ for all $t \in T$.

In this thesis we are primarily concerned with systems whose behaviours can be represented as solution sets of systems of difference or differential equalities and inequalities. Let $w \in \mathbb{R}^q$ denote the variables whose time paths we try to describe, i.e. the *manifest* variables. A linear difference equality is given by

$$R_L w(t + L) + R_{L-1} w(t + L - 1) + \ldots + R_l w(t + l) = 0, \forall t \in T,$$

where $R_L, R_{L-1}, \ldots, R_l \in \mathbb{R}^{q \times q}$ with $W = \mathbb{R}^q, T = \mathbb{Z}$. We assume that $L \geq l$ ($L, l \in \mathbb{Z}$). $L - l$ is called the lag of the difference equality (2.1). Introduce the polynomial matrix

$$R(s, s^{-1}) = R_L s^L + R_{L-1} s^{L-1} + \ldots + R_1 s + R_0 \in \mathbb{R}^{q \times q}[s, s^{-1}],$$

where $s$ is an indeterminate.

The above system of difference equations can be written as

$$R(\sigma, \sigma^{-1})w = 0. \tag{2.2}$$

Note that $R \in \mathbb{R}^{q \times q}[s, s^{-1}]$ can be considered as inducing the linear map $R(\sigma, \sigma^{-1}) : (\mathbb{R}^q)^Z \to (\mathbb{R}^q)^Z$, so $B = \ker R(\sigma, \sigma^{-1})$. We will refer to a representation as in equation (2.2) as a *kernel representation*. 

2.2: Behavioural equalities

In a kernel representation of a dynamical system the number of columns is fixed as this equals the number of manifest variables. However, the number of rows is variable as this equals the number of equations used to describe the behaviour.

The behavioural difference equation (2.2) describes a dynamical system \( \Sigma := (T, \mathbb{W}, \mathcal{B}) \) with time-axis \( T = \mathbb{Z} \), signal space \( \mathbb{W} = \mathbb{R}^q \), and behaviour \( \mathcal{B} = \{ w : \mathbb{Z} \to \mathbb{R}^q \mid R(\sigma, \sigma^{-1})w = 0 \} \).

The question arises which properties of the behaviour of a dynamical system allow the system to be represented by a difference equation (2.2).

We introduce some notation. Let \( t_1, t_2 \in T \), \(-\infty < t_1 \leq t_2 < \infty\). Define \( \mathcal{B}[\cap [t_1, t_2] := \{ w|_{\cap [t_1, t_2]} \mid w \in \mathcal{B} \} \). To shorten notation we denote \( \mathcal{B}[_{[t_1, t_2]} := \mathcal{B}[\cap [t_1, t_2] \).

**Definition 2.2.4** A dynamical system \( \Sigma := (T, \mathbb{W}, \mathcal{B}) \) is said to be complete if \( \{ w \in \mathcal{B} \} \Leftrightarrow \{ w|_{[t_1, t_2]} \in \mathcal{B}[_{[t_1, t_2]}, \forall t_1, t_2 \in T, -\infty < t_1 \leq t_2 < \infty \} \).

Let \( L^q \) denote the space of time series \( w : \mathbb{Z} \to \mathbb{R}^q \), equipped with the topology of pointwise convergence: \( \{ w_n \overset{\text{p-w}}{\to} w \} \Leftrightarrow \{ w_n(t) \overset{\text{p-w}}{\to} w(t), \forall t \in T \} \) (this last convergence should be understood in the usual norm topology on \( \mathbb{R}^q \)). \( L^q \) is a separable, metrizable topological space [127]. We will denote by \( L^q \) the collection of all linear, closed, shift-invariant subspaces of \( L^q \).

**Lemma 2.2.5** Let \( \Sigma = (T, \mathbb{W}, \mathcal{B}) \) be a linear time-invariant dynamical system. Then: \( \Sigma \) is complete if and only if \( \mathcal{B} \) is closed.

The next result follows.

**Theorem 2.2.6** Let \( \Sigma := (T, \mathbb{W}, \mathcal{B}) \) be a dynamical system. The following statements are equivalent:

(i) \( \Sigma \) is linear time-invariant and complete;
(ii) \( \exists g \in \mathbb{N} \text{ and } \exists R \in \mathbb{R}^{g \times q} [s, s^{-1}] \text{ such that } \mathcal{B} = \ker R(\sigma, \sigma^{-1}) \);
(iii) \( \mathcal{B} \in L^q \).

The continuous time analogon of equation (2.1) reads

\[
R_L \frac{d^L}{dt^L} w + R_{L-1} \frac{d^{L-1}}{dt^{L-1}} w + \ldots + R_0 w = 0,
\]

or, for short,

\[
R \left( \frac{d}{dt} \right)^L w = 0.
\]

A characterization as in theorem 2.2.6 for the continuous-time case is more difficult [124, 144]. The problem is what sort of solution one wants to use for equations (2.3) and (2.4). In [145] a distinction is made between strong and weak solutions of (2.3). A time-trajectory
\( w \) is called a strong solution of (2.3) if \( w \) is \( L \)-times differentiable and satisfies

\[
R_L \frac{d^L w}{dt^L}(t) + R_{L-1} \frac{d^{L-1} w}{dt^{L-1}}(t) + \ldots + R_0 w(t) = 0,
\]

for all \( t \in \mathbb{R} \).

In many applications this solution concept is too restrictive as it excludes for example the step-response as a signal. A time-trajectory \( w \) is called a weak solution of (2.3) if \( w \in L_t^{\infty}(\mathbb{R}, \mathbb{R}^q) \), the space of locally integrable functions, and if for all functions \( f \in C^\infty(\mathbb{R}, \mathbb{R}) \) of compact support, there holds

\[
\langle R^T (\frac{d}{dt}) f, w \rangle = 0,
\]

with \( \langle f_1, f_2 \rangle = \int_{-\infty}^{+\infty} f_1^T(t) f_2(t)dt \) [145]. In other words, in the sequel we take, if \( T = \mathbb{R}, w \in L_t^{\infty} \) and take the equalities in equations (2.3) and (2.4) in the sense of distributions. (We refer to [127] for theory on distributions.)

It follows that (2.5) defines a dynamical system \( \Sigma := (\mathbb{R}, \mathbb{R}^q, \ker(R)) \) with \( \ker(R) \) the set of weak solutions of (2.5). The following properties hold [145]:

(i) \( w \) is a weak solution of (2.5) if and only if \( R \left( \frac{d}{dt} \right) w \) is zero as a vector of distributions.

(ii) \( \ker(R) \cap C^\infty(\mathbb{R}, \mathbb{R}^q) \) is dense (in the topology of \( L_t^{\infty}(\mathbb{R}, \mathbb{R}^q) \)) in \( \ker(R) \).

The last property states that every weak solution of (2.5) can be approximated (in the topology of pointwise convergence) by a sequence of strong solutions. It can be shown that \( \Sigma := (\mathbb{R}, \mathbb{R}^q, \mathcal{B}) \), with \( \mathcal{B} = \{w \in L_t^{\infty}(\mathbb{R}, \mathbb{R}^q) | \text{equation (2.4) is satisfied}\} \), is a linear time-invariant system [145]. In the sequel we will sometimes write \( L_t^{\infty} \) instead of \( L_t^{\infty}(\mathbb{R}, \mathbb{R}^q) \), and \( L_{1,+}^{\infty} \) instead of \( L_{1,+}^{\infty}(\mathbb{R}, \mathbb{R}^q) \), when the dimension \( q \) in \( \mathbb{R}^q \) is clear from the context.

In the remainder of this chapter we concentrate on discrete-time dynamical systems. Many results also hold, mutatis mutandis, for continuous-time systems [120, 124, 145].

### 2.3 Interconnections and latent variables

Usually complex systems contain many subsystems. The system can then be viewed as the total of subsystems and interfaces that connect these subsystems. In [144, 145] the notion of interconnection is introduced to deal with this situation.

**Definition 2.3.1** Let \( \Sigma_1 := (T, \mathcal{W}, \mathcal{B}_1) \) and \( \Sigma_2 := (T, \mathcal{W}, \mathcal{B}_2) \) be two dynamical systems with the same time axis and the same signal space. The **interconnection** of \( \Sigma_1 \) and \( \Sigma_2 \), denoted by \( \Sigma_1 \wedge \Sigma_2 \), is defined as \( \Sigma_1 \wedge \Sigma_2 = (T, \mathcal{W}, \mathcal{B}_1 \cap \mathcal{B}_2) \).

It follows that the behaviour of \( \Sigma_1 \wedge \Sigma_2 \) consists of those trajectories \( w : T \rightarrow \mathcal{W} \) which are compatible with \( w \in \mathcal{B}_1 \) and \( w \in \mathcal{B}_2 \). The central role that interconnection plays in systems and control theory is further motivated and detailed in [145].
2.3: Interconnections and latent variables

The concept of interconnection will play an important role in the present thesis for manifest behaviours as well as for first-order representations. At this point it is worth remarking that in [88] it is stated, in a discussion on manifest behaviours, that before interconnecting two systems typically the state of both systems should be prepared. We will study this issue later.

Example 2.3.2 Consider two carts that are rigidly connected to each other. Let \( y_1 \) be a point on the left cart, and let \( y_2 \) be a point on the right cart (see figure 2.1). Consider the following model

\[
\text{Left Cart: } m_1 \frac{d^2 y_1}{dt^2} + d_1 \frac{dy_1}{dt} + k_1 y_1 = f, \tag{2.6}
\]

\[
\text{Right Cart: } m_2 \frac{d^2 y_2}{dt^2} + d_2 \frac{dy_2}{dt} + k_2 y_2 = -f, \tag{2.7}
\]

where \( m_i, d_i, \) and \( k_i, i = \{1, 2\} \), are the system parameters for the left cart, right cart, respectively, and \( f \) models the interaction force. Let \( \Sigma_1 := (\mathbb{R}, \mathbb{W}, \mathcal{B}_1) \), with \( \mathcal{B}_1 = \{(y_1, f) \in \mathcal{L}^{loc}_{1, +} \times \mathcal{L}^{loc}_{1, +} | \text{equation (2.6) is satisfied}\} \) and let \( \Sigma_2 := (\mathbb{R}, \mathbb{W}, \mathcal{B}_2) \), with \( \mathcal{B}_2 = \{(y_2, f) \in \mathcal{L}^{loc}_{1, +} \times \mathcal{L}^{loc}_{1, +} | \text{equation (2.7) is satisfied}\} \). The interconnection of \( \Sigma_1 \) and \( \Sigma_2 \) is given by \( \Sigma_1 \wedge \Sigma_2 = (\mathbb{R}, \mathbb{W}, \mathcal{B}_1 \cap \mathcal{B}_2) \). Elimination of the interaction force and using \( y_1(t) = y_2(t) \) for all \( t \in \mathbb{R}_+ \) gives

\[
(m_1 + m_2) \frac{d^2 y_1}{dt^2} + (d_1 + d_2) \frac{dy_1}{dt} + (k_1 + k_2)y_1 = 0 \tag{2.8}
\]

as a model of the dynamics of the combination of the two carts in terms of the variable \( y_1 \), i.e. \( \mathcal{B}_1 \cap \mathcal{B}_2 = \{y_1 \in \mathcal{L}^{loc}_{1, +} | \text{equation (2.8) is satisfied}\} \).

Apart from the variables whose behaviour we want to specify, in general one also works with auxiliary variables. These variables may for instance be introduced to facilitate the modelling process. In the behavioural theory this leads to the concept of latent variables.
Definition 2.3.3 A dynamical system with latent variables is defined as \( \Sigma_f = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_f) \) with \( \mathbb{W} \) the space of manifest variables, \( \mathbb{L} \) the space of latent variables and \( \mathfrak{B}_f \subset (\mathbb{W} \times \mathbb{L})^T \) the full behaviour.

Let \( \Sigma_f = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_f) \). The full behaviour defines a latent variable representation of the manifest dynamical system defined by \( \Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B}) \), with \( \mathfrak{B} := \{ w \in \mathbb{W}^T \mid \exists \ell \text{ such that } (w, \ell) \in \mathfrak{B}_f \} \). The following result is crucial to the development of the behavioural theory for the case \( \mathbb{T} = \mathbb{Z} \).

Theorem 2.3.4 Let \( \Sigma_f = (\mathbb{Z}, \mathbb{R}^q, \mathbb{R}^d, \mathfrak{B}_f) \) be linear time-invariant and complete. Then the manifest dynamical system it represents, \( \Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B}) \), is also linear time-invariant and complete.

In the context of interconnections this theorem states (for \( \mathbb{T} = \mathbb{Z} \)) that the interconnecting variables can be eliminated. The resulting representation is again similar to the one in equation (2.2). This 'elimination theorem' allows one to investigate many properties of systems by investigating kernel representations.

A general model which involves both manifest variables and latent variables is given by

\[
R(\sigma, \sigma^{-1})w = M(\sigma, \sigma^{-1})\ell, \tag{2.9}
\]

with \( M \in \mathbb{R}^{q \times d}[s, s^{-1}] \) and \( \ell : \mathbb{Z} \to \mathbb{R}^d \). A special class of representations is referred to as image representations, and is given by:

\[
w = M(\sigma, \sigma^{-1})\ell. \tag{2.10}
\]

It can be seen that representations (2.1) and (2.10) are at the two extremes of the model class (2.9). From the elimination theorem it follows that every image representation allows for a kernel representation. The reverse statement holds only under a special condition.

Definition 2.3.5 Let \( \Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B}) \) be a time-invariant dynamical system with \( \mathbb{T} = \mathbb{R} \), or \( \mathbb{T} = \mathbb{Z} \). \( \Sigma \) is said to be controllable if for all \( w_1, w_2 \in \mathfrak{B} \) there exist \( t^* \in \mathbb{T}, t^* \geq 0 \), and \( w^* : \mathbb{T} \cap [0, t^*] \to \mathbb{W} \) such that \( w' \in \mathfrak{B} \), with \( w' : \mathbb{T} \to \mathbb{W} \) defined by

\[
w'(t) = \begin{cases} 
    w_1(t) & \text{for } t < 0, \\
    w^*(t) & \text{for } 0 \leq t < t^*, \\
    w_2(t) & \text{for } t \geq t^*. 
\end{cases} \tag{2.11}
\]

In a controllable system one can connect a 'past' trajectory piece with any future trajectory piece using an intermediate finite-length trajectory piece.
There exists a concrete test for controllability in case of kernel representations of a dynamical system. Let \( \Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B}) \) be represented by the equations \( R(\sigma, \sigma^{-1})w = 0 \). Then \( \Sigma \) is controllable if and only if the rank of the matrix \( R(\lambda, \lambda^{-1}) \) (as an element of \( \mathbb{C}^{g \times q} \)) is independent of \( \lambda \) for \( \lambda \neq \lambda \in \mathbb{C} \). In [144] it is shown that a system \( \Sigma \) is controllable if and only if it allows an image representation as in equation (2.10).

**Definition 2.3.6** Let \( \Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B}) \) be a time-invariant dynamical system with \( \mathbb{T} = \mathbb{R} \) or \( \mathbb{T} = \mathbb{Z} \). Then \( w_2 \) is said to be observable from \( w_1 \) in \( \Sigma \) if \( \{(w'_1, w'_2), (w''_1, w''_2) \in \mathfrak{B} \) and \( w'_1 = w''_1 \} \Rightarrow \{w'_2 = w''_2 \} \).

In the behavioural theory, observability is thus a property of a system in which some of the variables should be deduced from others. In particular, for latent variable systems in which the latent variables should be deduced from the manifest variables. Also for observability a concrete test is available in case of kernel representations of a dynamical system. Let \( \Sigma = (\mathbb{T}, \mathbb{R}^q \times \mathbb{R}^d, \mathfrak{B}) \) be represented by the equations \( R(\sigma, \sigma^{-1})w = M(\sigma, \sigma^{-1})\ell \). Then \( \ell \) is observable from \( w \) in \( \Sigma \) if and only if the rank of the matrix \( M(\lambda, \lambda^{-1}) \) (as an element of \( \mathbb{C}^{q \times d} \)) is equal to \( d \) for all \( \lambda \neq \lambda \in \mathbb{C} \).

Combining the notions of controllability and observability leads to the following result. Following Willems [144], in the following theorem we will call a latent variable model \( \Sigma_f = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B}_f) \) controllable if the full behaviour is controllable.

**Theorem 2.3.7** Assume that in the latent variable representation \( R(\sigma, \sigma^{-1})w = M(\sigma, \sigma^{-1})\ell \), \( \ell \) is observable from \( w \). Let the kernel representation \( R'(\sigma, \sigma^{-1})w = 0 \) describe its manifest behaviour. Then the full behaviour is controllable if and only if the manifest behaviour is.

### 2.4 On representations

Models for physical systems generally follow from certain principles, such as setting up the motion equations of motion control systems by use of the Lagrangian formalism. The resulting set of differential-algebraic equations is then usually a higher-order system. On the other hand, for simulation and controller design, classical first-order models are important. In this section we briefly discuss some issues in representations of linear dynamical systems. We start with the notion of minimal kernel representations.

**Definition 2.4.1** A system of difference equations (2.2) with \( R \in \mathbb{R}^{g \times q}[s, s^{-1}] \) is called minimal if \( R' \in \mathbb{R}^{g' \times q}[s, s^{-1}] \) and \( \Sigma(R) = \Sigma(R') \) imply \( g' \geq g \).

We will need some terminology. If \( U \in \mathbb{R}^{g \times q}[s, s^{-1}] \) has an inverse \( U^{-1} \in \mathbb{R}^{q \times g}[s, s^{-1}] \) then \( U(s, s^{-1}) \) is said to be unimodular. It can be shown that \( U(s, s^{-1}) \) is unimodular if and only if \( \det(U) = \alpha s^d \) with \( \alpha \neq 0 \) and \( d \in \mathbb{Z} \). We will say that \( R \in \mathbb{R}^{q \times g}[s, s^{-1}] \) has full
row-rank if there exist an \( g \times g \) submatrix \( R'(s, s^{-1}) \) of \( R(s, s^{-1}) \) such that \( \det(R'(s, s^{-1})) \) is not the zero polynomial.

Polynomial representations of a linear behaviour are related as follows.

**Proposition 2.4.2** The following holds:

(i) \( R \in \mathbb{R}^{g \times q}[s, s^{-1}], g \in \mathbb{N}, \) is minimal if and only if \( R \) has full row-rank.

(ii) \( \Sigma(R_1) = \Sigma(R_2) \) if and only if there exist polynomial matrices \( F_1(s, s^{-1}) \) and \( F_2(s, s^{-1}) \) of appropriate dimensions such that \( R_1(s, s^{-1}) = F_2(s, s^{-1})R_2(s, s^{-1}) \) and \( R_2(s, s^{-1}) = F_1(s, s^{-1})R_1(s, s^{-1}) \).

(iii) Let \( R_1, R_2 \in \mathbb{R}^{g \times q}[s, s^{-1}] \) both have full row-rank. Then \( \Sigma(R_1) = \Sigma(R_2) \) if and only if there exists an unimodular matrix \( U(s, s^{-1}) \) such that \( R_2(s, s^{-1}) = U(s, s^{-1})R_1(s, s^{-1}) \).

Starting from a polynomial matrix \( R(s, s^{-1}) \) one can always find a polynomial matrix \( R'(s, s^{-1}) \) such that \( \Sigma(R) = \Sigma(R') \) and \( R' \) has full row-rank (see [106] for an algorithm).

The following notion will also play a role in the sequel.

**Definition 2.4.3** Let \( \Sigma := (\mathbb{T}, \mathcal{W}, \mathcal{B}) \) be a time-invariant system. Then \( \Sigma \) is said to be autonomous if: \( \{w_1, w_2 \in \mathcal{B} \text{ and } w_1(t) = w_2(t) \text{ for } t < 0\} \Rightarrow \{w_1 = w_2\} \).

**Proposition 2.4.4** Let \( \Sigma := (\mathbb{Z}, \mathbb{R}, \mathcal{B}) \) be a time-invariant system. The following statements are equivalent:

(i) \( \Sigma \) is autonomous.

(ii) \( \mathcal{B} \) is finite-dimensional.

(iii) \( \Sigma \) admits a representation \( R(\sigma, \sigma^{-1})w = 0 \) having \( \det(R) \neq 0 \).

We will need the following result in our discussion on behavioural inequalities in chapter 4. The lemma given below states that if a time series satisfies a difference equality for a sufficient number of time points, then its "inner part" belongs to the behaviour specified by the difference equality.

**Lemma 2.4.5 ([110])** Let \( R(s) = R_{\Delta}s^\Delta + R_{\Delta-1}s^{\Delta-1} + \ldots + R_0, R \in \mathbb{R}^{g \times q}[s], \) \( R(s) \) has full row-rank, and \( \mathcal{B} = \ker R(\sigma) \). Then there exist \( k \in \mathbb{Z}_+, K \in \mathbb{Z}_+, k \leq K \), depending only on \( \Delta \), such that, for \( w : \mathbb{Z} \to \mathbb{R}^q, \{(R(\sigma)w)(t) = 0, \text{ for } -T \leq t \leq T, \text{ and } T \geq K\} \) implies \( \{w|_{-T+k, T-k}] \in \mathcal{B}|_{-T+k, T-k}\} \).

As a simple example consider \( r(s) = r_Ls^L + \ldots + r_0 \), with \( r_i \in \mathbb{R} \), and \( r_L \) and \( r_0 \) not equal to zero. Assume that \( (r(\sigma)w)(t) = 0 \) for \( -T \leq t \leq T, \text{ and } T = K \geq L. \) Define recursively

\[
w(t) = \begin{cases} \frac{1}{r_L}(-r_{L-1}w(t-1) - \ldots - r_0w(t-L)) & \text{for } t > T \\ \frac{1}{r_0}(-r_Lw(t+L) - \ldots - r_1w(t+1)) & \text{for } t < -T. \end{cases}
\]

(2.12)
Then \( w \in \text{ker} \tau(\sigma) \).

In the remainder of this section we will briefly discuss input/output and state-space dynamical systems. Formally, in an input/output dynamical system \( \mathbb{W} := \mathbb{U} \times \mathbb{Y} \), \( \mathbb{U} \) denotes the input space and \( \mathbb{Y} \) denotes the output space. An important result is that every kernel system \( \mathbb{R} = 0 \), with \( \mathbb{R} \in \mathbb{R}^{p,q}[s,s^{-1}] \), allows for an input/output representation

\[
P(\sigma,\sigma^{-1})y = Q(\sigma,\sigma^{-1})u, \quad \text{with } w = \Pi \begin{bmatrix} u \\ y \end{bmatrix}.
\]

Here \( P \in \mathbb{R}^{p,p}[s,s^{-1}] \) with \( \det(P) \neq 0 \), \( Q \in \mathbb{R}^{p,m}[s,s^{-1}] \), \( P^{-1}Q \) proper and \( m + p = q \). \( \Pi \) is a permutation matrix that serves to identify which components of \( w \) are inputs and which components are outputs. (We refer to [144] for a precise mathematical formulation needed to arrive at (2.13).) In representation (2.13) the specific choice of input variables and output variables is not invariant: the choice of the permutation \( \Pi \) may depend on the physics of the problem at hand and the application that we have in mind. However, the number of input variables and the number of output variables is invariant. (In [144] it is shown that the number of output variables equals the number of equations in a minimal representation.)

Historically, so called state-space representations of systems have been used as a means to study linear dynamical systems ([80]). Basic features of state-space models are that they are first-order in the state-space variable, denoted by \( x \), and zero-th order in manifest variable \( w \). An extensive treatment of first-order representations in a behavioural context can be found in [89, 144]. State-space representations of linear dynamical systems are required to satisfy the axiom of state. Assume that the state variables \( x \) take their values in \( X \).

**Definition 2.4.6** A state-space dynamical system is defined as a dynamical system \( \Sigma = (\mathbb{T}, \mathbb{W}, X, \mathfrak{B}_s) \) in which the full behaviour \( \mathfrak{B}_s \) satisfies the axiom of state. This axiom requires that \( \{(w_1,x_1),(w_2,x_2) \in \mathbb{B}_s, t \in \mathbb{T}, \text{ and } x_1(t) = x_2(t) \} \Rightarrow \{(w,x) \in \mathbb{B}_s\} \), where

\[
(w(t'),x(t')) = \begin{cases} (w_1(t'),x_1(t')) & \text{for } t' < t \\ (w_2(t'),x_2(t')) & \text{for } t' \geq t. \end{cases}
\]

In the continuous-time case the requirement \( x_1, x_2 \) continuous at \( t' \) must be added [124].

Starting from a kernel representation \( \mathfrak{B} = \{w : \mathbb{Z} \to \mathbb{R}^q | R_L w^L + \ldots + R_0 w = 0\} \) one can always define a state-space representation that still contains the manifest variables \( w \), and for which the projection of \( \mathfrak{B}_s \) on the space of the external variables gives again the behaviour \( \mathfrak{B} \). One possible choice is

\[
E \sigma x + Fx + Gw = 0,
\]
with \( x := \text{col}(w, \sigma w, \ldots, \sigma^Tw) \) and

\[
E = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
I & 0 & \ldots & 0 & 0 \\
0 & I & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & I & 0 \\
0 & 0 & \ldots & 0 & 0
\end{bmatrix}, \quad F = -\begin{bmatrix}
I & 0 & 0 & \ldots & 0 \\
0 & I & 0 & \ldots & 0 \\
0 & 0 & I & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & I \\
R_0 & R_1 & R_2 & \ldots & R_L
\end{bmatrix}, \quad G = \begin{bmatrix}
I \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]

A representation as in (2.14) is said to be minimal if both the dimension of the state vector and the number of equations are as small as possible.

A classical input/state/output representation is given by:

\[
\begin{align*}
\sigma x &= Ax + Bu, \\
y &= Cx + Du.
\end{align*}
\]

(2.16)

Here we assume \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p \) and \( w = \text{col}(u, y) \). For system (2.16) controllability is usually defined with respect to the state \( x \), opposed to controllability for manifest variables in definition 2.3.5.

We can write (2.16) also as a difference equality

\[
R(\sigma) \begin{bmatrix} x \\ u \\ y \end{bmatrix} = 0, \quad \text{with } R(s) = \begin{bmatrix} I_s - A & -B & 0 \\ -C & -D & I \end{bmatrix}.
\]

(2.17)

Since the matrix \( R \) is of full row-rank, the system of equations (2.16) is a minimal difference equation representation of its full behaviour.

For minimality as an input/state/output representation it is required that the dimension of the state vector is as small as possible and that the number of equations is as small as possible. In the case \( \mathbb{T} = \mathbb{Z} \), a necessary and sufficient condition for system (2.16) to be a minimal representation is that matrix \([C^T : (CA)^T : \ldots : (CA^{n-1})^T]^T \) has rank \( n \) and \( A\mathbb{R}^n + \text{im}(B) = \mathbb{R}^n \). Note that minimality does not imply controllability. For controllable systems we have the following result.

**Proposition 2.4.7** Assume that the manifest behaviour of system (2.16) is controllable. Then it is minimal if and only if \([B : AB : \ldots : A^{n-1}B] \) and \([C^T : (CA)^T : \ldots : (CA^{n-1})^T]^T \) both have rank \( n \).
2.4: On representations

2.5 Concluding remark

In this chapter we have touched upon some basic results from the behavioural approach to modelling dynamical systems as put forward in [143, 144, 145]. For additional motivation and a more extensive treatment we refer to these references. In the next chapter we will start our investigation on inequality systems with the simplest types: static inequality systems.