Sufficient Conditions for Minimality of a Nonlinear Realization via Controllability and Observability Functions

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Abstract

In this paper we develop a set of sufficient conditions in terms of controllability and observability functions under which a given state space realization of a formal power series is minimal. Specifically, it will be shown that positivity of these functions, plus a few technical conditions, implies minimality. In doing so, connections are established between Hamilton-Jacobi type optimal control theory and the well known necessary and sufficient conditions for minimality in terms of Kalman type rank conditions on the accessibility and observability distributions.

1 Introduction

The problem of determining when the dimension of a state space realization of a given input-output map is minimal is a fundamental problem in systems. Its connects to many other topics in realization theory like similarity invariance, controllability and observability properties, model reduction and balanced realizations. The theory is quite complete in the case of linear systems. For example, it is well known that minimality is equivalent to joint controllability and observability, and for stable systems, this is further equivalent to the positive definiteness of the controllability and observability Gramians. These Gramian matrices naturally appear in balanced realization theory, and are related to optimal control problems. In the nonlinear case, minimality theory is not nearly as well developed. For example, there are several existing theories for minimality depending on the exact nature in which the input-output mapping is described, i.e., in terms of a set of input-output differential equations (see [1] and the references therein), a Volterra series [6, 7, 8] or a formal power series/Chen-Fliess functional expansion [6]. At present, the exact connections between these different approaches are not completely understood. Furthermore, motivated by the linear case, we might expect that minimality should have connections to the nonlinear extensions of the Gramians, which have been developed for nonlinear balancing [1]-[4],[10]-[12]. But these connections are also largely unknown at present.

The specific purpose of this paper is to develop a set of sufficient conditions in terms of controllability and observability functions under which a given state space realization of a formal power series is minimal. Specifically, it will be shown that positivity of these functions, plus a few technical conditions, implies minimality. Of course there exists well known necessary and sufficient conditions for minimality in terms of Kalman type rank conditions on the accessibility and observability distributions. So the novelty of the approach taken here is in establishing a connection between these differential geometric type minimality conditions and properties of functions that are connected with Hamilton-Jacobi type optimal control theory. As an added benefit, it also seems possible to use this new minimality characterization to further develop the nonlinear notions of similarity invariance [13], the Kalman decomposition [12], and Hankel operators.

The paper is organized as follows. In Sections 2, the background material pertaining to all the relevant subjects is briefly reviewed, specifically: the main definitions and rank conditions associated with reachability and observability, the definitions and known properties of controllability and observability functions, and minimality theory for state space realizations of formal power series. In Section 3.1 we then develop a relationship between positivity of the controllability function and the accessibility rank condition. The analogous connections between positivity of the observability function and the observability rank condition are covered in Section 3.2. The main result of the paper involving minimality, plus some concluding remarks, are presented in the final section.

The mathematical notation used throughout is fairly standard. Vector norms are represented by $\|x\| = \sqrt{x^T x}$ for $x \in \mathbb{R}^n$. $L_2(a,b)$ represents the set of Lebesgue measurable functions, possibly vector-valued, with finite $L_2$ norm $\|x\|_{L_2} = \sqrt{\int_a^b |x(t)|^2 \, dt}$. If $L_2: \mathbb{R}^n \rightarrow \mathbb{R}$ is a differentiable function, then its partial derivative $\frac{\partial}{\partial t}$ will be the row vector of partial derivatives $\frac{\partial}{\partial t}$ where $i = 1, \ldots, n$.

2 Background

2.1 Controllability and observability functions

Controllability and observability functions play an important role in balancing and model reduction for stable nonlinear systems [10, 12]. In this section we give a brief review of the results that are important for the minimality theory presented in Section 3.

Consider a smooth, i.e., $C^\infty$, nonlinear system of the form

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

(1)

where $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$, $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$ and $x = (x_1, \ldots, x_n)$ are local coordinates for a smooth state space manifold denoted by $M$. Throughout we assume that the system has an equilibrium. Without loss of generality we take this equilibrium to be at $0$, i.e. $f(0) = 0$, and we also take $h(0) = 0$. 
The following definition is an addition to the well-known definitions of the (strong) accessibility distribution and observability codistribution [7].

**Definition 2.7** Consider the system (1).
- The zero-observability space \( \mathcal{O}_0 \) is the linear space of functions on \( M \) containing \( h_1, \ldots, h_p \) and all repeated Lie derivatives \( L_i^j h_i \), \( j \in 1, \ldots, p, k = 1, 2, \ldots \).
- The zero-observability codistribution \( d\mathcal{O}_0 \) is given by \( d\mathcal{O}_0(q) = \text{span}\{dH(q)\} \, H \in \mathcal{O}_0 \), where \( q \in M \).

**Theorem 2.8** [7] Consider the system (1). Let \( C \) and \( \mathcal{O}_0 \) denote the accessibility and strong accessibility distributions, respectively; and \( dC \) and \( d\mathcal{O}_0 \) be the observability and zero-observability codistributions, respectively.
- If \( \dim C(x_0) = n \), then the system is locally accessible from \( x_0 \).
- If \( \dim C_0(x_0) = n \), then the system is locally strongly accessible from \( x_0 \).
- If \( \dim d\mathcal{O}_0(x_0) = n \), then the system is locally observable at \( x_0 \).
- If \( \dim d\mathcal{O}_0(0) = n \), then the system is locally zero-state observable.

**Remark 2.9** A consequence of this theorem and Definition 2.7 is that local zero-state observability implies local observability at \( 0 \). Furthermore, it follows that local strong accessibility at \( x_0 \) implies local accessibility at \( x_0 \).

The following theorems are closely related to results that appear in [5] and [9]. They reveal important properties of the system (1) in terms of the relationships between zero-state observability, positive definiteness of the observability function, and asymptotic stability of the equilibrium at \( 0 \).

**Theorem 2.10** [10] Assume \( f(x) \) is asymptotically stable on a neighborhood \( W \) of \( 0 \). If the system (1) is zero-state observable on \( W \), then \( \mathcal{L}_x > 0, \forall x \in W, x \neq 0 \).

**Theorem 2.11** [10] If the system (1) is zero-state observable and \( \mathcal{L}_x \) is well-defined, smooth and positive definite, then the system is locally asymptotically stable. If \( \mathcal{L}_x \) is proper, then \( x = f(x) \) is globally asymptotically stable.

### 2.2 A more general observability function

Local zero-state observability is certainly more restrictive than local observability. In order to extend the previous results to a more general setting, a more general observability function needs to be considered in which the input plays a direct role. Given the system \((f, g, h)\), the corresponding homogeneous system is denoted by \((f, \tilde{g}, h)\), where \( \tilde{g}(x) = g(x) - g(0) \). It is easily shown that \((f, \tilde{g}, h)\) and its homogeneous counterpart always have the same observability spaces, and thus have basically the same observability properties.

**Definition 2.12** [1]-[4] The natural observability function for the system (1) is defined as

\[
\mathcal{L}_N(x_0) = \max_{u \in U_2(0, \infty), \|u\|_{\infty} \leq \alpha} \frac{1}{2} \int_{0}^{\infty} \|\tilde{y}(t)\|^2 \, dt,
\]

where \( \alpha \geq 0 \) is a fixed real number, and \( \tilde{y} \) is the output response of the corresponding homogeneous system.
Clearly $L_0^N(x_0)$ is the maximum output energy one could expect from initializing the homogeneous system at $x(0) = x_0$ and applying any input with energy bounded by $\alpha$. When $\alpha = 0$, we have the observability function given in Definition 2.1. The following theorem provides a defining equation for $L_0^N$ analogous to equation (4).

**Theorem 2.13** [1]-[4] For any fixed $x_0$ in a neighborhood $W$ of $0$, $L_0^N(x_0)$ is uniquely determined by evaluating the smooth solution of

$$\frac{dL_0}{dx} + 2 \mu h - \frac{1}{2} \mu^{-1}(x_0) \frac{dL_0}{dx} \frac{dL_0}{dx}^T \frac{dL_0}{dx} = 0,$$  \hspace{1cm} (7)

with $L_0(0, x_0) = 0$ at $x = x_0$ and under the assumption that a smooth solution $L_0(x, x_0)$ exists on $W$, and is an asymptotically stable equilibrium on $W$ of $\hat{f} = (f - \mu^{-1}(x_0) g g^T x, x_0)$ with $\mu(x_0) := \|g^T(x_0) g g^T x, x_0)\|_{L_2}$ a negative real number when $\phi = \hat{f}(\phi), \phi(0) = x_0$.

The following theorem describes a sufficient condition for having $L_0^N(x) > 0$ on $x \in W$, $x \neq 0$, in terms of an observability condition on the set of inputs $B_0 := \{u \in L_2[0, \infty) : \|u\|_{L_2} \leq \alpha\}$.

**Theorem 2.14** [1]-[4] Suppose $0$ is an asymptotically stable equilibrium of the system $(f, g, h)$ on a neighborhood $W$ of $0$ and $h(0) = 0$. If the system $(f, g, h)$ is observable with respect to $B_0$ then $L_0^N(x) > 0$ when $x \in W, x \neq 0$.

**Remark 2.15** It is clear that the property of zero-state observability in the previous section is playing the same role as observability with respect to the trivial input class $B_0$.

**Remark 2.16** [2, 4] It is also known that when the system $(f, g, h)$ is observable with respect to $B_0$ then $L_0^N(x)$ has a strong local minimum equal to zero at $x = 0$, i.e., $L_0^N(0) = 0$, with $\frac{dL_0}{dx}(0) = 0$ and the Hessian $\frac{d^2L_0}{dx^2}(0) > 0$. This last property is critical in the balancing transformation theory presented in [2].

### 2.3 Minimality via formal power series

In this section we briefly review a theory of minimal state space realizations for input-output systems that can be represented by a formal power series (Chen-Fliess functional expansion). A detailed treatment may be found in [6]. Ultimately this leads to rank conditions as in Theorem 2.8, which are necessary and sufficient conditions for a given realization to be minimal.

Let $S$ be a given input-output map represented by a convergent generating series

$$S : u \rightarrow y(t) = \sum_{\eta \in \ell^r} c(\eta) E_\eta[u](t, \theta),$$  \hspace{1cm} (8)

where $\ell^r$ is the set of multi-indices for the index set $I = \{0, 1, \ldots, m\}$, $c(\eta) \in \mathbb{R}^n$, and

$$E_{\eta-\zeta}[u](t, \theta) := \int_0^t u_\eta(\tau) E_{\eta-\zeta}[u](\tau, \theta) d\tau$$  \hspace{1cm} (9)

for $t \in [0, T]$ with $E_\eta[u](t, \theta) := 1$ and $u_\eta(0) := 1$. The mapping $S$ can then be represented by a formal power series in noncommuting monomials $Z = \{z_0, z_1, \ldots, z_n\}$ via

$$c = \sum_{\eta \in \ell^r} c(\eta) z_\eta,$$  \hspace{1cm} (10)

where $z_\eta = z_{\eta_0} \cdots z_{\eta_n}$ when $\eta = (\eta_0, \ldots, \eta_n)$. Now define the sets:

- $R \subset \mathbb{R} >$ : the set of polynomials in $Z$ over $\mathbb{R}$;
- $R^p \subset \mathbb{R} >$ : the set of formal power series in $Z$ over $\mathbb{R}^p$.

Then the (block) Hankel mapping associated with $c$ is defined to be the $\mathbb{R}$-vector space morphism $H : R \subset \mathbb{R} > \rightarrow R^p \subset \mathbb{R} >$, uniquely specified by the generalized shifting property $H(z_\eta)[(\eta) = c(\eta, \zeta)$, where $\eta, \zeta \in \ell^r$. In this context we have the following definition.

**Definition 2.17** The Lie rank of a formal power series $c$ is defined as $\rho_L(c) := \dim(H(L_2(Z)))$, where $L_2(Z)$ denotes the smallest Lie algebra containing $Z$.

An analytic state space realization $(f, g, h)$ defined locally about $x_0$ is said to realize a formal power series $c$ if

$$c(u_0, \ldots, u_n) = L_{x_0} [X_{x_1} \cdots X_{x_n} h(x_0)]$$  \hspace{1cm} (11)

for every $(u_0, \ldots, u_n) \in \ell^r$, where $X_i, i \in \ell^r$, in the set $\{f, g_1, \ldots, g_m\}$. It is well known that if a certain growth condition on the coefficients $\{c(\eta)\}_{\eta \in \ell^r}$ is satisfied, then there exists a realization of $c$ if and only if the Lie rank of $c$ is finite. The following results characterize minimality.

**Theorem 2.18** An analytic realization $(f, g, h)$ about $x_0$ of a formal power series $c$ is minimal if and only if its dimension is equal to the Lie rank $\rho_L(c)$.

**Theorem 2.19** An analytic realization $(f, g, h)$ about $x_0$ of a formal power series $c$ is minimal if and only if $\dim \mathcal{C}(x_0) = n$ and $\dim \mathcal{D}(x_0) = n$.

Finally any two minimal realizations $(f, g, h)$ about $x_0$ and $(\tilde{f}, \tilde{g}, \tilde{h})$ about $\tilde{x}_0$ are necessarily related by a diffeomorphism $T : \tilde{V} \rightarrow V$ where $V$ and $\tilde{V}$ neighborhoods of $z^0$ and $\tilde{z}^0$, respectively. Thus minimal realizations of formal power series are unique modulo a diffeomorphism.

### 3 Minimality and Energy Functions

#### 3.1 Accessibility

In this section we study the controllability function and characterizations that are important for obtaining the accessibility rank condition in order to use Theorem 2.19. It can be easily deduced that we have the following relation (following the lines of the proof of Theorem 13 in [9])

$$L_c(x_0) := L_c(x_0) := \inf_{\alpha \in L_2(0, 1)} \frac{1}{2} \int_0^1 \|u(t)\|^2 dt,$$  \hspace{1cm} (12)

and clearly reachability from $x_0$ implies $L_c$ is well-defined for all $x \in M$, and thus, likewise for $L_c$. However, reachability is not implied from a well-defined and positive definite $L_c$. For our application it is sufficient (as observed from Theorem 2.4) to consider only the anti-stabilizability of the solution of the Hamilton-Jacobi equation (5), which is a condition that can be seen as reachable from 0 in finite time (so called asymptotic reachability from 0). We formally define this notion below.

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Definition 3.1 A system (1) is said to be asymptotically reachable from $x_0$ on a neighborhood $W$ of $x_0$ if $\forall x \in W$ there exists a $u \in L_2[0,\infty)$ such that $\psi(t,0,x_0,u) \in W$ for all $t \geq 0$, and $\lim_{t \to \infty} \psi(t,0,x_0,u) = x$, i.e., $\forall x \in W, \forall t \geq 0, \exists K < \infty, \text{s.t. } \psi(t,0,x_0,u) \in W$ for all $t \leq K$, and s.t. $|\psi(K,0,x_0,u) - x| < \epsilon$.

A system (1) is said to be locally asymptotically reachable from $x_0$ if there exists a neighborhood $W$ of $x_0$ such that the system is asymptotically reachable from $x_0$ on every neighborhood $V \subset W$ of $x_0$. □

In the following theorem, we obtain the relation between local asymptotic reachability from $x_0$ and local accessibility from $x_0$.

Theorem 3.2 Assume that the accessibility distribution $C$ has constant dimension about $x_0$. Then local asymptotic reachability from $x_0$ implies that the system is locally accessible from $x_0$.

Proof Suppose that the system is not locally accessible from $x_0$, then we know from standard results in the literature (e.g., [7]) that $\dim(C(x_0)) = k < n$. Hence from Proposition 3.12 in [7] we know there exists a neighborhood $V$ of $x_0$ and local coordinates $x_1, \ldots, x_n$ such that the manifold $S_{x_0} = \{ q \in V | C_{x_0}(q) = x_1(x_0), i = 1, \ldots, n \}$ contains $R^k_{x_0}(x_0)$ (where $R^k_{x_0}(x_0)$ is the reachable set from $x_0$ at time $T > 0$, following the trajectories which remain in the neighborhood $V$ of $x_0$ for $t \leq T$, and then $R^k_{x_0}(x_0) := \cup_{t \leq T} R^k_{x_0}(x_0, t)$) for any neighborhood $V \subset V$ of $x_0$ and for all $T > 0$. This implies that all $q \in V$ such that $q \notin S_{x_0}$ are not asymptotically reachable from $x_0$ on $V$, and thus the local asymptotic reachability from $x_0$ is contradicted.

Our main aim is now to relate the positivity definiteness and well-definedness of the controllability function to the accessibility rank condition.

Lemma 3.3 Assume that $f$ is asymptotically stable on a neighborhood $W$ of $0$. Then the controllability function $L_c(x)$ is smooth, finite and satisfies $L_c(x) > 0$ for $x \in W$, $x \neq 0$, if and only if the system (1) is asymptotically reachable from $0$ on $W$.

Proof The result follows immediately from the definition of asymptotic reachability and Theorem 2.4. □

The previous two results, combined with Proposition 3.12 and Corollary 3.13 of [7], give rise to the following corollary:

Corollary 3.4 Assume that the accessibility distribution $C$ has constant dimension about $0$, and assume that $f$ is locally asymptotically stable. If there exists a neighborhood $W$ of $0$ such that the controllability function $L_c(x)$ is smooth, finite and satisfies $L_c(x) > 0$ for $x \in V$, $x \neq 0$, for all $V \subset W$, then $\dim(C(0)) = n$. □

Remark 3.5 The above corollary is restricted by local requirements on $L_c$, since we need local asymptotic reachability from $0$ in order to use Theorem 3.2. Only asymptotic reachability on a neighborhood $W$ of $0$ does not suffice. An example of a smooth system that is asymptotically reachable on a neighborhood $W$ of $0$ and that is not locally accessible is easy to construct. However, if we assume that the system (1) is analytic, then we can relax the local requirements on $L_c$ to requirements on a neighborhood $W$ of $0$. This is due to the fact that asymptotic reachability from $x_0$ implies local accessibility from $x_0$ for analytic systems, e.g., [15]. Analyticity is actually not a strong restriction in our setting, i.e., it is also an assumption for the realization theory presented in the previous section.

Corollary 3.6 Let the system (1) be analytic. Assume that the accessibility distribution $C$ has constant dimension about $0$, and assume that $f$ is asymptotically stable on a neighborhood $W$ of $0$. If the controllability function $L_c(x)$ is smooth, finite and satisfies $L_c(x) > 0$ for $x \in W$, $x \neq 0$, then $\dim(C(0)) = n$. □

So far, we have been concentrating on the concept of local accessibility. However, for state space analysis of similarity invariants, which is closely related to the analysis of this paper, see e.g. [14], the nonlinear counterpart of the Kalman decomposition is considered. Therefore, it is also of interest to consider the concept of local strong accessibility. The local strong accessibility version of Theorem 3.2 is given below.

Theorem 3.7 Assume that the strong accessibility distribution $C_0$ has constant dimension about $x_0$. Then local asymptotic reachability from $x_0$ implies that the system is locally strongly accessible from $x_0$.

Proof Suppose that the system is not locally strongly accessible from $x_0$, then we know from standard results in the literature (e.g., [7]) that $\dim(C_0(x_0)) = k < n$. Hence from Proposition 3.12 in [7] we know that there are two possibilities:

(i) If $f(x_0) \notin C_0(x_0)$, then it follows similar to the proof of Theorem 3.2.

(ii) If $f(x_0) \notin C_0(x_0)$, then by continuity $f(q) \notin C_0(q)$ for all $q \in \bar{U}$, $U \subset W$ neighborhood of $x_0$, and $\dim(C(q)) = \dim(C_0(q)) + 1$ for all $q \in U$. In this case, we can select the coordinates $\tilde{x}_1, \ldots, \tilde{x}_n$ in such a way that $S^m_{x_0} = \{ q \in \bar{U} | \tilde{x}_1 + 1(q) = T, \ldots, \tilde{x}_n(q) = 0 \}$ contains $R^k_{x_0}(x_0, t)$ for any $T > 0$. Again, we have two cases: (a) If $\dim(C_0(x_0)) = n - 1$, then this implies that all $q \in U$ such that $q \notin S^m_{x_0}$ are not asymptotically reachable from $x_0$ on $V$, and thus the local asymptotic reachability from $x_0$ is contradicted. (b) If $\dim(C_0(x_0)) = n - 1$, then all $q \in U$ such that $\tilde{x}_n = -K, K > 0$, are not asymptotically reachable from $x_0$ on $U$. This concludes the proof. □

This theorem gives rise to corollaries similar to Corollary 3.4 and 3.6, except with accessibility replaced by strong accessibility.

3.2 Observability

For the observability counterpart of the previous section we consider the observability functions as defined in (3) and (6). We begin with the observability function in (3) for which zero-state observability plays an important role.

Lemma 3.8 Assume that the observability function (3) is smooth and finite for system (1) on a neighborhood $W$ of $0$. Then $L_0(x) > 0$ for $x \in W$, $x \neq 0$ implies that the system (1) is zero-state observable on $W$.

Proof Assume that the system is not zero-state observable on $W$. Then there exists a trajectory $x(t) = \Psi(t, 0, x_0, u) \in W, t \geq 0$ such that $x(t) \neq 0$ for $0 \leq t \leq \bar{t}$ for some $0 < \bar{t} < \infty$ and such that $h(x(t)) = 0$, $u(t) = 0, \forall t \geq 0$. Hence, by definition of $L_0$, $L_0(x(t)) = 0$ for all $t$, which gives the contradiction with the positivity of $L_0$, and thus proves the lemma.

In the following lemma, we present the relationship between local zero-state observability and local observability at $0$.

Lemma 3.9 Assume that the zero-state observability codistribution has constant dimension about $0$. Then local zero-state observability implies local observability at $0$. □
Proof Along the same lines as for local observability in [7], it can be proven that local zero-state observability implies that $\dim dO_0(0) = n$. This implies that $\dim dO(0) = n$, and by Theorem 2.8 it follows directly that the system is locally observable at 0.

Motivated by the minimality conditions of Theorem 2.19, we obtain the following corollary.

**Corollary 3.10** Assume that the zero-observability codistribution $dO_0$ has constant dimension about 0. If the observability function (3) is smooth, finite and satisfies $L_{\omega}(x) > 0, x \in W, x \neq 0$, then $\dim dO(0) = n$.

In the event that system (1) is not zero-state observable, it still may be locally observable at 0. In which case the natural observability function given in Definition 2.12 gives the following result analogous to Lemma 3.8.

**Lemma 3.11** Let $L^N_{\omega}(x)$ be the natural observability function (6) for some $\alpha > 0$. Assume that $L^N_{\omega}(x)$ is smooth and finite for system (1) on a neighborhood $W$ of 0. Then $L^N_{\omega}(x) > 0$ for $x \in W, x \neq 0$ implies that the system (1) is locally observable at 0 with respect to $B_a$.

Proof Assume that the system (1) is not locally observable at 0 with respect to $B_a$. Then the corresponding homogeneous system is also not locally observable at 0 with respect to $B_a$. Hence there exists an initial state $x_0 \neq 0$ such that $h(\phi(t,0,0,u)) = h(\phi(t,0,x_0,u), t \geq 0, \forall u \in B_a$, where $\phi(\cdot)$ denotes the solution to homogeneous system. By definition of the natural observability function, we have that $L^N_{\omega}(0) = 0$ and by the positivity of $L^N_{\omega}$ it follows that $L^N_{\omega}(x_0) > 0$. However, from equation (6) it follows immediately that the maximum over $u \in B_a$ for both states 0 and $x_0$ results in the same optimal input $u$. This implies that $L_{\omega}(0) = L_{\omega}(x_0)$, and yields the desired contradiction to prove the lemma.

This lemma gives the analogue of Corollary 3.10 in terms of the general observability function as follows.

**Corollary 3.12** Assume that the observability codistribution $dO$ has constant dimension about 0. If the natural observability function (6) is smooth, finite and satisfies $L^N_{\omega}(x) > 0$ for $x \in W, x \neq 0$, then $\dim dO(0) = n$.

Remark 3.13 We may now compare the results of this section to the previous section. It is clear that they do not completely follow along similar or "dual" lines. Specifically, the results related to the observability functions as given by (3) and (6) are given in terms of the zero-state observability and observability rank condition, respectively. Starting with the rank conditions the converse of these results also holds by the Theorems 2.2, 2.8, and 2.10. However, for the controllability function, we are considering asymptotic reachability which implies local accessibility, and which can be related to the accessibility rank condition. The reverse direction is far less clear in this case, mainly because accessibility from 0 is not sufficient for asymptotic reachability from 0. However, if asymptotic reachability can somehow be assumed for a given system, then the converse of these results also follows for the controllability function.

4 Sufficient conditions for minimality

Briefly summarized we have obtained the following main result.

**Theorem 4.1** Assume that the observability codistribution $dO$ (or the zero-observability codistribution $dO_0$, respectively) and the accessibility distribution $C$ of a system $(f,g,h)$ each have constant dimension about 0. Furthermore, assume that the analytic system $(f,g,h)$ is a realization of the formal power series $c$. Then, if $0 < L_{\omega}(x) < \infty$ and $0 < L^N_{\omega}(x) < \infty$ (or $0 < L_{\omega}(x) < \infty$ for $x \in W, x \neq 0$, then $(f,g,h)$ is a minimal realization of $c$.

The necessity of these conditions is not obtained due to the fact that, contrary to the linear case, accessibility and controllability are not equivalent in general. Only under additional assumptions can a converse result be obtained.

The use for and relation with similarity invariants for balanced realizations of nonlinear systems may be found in [14].

Acknowledgement

This research was supported in part by the North Atlantic Treaty Organization through the NATO Collaborative Research Grant CRG-97113.

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