Singular Point-like Perturbations of the Bessel Operator in a Pontryagin Space

Aad Dijksma
Department of Mathematics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands
E-mail: a.dijksma@math.rug.nl

and

Yuri Shondin
Department of Physics, Pedagogical State University, Str. Ulyanova 1, Nizhny Novgorod 603600, Russia
E-mail: shondin@shmath.nnov.ru

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The spectral problem for the Bessel equation of order \( \nu \) on \((0, \infty)\) in the case \(0 < \nu < 1\) is closely related to the Nevanlinna function \( Q(z) = -\pi (z^2 + 2 \sin \nu \pi) \). If \( \nu > 1 \) and \( \nu \neq 2, 3, \ldots \), this function belongs to the generalized Nevanlinna class \( N_{m} \), \( m = \left[ \frac{\nu + 1}{2} \right] \). A natural question appears: To what spectral problem does this function correspond? We answer this and related questions using Pontryagin space operator realizations of suitable singular point-like perturbations of the Bessel operator. In this paper we discuss the spectra of these realizations and we derive eigenfunction expansions via related wave operators.

INTRODUCTION

Consider the Bessel differential expression

\[
\ell_\nu = \frac{d^2}{dy^2} + \frac{\nu^2 - 1/4}{y^2}, \quad y \in \mathbb{R}^+ := (0, \infty),
\]

(0.1)
of order \( \nu \) in the Hilbert space \( \mathcal{H}_0 = L^2(\mathbb{R}^+) \) with inner product

\[
\langle f, g \rangle_0 = \int_{\mathbb{R}^+} f(y) g(y)^* \, dy, \quad f, g \in \mathcal{H}_0
\]

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(* is used to denote the complex conjugate of a complex number as well as the adjoint of an operator). In this paper we study spectral problems associated with $\ell_v$ for $v > 1$ and $v \neq 2, 3, \ldots$, using methods developed in [S1, DT, S2, DLSZ].

To formulate these problems we first recall some results for the case $v \in (0, 1)$. Then (see, for example, [AG]) the minimal operator realization $S$ of $\ell_v$ in $H_0$ is symmetric, has defect numbers 1, 1, and the family of all canonical selfadjoint extensions of $S$ admits a one-parameter parametrization. The continuous spectrum of these extensions covers the positive real axis with multiplicity one and the corresponding generalized eigenfunctions (solutions of $\ell_v e(y, \lambda) = \lambda e(y, \lambda)$) are given by

$$ e(y, \lambda) = c(\lambda) \sqrt{y} (\cos \pi J_v(y \sqrt{\lambda}) + \sin \pi J_v(y \sqrt{\lambda})), \quad y, \lambda \in \mathbb{R}^+, \tag{0.2} $$

where $c(\lambda)$ is a suitable normalizing factor, $J_v$ is the Bessel function of order $v$, and $\pi \in [0, \pi)$ is the parameter describing the extensions; see [T] and Section 5 below. If we set $t = \pi \cot \pi/(2 \sin \pi v)$ then the one-parameter family of canonical selfadjoint extensions can also be described as the family of selfadjoint operators $H_t$, $t \in \mathbb{R} \cup \{\infty\}$, whose resolvents are given by M. G. Krein’s formula

$$ (H_t - z)^{-1} = (A_0 - z)^{-1} - \varphi(z) (Q(z) - t)^{-1} \langle \cdot, \varphi(z^*) \rangle_0. \tag{0.3} $$

Here (i) $A_0$ is the unique selfadjoint extension for which the generalized eigenfunctions are proportional to $-y J_v(y \sqrt{\lambda})$, (ii) the function $\varphi(z) = \varphi(\cdot, z)$ belongs to $\ker(S^* - z)$ and is given by

$$ \varphi(y, z) = \sqrt{y} (-z)^{v/2} K_v(y \sqrt{-z}), \quad K_v(y) = i \frac{\pi}{2} e^{i\pi/2} H^{(1)}_v(iy), \tag{0.4} $$

where $H^{(1)}_v(y)$ is the Hankel function of the first kind of order $v$, so $K_v(y)$ is the modified Bessel function of the third kind of order $v$ (also called the Basset or MacDonald function, see [E, LU]); and (iii) the function

$$ Q(z) = -\frac{\pi}{2} \frac{(-z)^{v}}{\sin \pi v} \tag{0.5} $$

is the so-called $Q$-function associated with $S$ and its canonical extension $A_0$. Its defining relation

$$ \frac{Q(z) - Q(w)^*}{z - w^*} = \langle \varphi(z), \varphi(w) \rangle_0 \tag{0.6} $$
determines \( Q(z) \) up to a real constant and corresponds to the formula (see [LU, p. 325, formula (8)])

\[
\frac{\pi}{2 \sin \pi \nu} \frac{a^{2v} - b^{2v}}{a^2 - b^2} = (ab)^\nu \int_0^\infty K_v(a'y) K_v(by) y \, dy.
\]

(0.7)

It follows from (0.6) that \( Q(z) \) is a Nevanlinna function, that is, it is holomorphic on \( \mathbb{C}^+ \) and for \( z \in \mathbb{C}^+ \), \( \text{Im} \, Q(z) \geq 0 \) and \( Q(z^*) = Q(z)^* \).

These results differ essentially from analogous results for the case \( \nu > 1 \): (a) The minimal realization of \( \ell \) in \( \mathcal{H}_0 \) is itself selfadjoint (see, for example, [AG]). So there are no other selfadjoint realizations of \( \ell \) in \( \mathcal{H}_0 \). In the sequel we shall denote this operator by \( A_0 \) also. (b) Since \( J_{-\nu}(y) \sim y^{-\nu} \) for \( y \) in a vicinity of zero, the functions \( e(y, \lambda) \), when \( \lambda \neq 0 \), cannot appear as generalized eigenfunctions of any selfadjoint operator in \( \mathcal{H}_0 \). (c) As will be shown in Section 2, the function \( Q(z) \) defined by (0.5) is not a Nevanlinna function in the sense described above, but belongs to the generalized Nevanlinna class \( N_m \) of functions with \( m = \lfloor \frac{\nu + 1}{2} \rfloor \) negative squares; see [KL]. Here \( \lfloor r \rfloor \) stands for the integral part of the real number \( r \geq 0 \); It is the largest integer \( \leq r \); its residu modulo 1 is defined as \( \{ r \} = r - \lfloor r \rfloor \).

So natural questions appear and in this note we will try to answer them: For which spectral problem are the functions \( e(y, \lambda) \) or adaptations of these functions generalized eigenfunctions and what are the corresponding expansions? What is the connection of this problem with \( Q \in N_m \)?

To indicate the direction in which we look for the answers to these questions, we again consider the case \( 0 < \nu < 1 \). It is a tradition in mathematical physics (see [AGHH]) to call the extensions \( \mathcal{H} \) determined by (0.3) singular perturbations of \( A_0 \) in \( \mathcal{H}_0 \). In general, these are operator realizations of formal expressions of the form

\[
A_0 + t^{-1} \langle \cdot, \chi \rangle_{A_0} \chi, \quad t \in \mathbb{R} \cup \{ \infty \},
\]

(0.8)

in which \( \chi \) is a generalized element from \( \mathcal{H}_{-\nu} \backslash \mathcal{H}_0 \), where \( \mathcal{H}_k, k = 0, \pm 1, \ldots \), are the scale spaces corresponding to \( A_0 \) which determine a rigging of \( \mathcal{H}_0 \) (we review this in Section 1). In the case of the Bessel operator with \( 0 < \nu < 1 \), we have in the sense of distributions,

\[
p(y) = (A_0 - \mu) \sqrt{y} (-\mu)^{\nu/2} K_{\nu/2}(y \sqrt{-\mu}),
\]

(0.9)
where $\mu$ is a point in $\mathbb{R}^- := (-\infty, 0)$. As $(\sqrt[2]{y} - \mu) \sqrt[2]{y} K_0(y \sqrt[2]{-\mu}) = 0$, the generalized function $\chi(y)$ is supported only in the point $y = 0$. This is why the $H'$'s are called singular point-like perturbations of $A_0$.

In the case $r > 1$, the function $\chi$ defined by (0.9) belongs to the set $\mathcal{H}_{-m-1} \setminus \mathcal{H}_{-m}$ and for formal expressions (0.8) with such $\chi$'s special realizations have been constructed in the papers [S1, DT, S2]. These realizations $H', t \in \mathbb{R} \cup \{ \infty \}$, are selfadjoint in a Pontryagin space $\Pi_m$ with (negative) index $m$. For $t \in \mathbb{R}$, $H'$ is an operator. But $H''$ is a selfadjoint relation with a nontrivial multivalued part. It plays a special role in the construction, namely a role similar to $A_0$ in $\mathcal{H}_0$ in the case $0 < r < 1$. For selfadjoint linear relations in Pontryagin spaces we refer to [AI, DS]. The answers to the questions posed above will be obtained by studying the usual spectral problems associated with the $H'$'s.

Similar questions but for the Laguerre operator are studied in the paper [D].

The rest of this paper consists of five sections and the references. In Section 1 we review the construction of the operators $H'$ in $\Pi_m$. We specialize in Section 2 to the model case, where $A_0$ is the operator in $L_2^2(\mathbb{R}^+)$ of multiplication by the independent variable $x$, say, and $\chi(x) = x^{\gamma/2}$, $\gamma \in \mathbb{C} \setminus \{0\}$. The construction of the realizations associated with the Bessel operator can be reduced to the model from Section 2 by using the Hankel transform. This we show in Section 3. In Section 4 we study the point spectrum of $H'$ and prove some results about related spectral shift functions. In Section 5 we derive the eigenfunction expansion associated with $H'$ by calculating the wave operators $W^\pm = W^\pm(H', A_0)$. These operators are integral operators and their kernels are the generalized eigenfunctions which appear in the expansion formulas. Finally, in Section 6 we determine a formula for the scattering operator $S' = W'^*W'$ associated with the model from Section 2.

Some of our results were announced by the second author at the M. G. Krein Conference in Odessa, 1997.

1. A PONTRYAGIN SPACE REALIZATION OF SINGULAR PERTURBATIONS OF A NONNEGATIVE OPERATOR

Consider an unbounded nonnegative selfadjoint operator $A_0$ in a Hilbert space $\mathcal{H}_0$ with inner product $\langle \cdot, \cdot \rangle_0$, a nonnegative integer $m$, and the corresponding scale of Hilbert spaces

$$\mathcal{H}_{m+1} \subset \mathcal{H}_m \subset \cdots \subset \mathcal{H}_{1} \subset \mathcal{H}_{0} \subset \mathcal{H}_{-1} \subset \cdots \subset \mathcal{H}_{-m} \subset \mathcal{H}_{-m-1},$$

(1.1)
where the inclusion mappings $\subseteq$ are contractions with a dense range. Recall (see [B]) that for $j = 1, 2, \ldots, m + 1$, $\mathcal{H}_j$ is the Hilbert space $\text{dom} A_0^j$ equipped with the inner product

$$\langle u, v \rangle_j = \langle (A_0 + 1)^j u, (A_0 + 1)^j v \rangle_0,$$  \hspace{1cm} u, v \in \mathcal{H}_j, \hspace{1cm} (1.2)$$

and that $\mathcal{H}_{-j}$ is the Hilbert space completion of $\mathcal{H}_0$ with respect to the norm

$$\|f\|_{-j} = \sup_{u \neq f \in \mathcal{H}_j} \frac{\langle f, u \rangle_0}{\|u\|_j}, \hspace{1cm} f \in \mathcal{H}_0.$$  \hspace{1cm} (1.1)$$

The scale (1.1) defines a rigging of $\mathcal{H}_0$ in the sense that the inner product $\langle f, u \rangle_0$ on $\mathcal{H}_0$ (viewed as a sesquilinear form on $\mathcal{H}_0 \times \mathcal{H}_0$) can be extended/restricted to a bounded sesquilinear form on $\mathcal{H}_{-j} \times \mathcal{H}_{-j}$ for each $f \in \mathcal{H}_{-j}$ there is a sequence $f_n \in \mathcal{H}_0$ converging to $f$ in the norm of $\mathcal{H}_{-j}$ such that for all $u \in \mathcal{H}_{-j}$ the limit

$$\langle f, u \rangle_0 := \lim_{n \to \infty} \langle f_n, u \rangle_0$$

exists (and is independent of the sequence $f_n$) and satisfies

$$\|\langle f, u \rangle_0\| \leq \|f\|_{-j} \|u\|_j, \hspace{1cm} f \in \mathcal{H}_{-j}, \hspace{1cm} u \in \mathcal{H}_j.$$  \hspace{1cm} (1.3)$$

Thus, alternatively, $\mathcal{H}_{-j}$ can be viewed as the space of antilinear functionals on $\mathcal{H}_j$, $j = 1, 2, \ldots, m + 1$. We set

$$\langle u, f \rangle_0 = \langle f, u \rangle_0*.$$  \hspace{1cm} (1.4)$$

By induction the resolvent $R_0(z) = (A_0 - z)^{-1}$ can be extended to a mapping from $\mathcal{H}_{-j-1}$ to $\mathcal{H}_{-j}$ via

$$\langle R_0(z) f, u \rangle_0 = \langle f, R_0(z^*) u \rangle_0, \hspace{1cm} f \in \mathcal{H}_{-j-1}, \hspace{1cm} u \in \mathcal{H}_j.$$  \hspace{1cm} (1.5)$$

This extended resolvent is continuous, satisfies the resolvent identity, and (1.2) holds true for negative $j$’s also.

In this section we describe the construction of an operator realization of the singular perturbations of $A_0$ formally represented as

$$A_0 + t^{-1} \langle \cdot, \chi \rangle_0 Z, \hspace{1cm} t \in \mathbb{R} \cup \{\infty\}, \hspace{1cm} (1.3)$$
where \( \chi \in \mathcal{H}_{m-1} \setminus \mathcal{H}_m \). This construction is a slight modification of the one in [S1, DT, S2]: we now use two auxiliary level points (\( \mu \) and \( \mu_0 \), see below) instead of one. This allows us to be more flexible when we apply the realization to the Bessel operator. When \( \chi \in \mathcal{H}_0 \) or when \( \chi \in \mathcal{H}_{-1} \) (the case \( m = 0 \)) the formal expressions (1.3) are realizable in the Hilbert space \( \mathcal{H}_0 \).

Indeed, in the first case they are rank one perturbations and in the second case they are interpreted (see, for example, [AGHH]) as the canonical selfadjoint extensions of the one-dimensional restriction \( S_0 \) of \( A_0 \),

\[
S_0 = A_0 \mid_{\text{dom } S_0}, \quad \text{dom } S_0 = \{ u \in \text{dom } A_0 \mid \langle u, \chi \rangle_0 = 0 \}.
\]

Evidently, \( S_0 \) is symmetric and has defect indices 1, 1.

If \( \chi \in \mathcal{H}_{-m-1} \setminus \mathcal{H}_{-m} \) and \( m \geq 1 \), \( S_0 \) becomes essentially selfadjoint in \( \mathcal{H}_0 \) and Berezin [BE] was the first to propose that non-trivial perturbations of \( A_0 \) generated by \( \chi \) should be realized in an indefinite metric space. To construct such a realization one associates with \( \chi \) a selfadjoint relation \( H^\infty \) in a Pontryagin space \( \Pi_m \) with (negative) index \( m \) and considers a suitable one-dimensional restriction \( S \) of \( H^\infty \). The realization or the interpretation of (1.3) is then by definition the family of all canonical selfadjoint extensions \( H^t \) of \( S \). The construction will be given below in three steps: first we build the space \( \Pi_m \), then the relation \( H^\infty \), and then we define \( S \) and the extensions \( H^t \).

For proofs of the statements we refer to the proofs of analogous results in the previously cited papers.

1. The Pontryagin Space \( \Pi_m \)

We fix two negative points \( \mu, \mu_0 \in \mathbb{R}^- \) and define the elements \( \chi_j \in \mathcal{H}_j \setminus \mathcal{H}_{j+1} \) by

\[
\chi_j = \begin{cases} 
(A_0 - \mu)^{-(m + 1 + j)} \chi, & j = -m, -m + 1, \ldots, -1, \\
(A_0 - \mu_0)^{-(j+1)} (A_0 - \mu)^{-m} \chi, & j = 0, 1, \ldots, m.
\end{cases}
\]

Often it is more convenient to change the order of the indices and to use the notation

\[
e_j = \chi_{j-m-1}, \quad j = 1, \ldots, 2m.
\]

Denote by \( \mathcal{P}_m \) the linear space

\[
\mathcal{P}_m = \mathcal{H}_m + \text{span}\{ e_1, e_2, \ldots, e_{2m} \}, \quad \text{direct sum},
\]
with elements written in the form

\[ f = f_m + \sum_{j=1}^{2m} F_j e_j, \quad f_m \in \mathcal{H}_m, \quad (F_j)_{j=1}^{2m} \in \mathbb{C}^{2m}. \]

On \( \mathcal{H}_m \) we define the inner product

\[ \langle f, f' \rangle = \langle f_m, f'_m \rangle_0 + \sum_{j=1}^{2m} (F_j \langle e_j, f'_m \rangle_0 + F_j^* \langle f_m, e_j \rangle_0) \]

\[ + \sum_{j,k=1}^{2m} g_{j,k} F_j F_k^*, \]

where real numbers \( g_{j,k} \) have been substituted for the formal quantities \( \langle e_j, e_k \rangle_0, j, k = 1, ..., 2m \), and these numbers are defined as

(a) If \( j + k \geq 2m + 2 \), then \( g_{j,k} = \langle e_j, e_k \rangle_0 \).

(b) If \( j, k = 1, ..., m \), then \( g_{j,k} = g_{j+k} \), where \( g_k, k = 2, ..., 2m \), are arbitrary real numbers.

(c) If \( 1 \leq j \leq m, m + 1 \leq k \leq 2m \) and \( j + k < 2m + 2 \), we first consider two cases.

(c)i The case \( \chi \in \mathcal{H}_{m-1/2} \setminus \mathcal{H}_m \). Then

\[ g_{m+1,m} = g_{m+1,m} = g_{2m+1} =: \langle (A_0 - \mu_0)^{-1/2} \chi_{-1}, (A_0 - \mu_0)^{-1/2} \chi_{-1} \rangle_0. \]

(c)ii The case \( \chi \in \mathcal{H}_{m-1} \setminus \mathcal{H}_{m-1/2} \). We choose an additional real parameter \( g_{2m+1} \) and define \( g_{m+1,m} = \mathbb{g}_m, m + 1 = 2m + 1 \).

The numbers \( g_{j,k} \) for the remaining values of \( j \) and \( k \) are defined by the rule

\[ g_{j+m,n} = g_{m+n,j} = \sum_{l=n}^{m-j+1} (l-1) \binom{l-1}{n-1} \langle e_m, e_{m+l} \rangle_0, \quad \sum_{i=2}^{m-j}(m-j) \binom{m-j}{n-i} \langle e_m, e_{m+i} \rangle_0, \quad n + j \leq m + 1. \]

The space \( \Pi_m \) is now defined as the space \( \Pi_m = \mathcal{H}_0 \oplus \mathbb{C}^m \oplus \mathbb{C}^m \) whose elements are column vectors of the form

\[ f = \text{column}(f_0, F_0, F), \quad f_0 \in \mathcal{H}_0, \quad F_0 \in \mathbb{C}^m, \quad F \in \mathbb{C}^m, \]
with inner product \( \langle \cdot , \cdot \rangle = \langle \cdot , G \cdot \rangle_{\mathcal{H}_0 \otimes \mathbb{C}^m \otimes \mathbb{C}^m} \), where the Gram operator \( G \) is of the form

\[
G = \begin{pmatrix}
1_0 & 0 \\
0 & G_m
\end{pmatrix}, \quad G_m = \begin{pmatrix}
0 & 1_m \\
m & G
\end{pmatrix},
\]

where the Gram operator \( G_m \) is of the form

\[
G_m = \begin{pmatrix}
0 & 1_m \\
m & G_m + 1
\end{pmatrix}, \quad G_m = \begin{pmatrix}
0 & 1_m \\
m & G_m
\end{pmatrix}.
\]

The mapping \( \tau_x \) defined by the rule

\[
\tau_x \left( f_m + \sum_{j=1}^{2m} F_j \varepsilon_j \right) = \begin{pmatrix}
f_m + \sum_{k=m+1}^{2m} F_k \varepsilon_k \\
\langle f_m, \varepsilon_j \rangle_0 + \sum_{k=m+1}^{2m} g_{j,k} F_k \rangle_j \sum_{j=1}^{m}
\end{pmatrix}
\]

maps \( \mathcal{P}_m \) isometrically in \( \Pi_m \) and has a dense range. Hence \( \mathcal{P}_m \) is a pre-

Pontryagin space and \( \Pi_m \) is its Pontryagin space completion.

2. The Selfadjoint Linear Relation \( H^\infty \) in \( \Pi_m \)

The operator \( A_0 \) is naturally embedded in \( \mathcal{P}_m \) as the operator \( \tilde{A}_0 \) defined on the domain

\[
\text{dom}(\tilde{A}_0) = \left\{ f = f_m + \sum_{j=1}^{2m} F_j \varepsilon_j \in \mathcal{P}_m \left| f_m = f_{m+1} + (A_0 - \mu)^{-1} e_{2m+1} F_{2m+1}, \right. \right. \\
\left. \left. f_{m+1} \in \mathcal{H}_{m+1}, F_{2m+1} \in \mathbb{C}, F_1 = 0 \right\}
\]

by

\[
\tilde{A}_0 f = A_0 f_{m+1} + \mu \langle A_0 - \mu \rangle^{-1} e_{2m+1} F_{2m+1} + \sum_{j=1}^{m} (F_{j+1} + \mu F_j) \varepsilon_j
\]

\[
+ \sum_{j=m+1}^{2m} (F_{j+1} + \mu F_j) \varepsilon_j.
\]

We define \( H^\infty \) as the closure in the Pontryagin space \( \Pi_m \) of \( \tau_x \tilde{A}_0 \tau_x^{-1} \), the isometric copy under \( \tau_x \) of \( \tilde{A}_0 \) in \( \Pi_m \). It can be shown that \( H^\infty \) is a selfadjoint
linear relation in $\Pi_m$ and its multivalued part, that is, its eigenspace corresponding to the eigenvalue $\infty$, is the subspace

$$H^\infty(0) = \text{span}\{\text{column}(0, e_1, 0)\},$$

where $e_j$ is the $j$th standard unit vector in $C^m$, $j = 1, \ldots, m$. The relation $H^\infty$ can be considered as a kind of "lifting" of $A_0$ from $\mathcal{K}_0$ to $\Pi_m$. Furthermore, it can be shown that $\rho(H^\infty) = \rho(A_0)$ and that the resolvent operator $R^\infty(z) = (H^\infty - z)^{-1}$ is given by

$$f R^\infty(z) = \left( \begin{array}{c} \sum_{j=1}^m (z - \mu)^{-1-j-1} F_{0j} + q_j(f, z) \end{array} \right) \left( \begin{array}{c} p_j(z) \end{array} \right)^m_{j=1},$$

where $f \in \mathcal{K}_0$, $(F_{0j})^m_{j=1}, (F_j)^m_{j=1} \in C^m$, $R_0(z) = (A_0 - z)^{-1}$, and

$$p_j(z) = \sum_{k=1}^{j-1} F_k(z - \mu)^{-j-k}, \quad j = 1, \ldots, m + 1$$

$$q_j(f, z) = (z - \mu)^{m-j} \langle R_0(z) f, e_m \rangle_0 + p_m(z) \langle R_0(z) e_m, e_j \rangle_0, \quad j = 1, \ldots, m.$$

Evidently, $H^\infty = \{ \{ R^\infty(z) \hat{f}, \hat{f} + z R^\infty(z) \hat{f} \} \mid \hat{f} \in \Pi_m \}$, where the set on the right-hand side is independent of $z \in \rho(H^\infty)$.

3. The Selfadjoint Operators $H'$ in $\Pi_m$

These operators are defined as the canonical selfadjoint extensions of a one-dimensional restriction $S$ of $H^\infty$. $S$ is chosen such that the defect subspace $\ker(S^* - \mu) = \text{ran}(S - \mu)^\perp$ is the one-dimensional space generated by the vector $\tau_x e_1 = \text{column}(0, 0, e_1)$. Hence $S$ and its adjoint $S^*$ in $\Pi_m$ are given by

$$S = H^\infty \cap \{ \{ \tau_x e_1, \mu \tau_x e_1 \} \}_{\perp}, \quad S^* = H^\infty + \{ \{ \tau_x e_1, \mu \tau_x e_1 \} \}.$$

The canonical selfadjoint extension $H'$ of $S$ in $\Pi_m$ is a part of $S^*$ and can be written in the form

$$H' = \{ \{ f_\infty + F_1, f_\infty' + \mu F_1 \} \in S^* \mid \{ f_\infty, f_\infty' \} \in H^\infty, \quad F_1 \in C \}, \quad \Gamma'(f_\infty' - \mu f_\infty) = t F_1,$$

where

$$\Gamma: C \to \ker(S^* - \mu) \subset \Pi_m, \quad \Gamma e = \tau_x e_1.$$
The parameter \( t \in \mathbb{R} \cup \{ \infty \} \) distinguishes between the different canonical self-adjoint extensions of \( S \), and every canonical selfadjoint extension of \( S \) coincides with one of the \( H^t \)'s. \( H^\infty \) is the unique multivalued canonical extension, that is, linear relation rather than operator. The family of operators \( H^t \); \( t \in \mathbb{R} \cup \{ \infty \} \), is by definition the operator realization of the formal expression (1.3).

We now describe Krein's formula for the resolvents \( R(z) = (H^t - z)^{-1} \) of \( H^t \). For \( z \in \rho(A_0) \), we define the operators \( \Gamma_z : \mathbb{C} \rightarrow \text{ker}(S^* - z) \subseteq \mathcal{H}_m \) by \( \Gamma_z = (I + (z - \mu) R^\infty(z)) \Gamma \). They are related via the formula

\[
\Gamma_z - \Gamma_w = (z - w) R^\infty(z) \Gamma_w, \quad z, w \in \rho(A_0).
\]

By definition the \( Q \)-function \( Q(z) \) associated with the symmetric operator \( S \) and its selfadjoint extension \( H^\infty \) in \( \mathcal{H}_m \) is determined by the formula

\[
\frac{Q(z) - Q(w)^*}{z - w^*} = \Gamma_{z, w}^* \Gamma_{z, w}.
\] (1.5)

If we set \( \varphi(z) = \Gamma_z^* \) 1, then the righthand side here can be written as \( \langle \varphi(z), \varphi(w) \rangle \). The relation (1.5) defines \( Q(z) \) up to a real constant; we write \( g_1 \) for \( Q(\mu) \) (so \( g_1 \) is in fact arbitrary) and obtain

\[
Q(z) = (z - \mu)^{2m} Q_0(z) + p_{2m-1}(z),
\] (1.6)

where

\[
Q_0(z) = (z - \mu_0)^2 \langle R_0(z) \chi_{-1}, \chi_0 \rangle_{\mathcal{H}_m} + g_{2m+1},
\] (1.7)

and \( p_{2m-1} \) is the selfadjoint polynomial \( (p_{2m-1}(z)^* = p_{2m-1}(z)) \) of degree at most \( 2m - 1 \) given by

\[
p_{2m-1}(z) = \sum_{j=0}^{2m-1} (z - \mu)^j g_{j+1}.
\] (1.8)

In the next section we will use that

\[
\lim_{x \to -\infty} \frac{Q(x)}{x^{2m}} = \begin{cases} 0, & \text{if } x \in \mathcal{H}_{-m-1/2} \setminus \mathcal{H}_{-m}, \\ -\infty, & \text{if } x \in \mathcal{H}_{-m-1} \setminus \mathcal{H}_{-m-1/2}. \end{cases}
\] (1.9)

To show this we consider the function \( Q_0(x) \) from (1.7) for \( z = x \in \mathbb{R}^- \). Since

\[
\frac{d}{dx} Q_0(x) = \langle R_0(x) \chi_{-1}, R_0(x) \chi_{-1} \rangle_0 > 0,
\]
\( Q_d(x) \) is monotonically decreasing when \( x \rightarrow -\infty \), hence the (finite or infinite) limit \( \lim_{x \rightarrow -\infty} Q_d(x) \) exists. Introducing the element \( \mathcal{X}_{-1/2} := (A_0 - \mu_0)^{-1/2} \mathcal{X}^{-1} \) which belongs to \( \mathcal{H}_{-1/2} \), we can rewrite \( Q_d(x) \) in the form

\[
Q_d(x) = \left\langle (R_d(x) - R_d(\mu_0)) \mathcal{X}^{-1}, \mathcal{X}^{-1} \right\rangle_0 + \varepsilon_{2m+1}
\]

\[
= \left\langle (A_0 - \mu_0)(R_d(x) - R_d(\mu_0)) \mathcal{X}^{-1}, \mathcal{X}^{-1} \right\rangle_0 + \varepsilon_{2m+1}.
\]

Hence, if \( \mathcal{X}_{-1/2} \in \mathcal{M}_0 \) then \( \lim_{x \rightarrow -\infty} Q_d(x) = -\left\langle \mathcal{X}^{-1}, \mathcal{X}^{-1} \right\rangle_0 + \varepsilon_{2m+1} \), on the other hand if \( \mathcal{X}_{-1/2} \in \mathcal{H}_{-1/2} \setminus \mathcal{M}_0 \) then \( \lim_{x \rightarrow -\infty} Q_d(x) = -\infty \). Applying the rules (c), and (c)_o for choosing the number \( \varepsilon_{2m+1} \) we obtain that

\[
\lim_{x \rightarrow -\infty} Q_d(x) = \begin{cases} 
0, & \text{if } \mathcal{X} \in \mathcal{H}_{-m-1/2} \setminus \mathcal{H}_{-m}, \\
-\infty, & \text{if } \mathcal{X} \in \mathcal{H}_{-m-1} \setminus \mathcal{H}_{-m-1/2}.
\end{cases}
\]

This together with the formula (1.6) for \( Q(z) \) implies (1.9).

The function \( Q_d(z) \) is a Nevanlinna function as defined in the Introduction and \( Q_d(\mu_0) = g_{2m+1} \). But the function \( Q(z) \) belongs to the generalized Nevanlinna class \( \mathcal{N}_m \) of functions with \( m \) negative squares. This means that (a) \( Q(z) \) is meromorphic on \( \mathbb{C} \setminus \mathbb{R} \), (b) \( Q(z^*) = Q(z)^* \) for all \( z \in \rho(Q) \), the set of all points from \( \mathbb{C} \setminus \mathbb{R} \) in which \( Q \) is holomorphic, and (c) the kernel

\[
K_d(z, w) = \frac{Q(z) - Q(w)^*}{z - w^*},
\]

has \( m \) negative squares: All hermitian matrices of the form \( (K_d(z_j, z_k))^{n}_{j,k=1} \), where \( n \) is an arbitrary integer \( \geq 1 \) and the \( z_j \)'s are arbitrary points in \( \rho(Q) \), have at most and at least one of them has exactly \( m \) negative eigenvalues counted with multiplicity; see [KL]. (The class \( \mathcal{N}_0 \) coincides with the class of Nevanlinna functions; thus for example, the function \( Q_d(z) \) belongs to \( \mathcal{N}_0 \).) This property of \( K_d(z, w) \) follows from the representation (1.6) by comparing this representation with the integral representation of \( \mathcal{N}_m \)-functions given in [DL], but also directly from (1.5) and the fact that without loss of generality we may assume that \( \{ \mathcal{M}_0, A_0, \mathcal{X} \} \) is minimal in the sense that

\[
\mathcal{M}_0 = \text{span} \{ \text{ran} \Gamma_z | z \in \rho(A_0) \};
\]

for a detailed discussion, see [DLSZ, Subsect. 3.6].

M. G. Krein’s formula for the resolvent of \( H' \) takes the form

\[
R'(z) = R^*(z) - \frac{\Gamma_z \Gamma_z^*}{Q(z) - Q(\mu) - l}, \quad z \in \rho(A_0) \cap \rho(H').
\]
Indeed, since ran($H' - z$) = $\Pi_\infty$, every element in $\Pi_\infty$ can be written in the form $f_\infty - z f'_\infty + (\mu - z) F_1$ with $\{f_\infty + \mu F_1, f'_\infty + \mu F_1\} \in H'$, and we have

$$R'(z)(f_\infty - z f'_\infty + (\mu - z) F_1) = f_\infty + \mu F_1,$$

$$R^\ast(z)(f_\infty - z f'_\infty + (\mu - z) F_1) = f_\infty + (\Gamma - \Gamma_\infty) F_1,$$

and

$$\Gamma^*_2 \Gamma^*_2(z)(f_\infty - z f'_\infty + (\mu - z) F_1)
\quad= \Gamma^*_2(I + (z - \mu) R^\ast(z))(f_\infty - z f'_\infty + \Gamma^*_2 \Gamma F_1)
\quad= \Gamma^*_2(I + (z - \mu) R^\ast(z)) + (Q(\mu) - Q(z^\ast) F_1)
\quad= - \Gamma^*_2(Q(z) - Q(\mu) - t) F_1,$$

and these equalities readily imply (1.10). Note that the numerator on the righthand side of (1.14) is a form of rank 1,

$$\Gamma^*_2(z) = \langle \cdot, \varphi(z^\ast) \rangle \varphi(z),$$

where, as above, $\varphi(z) = \Gamma_\infty 1.$

There is an alternative interpretation of the operators $H'$: $H'$ is a non-canonical selfadjoint extension with exit to $\Pi_\infty$ of the one-dimensional restriction $S_0$ of $A_0$ in $\mathcal{H}_0$ to the domain

$$\text{dom}(S_0) = \{ f \in \text{dom}(A_0) | \langle f, e_0 \rangle_0 = 0 \}.$$

Clearly, $S_0$ is symmetric and has defect indices 1, 1. The extension $H'$ is determined by (that is, can be recovered up to unitary equivalence from) its generalized resolvent $P_{\mathcal{H}_0} R(z)|_{\mathcal{H}_0}$; see, for example, [DLS]. Krein’s formula for generalized resolvents in this case reads as

$$P_{\mathcal{H}_0} R(z)|_{\mathcal{H}_0} = R_0(z) - \frac{\Gamma^*_0 \Gamma^*_0 + \Gamma^*_0}{Q_0(z) + t(z)}, \quad z \in \rho(A_0) \cap \rho(H').$$

Here (i) the function $Q_0(z)$ is given by (1.7) (it is the $Q$-function associated with $S_0$ and its extension $A_0$); (ii) $\Gamma^*_0 : C \rightarrow \ker(S_0^2 - z) \subset \mathcal{H}_0$ is the operator function $\Gamma^*_0 c = \varphi_0(z) c$ with

$$\varphi_0(z) = (A_0 - z)^{-1} \chi_{-1} = (I + (z - \mu_0)(A_0 - z)^{-1}) \chi_0;$$

and (iii) $t(z)$ is the rational $N_m$-function

$$t(z) = \frac{P_{2m-1}(z) - Q(\mu) - t}{(z - \mu)^{2m}}.$$
2. A MODEL FOR SINGULAR PERTURBATIONS OF A NONNEGATIVE OPERATOR

In this section we consider a special case of the results in Section 1. We refer to this case as the model. We take \( H_0 = L^2(\mathbb{R}^+) \) and consider in \( H_0 \) the operator \( A_0 \) of multiplication by the independent variable \( A_0 h(x) = xh(x) \).

We take for \( \gamma \) the function \( \gamma(x) = \gamma x^{v/2} \), where \( v \) is a nonnegative real number and \( \gamma \in \mathbb{C}\setminus \{0\} \); in the next sections \( \gamma \) will always be equal to \( 1/\sqrt{2} \). The action of \( \gamma \) is that of a distribution, that is,

\[
\langle h, \gamma \rangle_0 = \gamma \left. \lim_{\epsilon \to 0} \int_0^\infty x^{v/2} h(x) \, dx \right|_{x = \epsilon}.
\]

It belongs to \( \mathcal{H}_{m-1} \backslash \mathcal{H}_m \) if and only if \( m = \lceil \frac{v}{2} \rceil \). More specifically, \( \gamma \in \mathcal{H}_{m-1/2} \backslash \mathcal{H}_m \) if \( \lceil v \rceil = 2m - 1 \) and \( \gamma \in \mathcal{H}_{m-1} \backslash \mathcal{H}_m - 1/2 \) if \( \lceil v \rceil = 2m \).

Only if \( 0 \leq v < 1 \) the expression (1.3) can be given a meaning in the framework of Hilbert space extension theory. We are interested in the case \( v \geq 1 \), that is, \( m \geq 1 \), and for this case we want to describe the singular perturbations \( H' \) of \( A_0 \) corresponding to the formal expression (1.3). In general, the model is described by \( t \) and the \( \lceil v \rceil \) real parameters, \( g_{k+1}, \ k = 1, 2, \ldots, \lceil v \rceil \).

The numbers \( g_{k+1} \) determine the inner product of \( \Pi_m \) via the Gram operator (1.4) and can be expressed in terms of the \( Q \)-function \( Q(z) \) by the formula

\[
g_{k+1} = \frac{1}{k!} \frac{d^k}{dz^k} Q(z)|_{z = \mu}, \quad k = 1, 2, \ldots, m - 1.
\]

We calculate the \( Q \)-function of the model.

**Proposition 2.1.** The \( Q \)-function associated with \( H^\infty \) and its one-dimensional restriction \( S \) in \( \Pi_m \), determined by \( \gamma(x) = \gamma x^{v/2} \), belongs to the class \( N_m, m = \lceil \frac{v}{2} \rceil \), and is for \( z \in \mathbb{C}^\ast \) given by

\[
Q(z) = \begin{cases} 
-\pi |\gamma|^2 \frac{(-z)v}{\sin \pi v} + \hat{p}_v(z), & \text{if } v \neq 0, 1, 2, \ldots, \\
-\pi |\gamma|^2 \frac{z}{\sin \pi v} \ln(-z) + \hat{p}_v(z), & \text{if } v = k = 0, 1, 2, \ldots,
\end{cases}
\]

where \( \hat{p}_v(z) \) is a selfadjoint polynomial of degree at most \( n \), whose coefficients can be expressed in terms of the \( g_{k+1} \)'s via formula (2.1). Here

\[
\frac{(-z)^v}{\sin \pi v} = \left( \frac{(-\gamma)^v}{\sin \pi v} \right) z^v,
\]

where the first factor on the righthand side is a Nevanlinna function.
The last remark allows us to write for 
\[ z = \rho e^{i\varphi}, \; \rho \in \mathbb{R}^+, \; 0 < \varphi < 2\pi, \]
\[ \frac{(-z)^v}{\sin \pi v} = \frac{\rho^v}{\sin \pi v} e^{i(1-v)\pi + iv\varphi} = \frac{\rho^v}{\sin \pi v} e^{i\varphi + iv\pi - \pi}. \]

**Proof.** That \( Q(z) \in N_m \) has been shown already in the previous section. Substituting \( \chi = \gamma x^{\alpha/2} \) in the expression (1.6) for \( Q \) we obtain
\[ Q(z) = |\gamma|^2 (z - \mu)^{2m} (z - \mu_0) \int_0^\infty \frac{x^v dx}{(x-z)(x-\mu)^2m (x-\mu_0)} \]
\[ + (z - \mu)^{2m} g_{2m+1} + p_{2m-1}(z). \]

To calculate this function we use analytic continuation in the parameter \( v \) of the function
\[ F(v, z) = \int_0^\infty \frac{x^v dx}{x-z}, \]
which is well defined and analytic in both variables for \( z \in \mathbb{C} \setminus \mathbb{R}^+ \) and \(-1 < \text{Re } v < 0\). For \( z \in \mathbb{R}^- \) and \(-1 < \text{Re } v < 0\) we have
\[ F(v, z) = -\pi \frac{(-z)^v}{\sin \pi v}, \quad (2.4) \]
and the righthand side is analytic in \( z \in \mathbb{C} \setminus \mathbb{R}^+ \) and \( v \in \mathbb{C} \setminus \{0, \pm 1, \pm 2, ...\} \).

We employ the same analytic continuation method as in [GS], but instead of the Taylor expansion we use the expansion obtained by repeatedly invoking the resolvent identity,
\[ \frac{1}{x-z} = \frac{1}{x-\mu} + \frac{z-\mu}{(x-z)(x-\mu)}. \]

We get
\[ F(v, z) = (z - \mu) \int_0^\infty \frac{x^v dx}{(x-z)(x-\mu)} + a_1(v) \]
\[ = (z - \mu)^{2m} \int_0^\infty \frac{x^v dx}{(x-z)(x-\mu)^{2m}} + \sum_{j=0}^{2m-1} a_j(v)(z-\mu)^j \]
\[ = (z - \mu)^{2m} (z-\mu_0) \int_0^\infty \frac{x^v dx}{(x-z)(x-\mu)^{2m} (x-\mu_0)} \]
\[ + \sum_{j=0}^{2m} a_j(v)(z-\mu)^j. \quad (2.5) \]
Here the integral is well defined for \(-1 < \Re v < 2m + 1\) and the \(a_j(v)\)'s are meromorphic functions in \(v\) which can be calculated explicitly. Using (2.4) we find for \(j = 0, 1, ..., 2m - 1,
\[
a_{j+1}(v) = \frac{1}{\pi} \int_0^{\infty} \frac{x^j \, dx}{(x - \mu)^{r+1}} = \frac{d^j}{d\mu^j} a_1(v) = \frac{(-1)^{j+1} (v)_j (-\mu)^{-j}}{j! \sin \pi v},
\]
where \((v)_0 = 1\), and \((v)_j = v(v - 1) \cdots (v - j + 1)\), \(j = 1, 2, \ldots\) It follows that the finite sum on the right-hand side of (2.5) has a pole of order 1 at \(v = k\) with residue equal to the residue of the sum \(\sum_{j=0}^{2m-1} a_{j+1}(v)(z - \mu)^j\) and straightforward calculation shows that the latter equals \(-z^k\).

From their formulas it follows that \(|y|^2 F(v, z)\) and \(Q(z)\) only differ by a polynomial \(r_{2m}(z)\) of degree at most \(2m\). Assume first that \([v]\) is even. Then \(2m = [v]\) and \(Q(z)\) has the representation as described in the proposition with \(\hat{r}_{[v]j}(z) = r_{[v]j}(z)\). When \([v]\) is odd then \(2m = [v] + 1\), \(z \in \mathbb{C}_{-m-1/2}\), and
\[
\lim_{x \to \infty} x^{-2m} Q(x) = \lim_{x \to \infty} x^{-[v] - 1} |y|^2 F(v, x) + r_{[v]+1}(x))
= \lim_{x \to \infty} x^{-[v]-1} r_{[v]+1}(x).
\]
By (1.9), the last limit equals zero, hence the polynomial \(\hat{r}_{[v]j}(z) = r_{[v]j}(z)\) has degree at most \([v]\). This proves the noninteger case in the proposition.

In the integer case, the points \(v = k\) are poles of order 1 of the function \(F(v, z)\),
\[
F(v, z) = -\frac{z^k}{(v - k)} - z^k \ln(-z) + \cdots,
\]
where the dots denote a power series in \(v - k\) which vanishes at \(v = k\). As we have seen the finite sum on the right-hand side of (2.5) as a function of \(v\) has the same singularity at \(v = k\) and is a polynomial in \(z\). The difference between \(F(v, z)\) and this finite sum is regular at \(v = k\), and \(|y|^2 \times this difference, modulo a selfadjoint polynomial \(r_{2m}(z)\) of degree at most \(2m\), coincides with \(Q(z)\). As in the noninteger case we distinguish between the cases \([v]\) is even and odd and repeating in the odd case the same reasoning using (1.9) we find that \(Q(z)\) has the representation (2.2).

3. POINT-LIKE PERTURBATION OF THE BESSEL OPERATOR

We now consider the Bessel differential expression \(\ell_v\) (0.1) in \(\mathcal{H}_0 = L^2(\mathbb{R}^+)\) and we assume \(v \geq 1\). As mentioned in the Introduction in this case the minimal
realization is selfadjoint; it will be denoted by $A_0$. We shall study the realizations $H'$ of the singular perturbations

$$A_0 + t^{-1} \langle \cdot, \chi \rangle_{A_0} Z, \quad t \in \mathbb{R} \cup \{ \infty \},$$

where in the sense of distributions

$$\chi(y) = (A_0 - \mu) \sqrt{y} (-\mu)^{\nu/2} K_\nu(y \sqrt{-\mu}).$$

Using a modified Hankel transform on $L^2(\mathbb{R}^+)$ we relate this singular perturbation to the model considered in Section 2. The modified Hankel transform is given by

$$(H_x f)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \sqrt{y} J_\nu(y \sqrt{x}) f(y) \, dy.$$

It is the composition of the standard unitary Hankel transform on $L^2(\mathbb{R}^+)$ (see, for instance, [AG]),

$$f(y) \rightarrow \tilde{f}(k) = \int_0^\infty \sqrt{ky} J_\nu(ky) f(y) \, dy,$$

which transforms $A_0$ to the operator of multiplication by $k^2$, and the unitary transform on $L^2(\mathbb{R}^+)$,

$$\tilde{f}(k) \rightarrow \hat{f}(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x^{1/4}} \tilde{f}(\sqrt{x}).$$

It follows that $H_x$ is a unitary mapping of $L^2(\mathbb{R}^+)$ onto itself which transforms $A_0$ to the operator of multiplication by $x$.

By duality (see, for instance, [Z]) the Hankel transform can be defined also on the negative spaces $H_{-k}$. In the table below we list the unitary correspondence between the point-like perturbations for the Bessel operator and the model in Section 2.

<table>
<thead>
<tr>
<th>$f(y) \in L^2(\mathbb{R}^+)$</th>
<th>$H_x \rightarrow \hat{f}(x) \in L^2(\mathbb{R}^+)$,</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0 \in H_x \rightarrow \text{multiplication by } x$,</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon_1(y) \in H_x \rightarrow \hat{\varepsilon}_1(x) = \frac{x^{\nu/2} \sqrt{2}}{x - \mu}$,</td>
<td></td>
</tr>
<tr>
<td>$\chi(y) \in H_x \rightarrow \hat{\chi}(x) = \frac{x^{\nu/2} \sqrt{2}}{x - \mu}$.</td>
<td></td>
</tr>
</tbody>
</table>

Here, for example, the last line means $\langle \hat{\chi}, \hat{u} \rangle_0 = \langle \chi, H_x^* \hat{u} \rangle_0 = \langle \chi, H_{-1}^{-1} \hat{u} \rangle_0$, where the expression on the left is related to the rigging associated with the
multiplication operator in the model and the element \( H_{m+1}^{-1} \) on the right belongs to \( H_m \). The fourth correspondence in the table follows from the third, \( \mathcal{Z} \in H_{m-1} \) and \( \varepsilon_1 \in H_m \) are related in the sense of distributions by 
\[
\mathcal{Z} = (A_0 - \mu) \varepsilon_1,
\]
where
\[
\varepsilon_1(y) = \sqrt{y} (-\mu)^{\nu} \sqrt{K_{\nu}(y \sqrt{-\mu})}.
\]
Note that \( \mathcal{Z} \) does not depend on the level point \( \mu \). The third line in the table follows from the formula (see [LU, p. 325, formula (7); E, p. 93, formula (39), together with p. 5, formula (14)])
\[
\int_{a}^{b} K_{\nu}(ay) J_{\nu}(by) \, y \, dy = \frac{b^{\nu}}{a^{\nu}(a^{2} + b^{2})},
\]
(3.1)

We want to choose the parameters in the realization in such a way that the \( Q \)-function in Proposition 2.1 for \( z \in \mathbb{R}^- \) and \( \nu > 1 \) and up to a real constant, is an analytic continuation in \( \nu \) of the \( Q \)-function given initially for \( 0 < \nu < 1 \) by formula (0.5),
\[
Q(z) := \begin{cases} 
-\frac{\pi}{2} (-z)^{\nu} \sin \pi \nu + g_1, & \text{if } \nu \neq 0, 1, 2, ..., \\
-\frac{\pi}{2} \ln(-z) + g_1, & \text{if } \nu = k = 0, 1, 2, ..., 
\end{cases}
\]
(3.2)

where \( g_1 \) is a real number. If \( \nu > 1 \) and \( \nu \neq 3, 5, ... \), then by (2.1), \( \lim_{\nu \to -0} g_{k+1} = 0, k = 1, 2, ..., 2m - 1 \). If \( \nu = 1, 3, 5, ... \), then \( \lim_{\nu \to -0} g_{2m} = \infty \); in this case we cannot choose \( \mu = -0 \), but have to stick to the choice \( \mu \in \mathbb{R}^- \), which complicates the formulas. In what follows we shall make the following assumptions concerning the parameters in the realization.

**HYPOTHESES 3.1.** Assume \( \nu > 1, \nu \neq 2, 3, ..., \) and set \( m = \lfloor \frac{\nu - 1}{2} \rfloor \). Choose \( \mu = -0, \) so that \( g_2 = g_3 = \ldots = g_{2m} = 0 \), and choose
\[
g_{2m+1} = -\frac{\pi}{2} \sin \pi (\nu - 2m).
\]

In accordance with the formulas (3.2) and (1.6) for \( Q(z) \) and the formula (1.7) for \( Q_{d}(z) \), the latter is given by
\[
Q_{dl}(z) = -\frac{\pi}{2} \sin \pi (\nu - 2m),
\]
(3.3)
and \( g_{2m+1} = Q_d(\mu_0) \). Later in this section and in the proof of Theorem 4.2 below we use that
\[
z^{2m}Q_d(z) = Q(z) - Q(-0), \tag{3.4}
\]
which can easily be verified.

As we have seen, \( Q(z) \) belongs to the class \( N_m \), and via Krein’s formula it fully characterizes the realizations of the singular point-like perturbations of the Bessel operator as selfadjoint operators in the Pontryagin space \( \Pi_m = \mathcal{H}_0 \oplus \mathbb{C}^m \oplus \mathbb{C}^m \) with inner product in which the Gram operator \( (1.4) \) is reduced to the form
\[
G = \begin{pmatrix} 1 & 0 \\ 0 & G_m \end{pmatrix}, \quad G_m = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix}. \tag{3.5}
\]
The operators \( H' \) which represent the one-parameter family of point-like perturbations in this situation have an especially simple form
\[
H' = \begin{pmatrix} f_1 + F_{m+1}X_0 \\ (F_0)^m_{j=1} \\ (F_j)^m_{j=1} \\ F_m + 1 \end{pmatrix}, \quad \begin{pmatrix} A_0 f_1 + \mu_0 F_{m+1}X_0 \\ tF_1 \\ (F_{j-1})^m_{j=2} \\ (F_j)^m_{j=1} \end{pmatrix}, \tag{3.6}
\]
where \( f_1 \in \text{dom } A_0, F_1, \ldots, F_{m+1}, F_{01}, \ldots, F_{0m-1} \in \mathbb{C} \), and
\[
F_{0m} = \langle f_1, \chi_{-1} \rangle_0 + g_{2m+1}F_{m+1}.
\]
To get the formula for \( H'' \), replace in (3.6) \( F_1 \) in the domain and the range of \( H' \) by \( F_1/t \) and formally let \( t \to \infty \). The resolvent \( R'(z) \) of \( H' \) is described by Kreins formula (1.10). We have
\[
R'(z) \begin{pmatrix} f \\ (F_0)^m_{j=1} \\ (F_j)^m_{j=1} \end{pmatrix} = \begin{pmatrix} R_d(z) f + F_{m+1}(z) R_d(z) \chi_{-1} \\ \langle z^{m-1} \langle R_d(z) f, \chi_{-1} \rangle_0 + g_j(z) F_{m+1}(z) + F_{0j}(z) \rangle_{j=1}^m \\ (F_j(z))_{j=1}^m \end{pmatrix}, \tag{3.7}
\]
where

\[ F_j(z) = \sum_{k=1}^{j-1} z^{j-k-1} F_k, \quad j = 1, \ldots, m+1, \]

\[ F_0(z) = \sum_{k=j+1}^{m} z^{k-1} F_k, \quad j = 1, \ldots, m, \]

\[ q_j(z) = z^{m-j} Q_j(z), \quad j = 1, \ldots, m, \]

\[ \Gamma_j = \varphi(z) = (I + z R^\infty(z)) \begin{pmatrix} 0 \\ z^{-m} R_0(z) \chi_{-1} \\ \vdots \\ z^{-m} q_j(z)_{m-1} \\ (z^{j-1})_{m-1} \end{pmatrix}, \]

and

\[ Q(z, t) := Q(z) - Q(0) - t = -\frac{\pi \varepsilon(z)}{2 \sin \pi \varepsilon} - t. \quad (3.8) \]

These formulas and (3.4) imply for \( f \in \mathcal{H}_0 \)

\[ (H' - z)^{-1} \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} R_0(z) f \\ 0 \\ 0 \end{pmatrix} - \frac{\langle R_0(z) f, \chi_{-1} \rangle}{Q(z, t)} \begin{pmatrix} z^{-m} R_0(z) \chi_{-1} \\ (z^{-m})_{m-1} \\ (z^{j-1})_{m-1} \end{pmatrix}. \quad (3.9) \]

4. SPECTRAL PROPERTIES OF \( H' \)

In this section we assume throughout the assumptions in Hypotheses 3.1 and discuss the spectrum of the realizations \( H', \, t \in \mathbb{R} \cup \{ \infty \} \), considered at the end of the previous section. The selfadjoint realization \( A_0 \) of the Bessel operator \( l \) in \( \mathcal{H}_0 \) does not have any eigenvalues: the point spectrum \( \sigma_p(A_0) \) is empty. More specifically,

\[ \sigma(A_0) = \sigma_c(A_0) = \sigma_{ac}(A_0) = \sigma_{ess}(A_0) = [0, \infty), \]

where from left to right the sets stand for the spectrum, the continuous, the absolutely continuous and the essential spectrum of \( A_0 \). We noted before that \( \rho(H^\infty) = \rho(A_0) \), hence \( \rho(H^\infty) = \mathbb{C} \setminus \mathbb{R}^+ \) and \( \sigma_\rho(H^\infty) = \{ \infty \} \). From Krein’s formula it follows that for \( t \in \mathbb{R} \), \( \sigma_\rho(H') \) is determined by the zeros in \( \mathbb{C} \setminus \mathbb{R} \) and the generalized zeros of negative type in \( \mathbb{R} \cup \{ \infty \} \) of the \( \bar{N}_m \) function in (3.8),

\[ Q(z, t) = Q(z) - Q(0) - t = -\frac{\pi \varepsilon(z)}{2 \sin \pi \varepsilon} - t. \quad (4.1) \]
Recall from [KL, L] that a point $x_0 \in \mathbb{R} \cup \{ \infty \}$ is a generalized zero (pole) of negative type and multiplicity $n(x_0)$ of the function $N \in N_m$ if for each sufficiently small neighbourhood $\mathcal{U}$ of $x_0$ there exists a real number $n(\mathcal{U}) > 0$ such that for $0 < \delta < n(\mathcal{U})$ ($\delta > n(\mathcal{U})$) the function $N(z) + i\delta$ has zeros of total multiplicity $n(x_0)$ in $\mathcal{U} \cap \mathbb{C}^+$. For example, the function (3.8) has a generalized pole at infinity of negative type and multiplicity $m$. This follows from the statement in [KL] that the sum of the multiplicities of the poles in $C^+$ and of the generalized poles of negative type of $Q \in N_m$ equals $m$. Since $Q$ in (3.8) has only one such pole at infinity the statement follows. This fact can also be obtained by applying a criterion of [L]. We denote by $\mathcal{Z}(Q(\cdot, t))$ the set of zeros and generalized zeros of the function in (4.1).

**Proposition 4.1.** Assume Hypotheses 3.1. The set $\mathcal{Z}(Q(\cdot, t))$ lies on the circle $|z| = s$, $s = (\frac{2|m| \sin \pi/2}{n(x_0)})^1$, is symmetric with respect to the real axis, consists of at most $2m + 1$ points, and $\mathcal{Z}(Q(\cdot, t)) \cap \mathbb{R}^+ = \emptyset$.

(i) If $t \neq 0$, $\mathcal{Z}(Q(\cdot, t))$ consists of simple zeros only and there are two possibilities:

(a) if $\text{sign } t = (-1)^{|x|}$, then $\mathcal{Z}(Q(\cdot, t))$ consists of $m$ different zeros in $C^+$ and their conjugates in $C^-$;

(b) if $\text{sign } t = (-1)^{|x|+1}$, there are again two possibilities:

(b1) if $|v|$ is odd, then $\mathcal{Z}(Q(\cdot, t))$ consists of $m - 1$ different zeros in $C^+$ and their conjugates in $C^-$ and of $z = -s$, and $Q(-s) < 0$

(b2) if $|v|$ is even, then $\mathcal{Z}(Q(\cdot, t))$ consists of $m$ different zeros in $C^+$ and their conjugates in $C^-$ and of $z = -s$, and $Q(-s) > 0$.

(ii) For $t = 0$, $\mathcal{Z}(Q(\cdot, 0)) = \{0\}$ and $z = 0$ is a generalized zero of negative type and multiplicity $m$.

**Proof.** The function does not have any zero in $\mathbb{R}^+$ since

$$\text{Im } Q(x + i0, t) = \frac{\pi}{2} x^s \neq 0, \text{ if } x > 0.$$ 

Part (i) follows easily from direct calculations and the choice of the branch of $-(\cdot)^s$ fixed by Proposition 2.1. Part (ii) follows from the statement in [KL] that the sum of the multiplicities of the zeros in $C^+$ and of the generalized zeros of negative type of a function in $N_m$ equals $m$. In our case $z = 0$ is the only zero. (Part (ii) can also be proved by applying a criterion from [L].)
In the following theorem we describe the point spectrum $\sigma_p(H')$ of $H'$.
We denote by $L'_t$ the span of the root subspaces of $H'$ corresponding to the eigenvalues in $\sigma_p(H')$.

**Theorem 4.2.** Assume Hypotheses 3.1.

(a) $\sigma_p(H') = \mathcal{Z}(Q(\cdot, t))$; in particular, the eigenvalues lie on the circle $|z| = s$ considered in Proposition 4.1.

(b) For $t \neq 0, \infty$, the eigenvalues of $H'$ are simple,

$$L'_t = \text{span}\{I_z 1 | z \in \mathcal{Z}(Q(\cdot, t))\},$$

$L'_t$ is a Pontryagin subspace of $\Pi_m$ with negative index $m$, and

$$\dim L'_t = \begin{cases} 2m & \text{if } \text{sign} t = (-1)^{|v|} \\ 2m - 1 & \text{if } \text{sign} t = (-1)^{|v| + 1}, \ [v] \text{ odd}, \\ 2m + 1 & \text{if } \text{sign} t = (-1)^{|v| + 1}, \ [v] \text{ even}. \end{cases}$$

If $\text{sign} t = (-1)^{|v| + 1}$ then $z = -s$ is a simple eigenvalue of $H'$ with a negative/positive eigenvector if $[v]$ is odd/even.

(c) $\sigma_p(H^0) = \{0\}$, $z = 0$ is a singular critical point of $H^0$, and $L_0$ coincides with the maximal neutral subspace $\{0\} \oplus \{0\} \oplus \mathbb{C}^m$ of $\Pi_m$.

(d) $\sigma_p(H^\infty) = \{\infty\}$ and $L_\infty$ coincides with the maximal neutral subspace $\{0\} \oplus \mathbb{C}^m \oplus \{0\}$ of $\Pi_m$.

**Proof.** (a) and (b). Consider $t \neq 0, \infty$ and $\lambda \in \rho(A_0)$. From (3.6) it easily follows that $\lambda \in \sigma_p(H')$ if and only if

$$\exists \neq 0 \in \text{dom } H' : (H' - \lambda) h = 0 \iff \exists \neq 0 \in C : (z^{2m}Q_0(\lambda) - t) c = 0.$$ 

On account of (3.4), the righthand side holds true if and only if $Q(\lambda) - Q(0) - t = 0$. The solution $h$ on the lefthand side is given by $h = I_\lambda c$. If $\text{sign} t = (-1)^{|v| + 1}$ and $h$ corresponds to the eigenvalue $z = -s$, then $\langle h, h \rangle = \text{sign} z Q(\cdot, t)|_{z=-s} = Q'(-s)$ which implies the last statement in (b). The statements in (b) concerning $L'_t$ now follow for example from [IKL, Lemma 12.1'] and the fact that the eigenfunctions corresponding to nonreal points $z_1$ and $z_2$ with $z_1 \neq z_2$ are orthogonal.

(c) Assume $t = 0$, then from the explicit expression of $H^0$ we have

$$H^0 u_1 = 0, \quad H^0 u_j = u_{j-1}, \quad j = 2, \ldots, m,$$

where $u_j = \text{column}(0, 0, e_j)$, $e_j$ being the $j$th element in the standard basis of $\mathbb{C}^m$. The relations mean that $z = 0 \in \sigma_p(H^0)$ and the vectors $u_m, \ldots, u_1$ form a Jordan chain in the root subspace corresponding to $z = 0$ and span the
invariant subspace \( \{0\} \oplus \{0\} \oplus \mathbb{C}^m \). This subspace is maximal neutral owing to the structure of the Gram operator (3.5). To show that \( \mathcal{L}_c = \{0\} \oplus \{0\} \oplus \mathbb{C}^m \) it is enough to prove that if an element \( f \in \text{dom}(\mathcal{H}_0) \) satisfies \( \mathcal{H}_0f = c \ u_m \), \( c \in \mathbb{C} \), then \( f \in \{0\} \oplus \{0\} \oplus \mathbb{C}^m \) and \( c = 0 \). Let \( f \) be such an element. Then by (3.6) with \( t = 0 \), \( f = \text{column}(f_1 + \chi_0 c, F_{0m}, 0) \) modulo \( \mathcal{L}_c \), where \( f_1 \in \text{dom}(A_0) \), \( c \in \mathbb{C} \), \( F_{0m} = (f_1, \chi_{-1} \chi_0 c + c \mathbb{C}_{2m + 1}, 1) \), and \( A_0 f_1 + \mu_0 \chi_0 c = 0 \). Choosing \( \rho \in \mathbb{R} \) we rewrite the last equality as \( (A_0 - \rho) f_1 + \rho f_1 + \mu_0 \chi_0 c = 0 \). Multiplying both sides by \( \text{ran}(\rho) \) and using the relation \( \text{ran}(\rho) \chi_0 = (\mu_0 - \rho)^{-1} (\rho - \text{ran}(\rho) \chi_{-1}) \) we obtain

\[
f_1 + \rho \text{ran}(\rho) f_1 + \mu_0 (\mu_0 - \rho)^{-1} \chi_0 c = \mu_0 (\mu_0 - \rho)^{-1} \text{ran}(\rho) \chi_{-1} c.
\]

As the spectrum of \( A_0 \) is continuous, \( \lim_{\rho \to 0} \rho \text{ran}(\rho) f_1 = 0 \). Thus the left-hand side of (4.2) has the finite limit \( f_1 + \chi_0 c \) when \( \rho \to 0 \), whereas the right-hand side has a finite limit only if \( c = 0 \). Indeed, \( \text{ran}(\rho) \chi_{-1} = \mathcal{Q}(\rho) \sim |\rho|^{v - 2m - 1} \) and as \( v - 2m - 1 < 0 \) the limit of this quantity is infinite. Therefore we have that \( f = 0 \) modulo \( \{0\} \oplus \{0\} \oplus \mathbb{C}^m \) and \( c = 0 \). This proves \( \mathcal{L}_c = \{0\} \oplus \{0\} \oplus \mathbb{C}^m \).

(d) The formula for \( \mathcal{H}_0 \) is described just after (3.6). It implies

\[
R^c(z) w_1 = 0,
\]

\[
R^c(z) w_j = w_{j-1} + z w_{j-2} + \cdots + z^{j-1} w_1, \quad j = 2, \ldots, m,
\]

where \( w_j = \text{column}(0, e_j, 0) \). Therefore \( R^c(z)^m \mathcal{L} = 0 \), which implies that \( \{0\} \oplus \mathbb{C}^m \oplus \{0\} \) is invariant under \( R^c(z) \) and contained in \( \mathcal{L}_c \). By the form of the Gram operator (3.5), \( \{0\} \oplus \mathbb{C}^m \oplus \{0\} \) is a maximal neutral subspace. To show that \( \{0\} \oplus \mathbb{C}^m \oplus \{0\} = \mathcal{L}_c \) we use the direct sum decomposition \( \Pi_m = \mathcal{H}_0 \oplus (\mathcal{L}_c + \{0\}) \oplus \mathbb{C}^m \oplus \{0\} \). Suppose that there exists an element \( f = h + u \) with \( h \in \mathcal{H}_0 \) and \( u \in \mathcal{L}_c \) such that \( R^c(z) f \in \{0\} \oplus \mathbb{C}^m \oplus \{0\} \). From \( \text{ran}(\mathcal{H}_0 - z) = \Pi_m \), we can write \( h \) and \( u \) in the notation of (3.6) as

\[
h = \text{ran}(z) f_1 + F_{2m+1} \chi_0,
\]

\[
u = F_2 e_1 + (F_1 - z F_2) e_2 + \cdots + (F_{m+1} - z F_m) e_m.
\]

and then

\[
R^c(z) f = \text{column}(f_1 + F_{m+1} \chi_0, w, F_2 e_2 + F_3 e_3 + \cdots + F_m e_m)
\]

for some \( w \in \mathbb{C}^m \). The condition \( R^c(z) f \in \{0\} \oplus \mathbb{C}^m \oplus \{0\} \) implies \( f_1 + F_{m+1} \chi_0 = 0 \) and \( F_2 = F_3 = \cdots = F_m = 0 \). It readily follows that \( h = 0 \) and \( u = 0 \). Hence the inclusion \( (R^c(z)) f \in \{0\} \oplus \mathbb{C}^m \oplus \{0\} \) implies \( f \in \mathcal{L} \). This proves \( \mathcal{L}_c = \{0\} \oplus \mathbb{C}^m \oplus \{0\} \).
Now we can describe the spectral properties of $H^t$ in its invariant subspace $L^\perp_t$. If $t \neq 0, \infty$, then by Theorem 4.2(b), $L^\perp_t$ is a Pontryagin space with negative index $m$. Hence $L^\perp_t$ is a Hilbert space. By Krein's formula the discontinuities of $R(z)|_{L^\perp_t}$ occur only on $\mathbb{R}^+$ and are determined by spectral functions of $A_0$ and $Q(z, t)^{-1}$ which are absolutely continuous on this set. Thus the spectrum of $H^t|_{L^\perp_t}$ is absolutely continuous, $\sigma_a(H^t) = \sigma(H^t|_{L^\perp_t})$, and the decomposition $H^m = L^\perp_t \oplus L^\perp_t$ is a spectral decomposition: the two spaces on the right correspond to $\sigma_p(H^t)$ and $\sigma_a(H^t)$ respectively, $t \neq 0, \infty$.

We now consider $t = 0$. Parts (c) and (d) of Theorem 4.2 imply that $H^m$ can be decomposed as

$$H^m = \mathcal{H}_0 \oplus (L_0 + L_m). \quad (4.3)$$

By the form of the Gram matrix (3.5) of the inner product of $L^\perp_0$, $L_0$, and $L_m$ are skewly linked neutral subspaces. It follows that $L^\perp_0 \cap L_0$ and $L^\perp_m$ are nonnegative but degenerate with isotropic part $L_0$. $\mathcal{H}_0$ can be considered as the quotient space $L^\perp_0 / L_0$ and in this case the operator $H^0$ induces a selfadjoint operator in $\mathcal{H}_0$ which we will denote by $A^0$. This operator can be calculated explicitly: The restriction of $H^0$ to $L^\perp_0 = \mathcal{H}_0 \oplus L_0$ is the mapping

$$\text{column}(f_1 + F_{m+1}^0, 0, (F_j)_{m+1})$$

$$\rightarrow \text{column}(A_0 f_1 + \mu_0 F_{m+1}^0, 0, (F_j)_{m+1}),$$

where $f_1 \in \text{dom}(A_0)$, $F_1, ..., F_{m+1} \in \mathbb{C}$, and $\langle f_1, \chi_0 \rangle = 0, 2m+1 F_{m+1} = 0$. In the quotient space $L^\perp_0 / L_0$ this restriction induces $A^0$ in $\mathcal{H}_0$ defined by

$$\text{dom}(A^0) = \{ f = f_1 + F_{m+1}^0 | f_1 \in \text{dom}(A_0), \langle f_1, \chi_0 \rangle = 0, -g_{2m+1} F_{m+1} = 0 \}.$$ 

$$A^0 f = A_0 f_1 + \mu_0 F_{m+1}^0.$$ 

$A^0$ is a canonical selfadjoint extension of the one-dimensional restriction $S_0$ of $A_0$ to the domain $\text{dom}(S_0) = \{ f_1 \in \text{dom}(A_0) | \langle f_1, \chi_0 \rangle = 0 \}$ considered in Section 1; see (1.11) and recall $\chi = \epsilon_m$. The canonical selfadjoint extensions of $S_0$ can be parametrized as $A^t$, $t \in \mathbb{R} \cup \{ \infty \}$, via Krein's formula for their resolvents

$$(A^t - \frac{1}{2} R_0(z))^{-1} = \frac{\Gamma_0 + \Gamma_0^*}{Q_0(z)} - t, \quad z \in \rho(A_0) \cap \rho(H^t). \quad (4.4)$$

The notation is consistent: the operator $A^0$ defined above coincides with the operator $A^t$ for $t = 0$. 

PERTURBATIONS OF THE BESSEL OPERATOR
Proposition 4.3. In Krein's formula (4.4), $Q_0$ is given by (3.3),

$$Q_0(z) = -\frac{\pi}{2} \frac{(-z)^{\nu-2m}}{\sin \pi(\nu-2m)},$$

and

$$F_0; 1(y) = (-1)^m (-z)^{\nu-2m} \sqrt{y} \left[ K_0(y \sqrt{-z}) \right],$$

where $\left[ K_0(y) \right]$ denotes the regular part of the function $K_0(y)$ obtained from $K_0(y)$ by removing the first $m$ singular terms in the standard expansion of $K_0(y)$ in powers of $y$, that is, if $K_0(y) = \sum_{j=0}^{m} c_j y^{\nu-2j} + \sum_{j=m+1}^{\infty} d_j y^{\nu-2j}$, then $\left[ K_0(y) \right] = \sum_{j=0}^{m} c_j y^{\nu-2j} + \sum_{j=m+1}^{\infty} d_j y^{\nu-2j}$.

Proof. We prove the formula for $\varphi_0(z) = F_0; 1$. In the model representation $\varphi_0(x) = x^{\nu-2m}/\sqrt{2}$ and $\varphi_0(z) = x^{\nu-2m}/\sqrt{2}(x-z)$. In the following $\nu > 0$, $\nu \neq 1, 2, ..., \left\lceil \frac{\nu}{2} \right\rceil$ and $\mu < 0$. For $i = 0$ and $i = m$ we introduce the generalized functions

$$G_i(y, \mu) = H^{-1}_y \frac{x^{\nu-2-i}}{\sqrt{2} (x-\mu)} = \frac{1}{2} \int_0^\infty J_y(x\sqrt{y}) \frac{x^{\nu-2-i} \, dx}{x-\mu}.$$ Substituting

$$\frac{x^{\nu-2-m}}{x-\mu} = \mu^{-m} \left( \frac{x^{\nu/2}}{x-\mu} - \sum_{j=0}^{m-1} \mu^j x^{\nu/2-j-1} \right)$$

in the defining relation for $G_m(y, \mu)$ we obtain the equality (in sense of distributions)

$$G_m(y, \mu) = \mu^{-m} \left( G_0(y, \mu) - \sum_{j=0}^{m-1} \mu^j a_j(y) \right),$$

(4.5)

where $a_j(y) = \frac{1}{2} \int_0^\infty \sqrt{y} J_y(x\sqrt{y}) x^{\nu/2-j-1} \, dx$. By (3.1) and the series expansion of $K_0(y)$ given in the proposition,

$$G_0(y, \mu) = (-\mu)^{\nu/2} \sqrt{y} \left[ K_0(y \sqrt{-\mu}) \right]$$

$$= (-\mu)^{\nu/2} \sqrt{y} \left[ K_0(y \sqrt{-\mu}) \right] + \sum_{j=0}^{m-1} (-\mu)^j c_j y^{1/2-\nu/2}.$$ Substituting this expression in (4.5), we get

$$G_m(y, \mu) = (-1)^m (-\mu)^{\nu-2m} \sqrt{y} \left[ K_0(y \sqrt{-\mu}) \right] + O(\mu^{-m}), \quad \mu \uparrow 0.$$
Near $\mu = 0$ the function $G_m(y, \mu)$ and the first function on the righthand side have the same asymptotic behavior, namely $\approx \text{constant } \mu^{y^{1/2-v+2m}}$, and hence
\[
G_m(y, \mu) = (-1)^m (-\mu)^{y^{1/2-m}} \sqrt{y} [K_v(y \sqrt{-\mu})].
\]
The proposition follows from $\varphi_d(y, \mu) = G_m(y, \mu)$ and analytic continuation.

From Krein's formula (4.4) and Proposition 4.3 it follows that $\sigma(A^0) = \sigma(\mathcal{A})$.

Formula (4.3) implies that $\mathcal{H}_0$ can also be identified with the quotient space $\mathcal{D}_0 \big/ \mathcal{D}_0$. If $P_0$ stands for the orthogonal projection in $\mathcal{H}_m$ onto $\mathcal{H}_0$, then by (3.7),
\[
P_0(H^\infty - z)^{-1} |_{\mathcal{H}_0} = R_0(z) = (A_0 - z)^{-1}.
\]
It follows that the relation $H^\infty$ induces the operator $A_0$ on $\mathcal{H}_0$, and we already know that $\sigma(A_0) = \sigma(\mathcal{A})(A_0)$. We summarize these results in a theorem.

**Theorem 4.4.** The spectra of $H^t \big|_{\mathcal{H}_m}$ in the Hilbert space $\mathcal{H}_m$, $t \neq 0, \infty$, and the spectra of $\mathcal{A}$ and $A_0$ induced on $\mathcal{H}_0$ by $H^0$ and $H^\infty$ are absolutely continuous and coincide with $[0, \infty)$.

The realizations $H^0$ and $H^\infty$ are especially interesting because of some extremal properties of the operators $\mathcal{A}$ and $A_0$ they induce on $\mathcal{H}_0$. For more complete and detailed results on $Q$-functions than provided in the proof below we refer to [HLS].

**Theorem 4.5.** The operator $A^0$ induced by $H^0$ on $\mathcal{H}_0$ and the operator $A_0$ (induced by $H^\infty$) are the Krein extension and the Friedrichs extension of $S_0$ when $[v]$ is even and vice versa when $[v]$ is odd.

**Proof.** Since
\[
v - 2m = \begin{cases} v - [v] > 0, & \text{if } [v] \text{ even}, \\ v - [v] - 1 < 0, & \text{if } [v] \text{ odd}, \end{cases}
\]
the following limits hold for the $Q$-function $Q_d(z)$ in Proposition 4.3,
\[
\lim_{x \to -\infty} Q_d(x) = -\infty, \quad \lim_{x \to 0} Q_d(x) = 0, \quad \text{if } [v] \text{ even},
\]
\[
\lim_{x \to -\infty} Q_d(x) = 0, \quad \lim_{x \to 0} Q_d(x) = \infty, \quad \text{if } [v] \text{ odd}.
\]
Recall that in the definition of a $Q$-function $Q(z)$ (determined up to a real additive constant) of a symmetric operator $S$ and a canonical selfadjoint
extension $A$ of $S$ a choice is made of a nonzero element $\varphi(\mu) := \Gamma_\mu 1$ in the eigenspace $\ker(S^* - \mu)$ for some $\mu \in \rho(A)$. If the defect numbers of $S$ are $1, 1$, another choice would lead to a $Q$-function of the form $aQ(z) + b$ where $a, b$ are real numbers and $a > 0$. The function $Q_d(\mu_0)$ above is the $Q$-function for $S_0$ and $A_0$ and the element $\varphi_d(\mu_0)$. So $\varphi_d(z) := \Gamma_{\mu_0} 1$ and $Q_d(z)$ are characterized by the relations

$$
\varphi_d(z) = (I + (z - \mu_0)(A_0 - z)^{-1}) \varphi_d(\mu_0),
\lefteqn{Q_d(z) = Q_d(\mu_0) + (z - \mu_0)\langle \varphi_d(z), \varphi_d(\mu_0) \rangle_0}.
$$

From Krein’s formula (4.4) we have

$$
(A^0 - z)^{-1} = (A_0 - z)^{-1} - \frac{\langle \cdot, \varphi_d(z^*) \rangle_0}{Q_d(z)} \varphi_d(z).
$$

We claim that this formula can be rewritten in a form in which the roles of $A_0$ and $A^0$ are interchanged,

$$
(A_0 - z)^{-1} = (A^0 - z)^{-1} - \frac{\langle \cdot, \varphi^0(z^*) \rangle_0}{Q^0(z)} \varphi^0(z),
$$

where for suitable choices of $\varphi^0(\mu_0) \in \ker(S^* - \mu_0)$ and $Q^0(\mu_0) \in \mathbb{R}$, the function

$$
Q^0(z) = -Q_d(z)^{-1}
$$

is the $Q$-function for $S_0$ and $A^0$, that is,

$$
\varphi^0(z) = (I + (z - \mu_0)(A^0 - z)^{-1}) \varphi^0(\mu_0),
\lefteqn{Q^0(z) = Q^0(\mu_0) + (z - \mu_0)\langle \varphi^0(z), \varphi^0(\mu_0) \rangle_0}.
$$

To prove the claim, we choose $\varphi^0(\mu_0) = \gamma \varphi_d(\mu_0)$ (the real number $\gamma$ will be determined later) and insert the formula (4.6) for $(A^0 - z)^{-1}$ into the formula for $\varphi^0(z)$. We obtain

$$
\varphi^0(z) = \varphi^0(\mu_0) + (z - \mu_0)(A_0 - z)^{-1} \varphi^0(\mu_0)
= (z - \mu_0)\langle \varphi^0(\mu_0), \varphi_d(z^*) \rangle_0 \varphi_d(z)
= \gamma \varphi_d(\mu_0) + \gamma (\varphi_d(z) - \varphi_d(\mu_0)) - \gamma \frac{Q_d(z) - Q_d(\mu_0)}{Q_d(z)} \varphi_d(z)
= \gamma \frac{Q_d(\mu_0)}{Q_d(z)} \varphi_d(z)
$$
and so
\[
\frac{Q^0(z) - Q^0(w)^*}{z - w^*} = \langle \varphi^0(z), \varphi^0(w) \rangle_0
\]
\[
= |\gamma Q_0(\mu_0)|^2 \left\langle \varphi_0(z), \frac{\varphi_0(w)}{Q_0(z) Q_0(w)^*} \right\rangle
\]
\[
= |\gamma Q_0(\mu_0)|^2 \frac{Q_0(z)^{-1} + Q_0(w)^{-1}}{z - w^*}.
\]

Hence
\[
Q^0(z) := -|\gamma Q_0(\mu_0)|^2 Q_0(z)^{-1}
\]
is a $Q$-function associated with $S_0$ and $A_0$. Formula (4.7) now easily follows from (4.6) and if we choose $\gamma = Q_0(\mu_0)^{-1}$ then also (4.8) holds. From the four limits for $Q_0(z)$ and (4.8), we readily obtain the limits
\[
\lim_{x \to 0} Q^0(x) = \infty, \quad \lim_{x \to \infty} Q^0(x) = 0, \quad \text{if } [v] \text{ even},
\]
\[
\lim_{x \to 0} Q^0(x) = 0, \quad \lim_{x \to \infty} Q^0(x) = -\infty, \quad \text{if } [v] \text{ odd}.
\]

The theorem now follows from [KO, Theorem 1] and these eight limits.

We end this section with a comparison of the spectral shift functions for the pairs $\{H^0, H^\infty\}$ and $\{A^0, A_0\}$. Recall from [K1, K2] that the shift function $\zeta$ for a given pair of operators $A, B$ in a Hilbert space for which the difference of the resolvents is trace class, is defined up to a constant by the conditions
\[
(1 + |\lambda|^2)^{-1} |\zeta(\lambda)| \in L^1(\mathbb{R}) \quad \text{and} \quad \text{Tr}[(A - z^{-1}) - (B - z^{-1})] = -\int_{\rho(A) \cap \rho(B)} \zeta(\lambda) \frac{d\lambda}{\lambda - z},
\]
\[
z \in \rho(A) \cap \rho(B).
\]
For nonnegative operators the ambiguity can be removed by requiring $\zeta$ to be zero at $-\infty$. We define the spectral shift function for the pair $\{H^0, H^\infty\}$ in $H_m$ by the same rule. This function exists since $R^0(z) - R^\infty(z)$ is a rank one operator. As the trace of the rank one operator $\beta < \cdot, e >_0 e$ with unit vector $e$ equals $\beta$, the Krein formulas yield
\[
\text{Tr}[(H^0 - z^{-1}) - (H^\infty - z^{-1})] = -(Q(z) - Q(-0))^{-1} \frac{d}{dz} Q(z) = -\frac{v}{z},
\]
\[
\text{Tr}[(A^0 - z^{-1}) - (A_0 - z^{-1})] = -(Q_0(z))^{-1} \frac{d}{dz} Q_0(z) = -\frac{v - 2m}{z}.
\]
From these equalities we obtain the following theorem. We denote by \( \theta(\lambda) \) the Heaviside function,
\[
\theta(\lambda) = \begin{cases} 
0, & \lambda < 0, \\
1, & \lambda \geq 0.
\end{cases}
\]

**Theorem 4.6.** The spectral shift function for the pair \( \{H^0, H^\infty\} \) is the function \( \xi(\lambda) = \nu(\theta(\lambda)) \) and that for the pair \( \{A^0, A_0\} \) is the function \( \xi(\lambda) = (\nu - 2m) \theta(\lambda) \).

### 5. EIGENFUNCTION EXPANSIONS

If \( 0 < \nu < 1 \) and \( \lambda \) is in the absolute continuous spectrum of the operator \( H^t \) in \( \mathcal{H}_0 \) then \( e(y, \lambda) \) in (0.2) is the generalized eigenfunction corresponding to \( \lambda \). We show that they are the kernels of the wave operators for the pair of operators \( H^t, A_0 \): We recall that the wave operators
\[
W^\pm = W^\pm(H^t, A_0) = s - \lim_{\nu \to \pm \infty} e^{iH^t + iA_0} P_{ac},
\]
where \( P_{ac} \) is the projection in \( \mathcal{H}_0 \) onto the subspace \( H_{ac} \) corresponding to \( \sigma(A_0) \) have the time-independent characterization
\[
W^\pm = s - \lim_{\eta \downarrow 0} e^{-\nu R(\lambda + \eta)} dE_0(\lambda) P_{ac},
\]
where \( E_0(\lambda) \) is the spectral family for \( A_0 \); see, for example, [AJS, MF]. We briefly repeat the proof. If \( f \) is a locally integrable function on \( \mathbb{R}^+ \) with values in a Banach space and \( \lim_{r \to \infty} f(r) = a \), then
\[
\lim_{\eta \downarrow 0} \int_0^\infty e^{-\nu \tau} f(\tau) d\tau = a.
\]
If we apply this so called Abel limit to the function \( f(\tau) = e^{\pm iH^t + iA_0} P_{ac} \), we obtain
\[
W^\pm = s - \lim_{\eta \downarrow 0} \int_0^\infty e^{\pm i(\lambda + \eta)} dE_0(\lambda) P_{ac}.
\]
We calculate the wave operators for the model and the Bessel operator in case $0 < v < 1$: If $t \geq 0$, then $\sigma(H') = [0, \infty)$ and the spectrum is absolutely continuous. But if $t < 0$, then $\sigma(H') = \{-s\} \cup [0, \infty)$, where $[0, \infty)$ is the absolutely continuous part of the spectrum and $z = -s$ is a normal eigenvalue of $H': Q(-s, t) = 0$ if $t < 0$. In the model the corresponding eigenvector $\varphi(-s) = \Gamma_{-s}1$ is the function

$$\varphi(x, -s) = \frac{x^{v/2}}{\sqrt{2(x+s)}};$$

in the setting of the Bessel operator it takes the form

$$\varphi(y, -s) = e(y, -s) := \sqrt{y^{v/2}}K_y(y\sqrt{s}).$$

It follows that if $t \geq 0$, then $P_{ac} = I$, the identity operator on $\mathcal{H}_a$, and if $t < 0$, then

$$P_{ac} = I - \frac{\langle \cdot, \varphi(-s) \rangle_0}{c(-s)} \varphi(-s),$$

where

$$c(-s) := \langle \varphi(-s), \varphi(-s) \rangle_0 = \frac{\partial}{\partial z} Q(z, t) \bigg|_{z = -s} = vs^{v-1}.$$

Since the wave operators are isometric, the eigenfunction expansions in terms of the wave operator take the well-known form,

$$P_{ac}f = W^+ \tilde{f}, \quad \tilde{f} = W^+ f, \quad f \in \mathcal{H}_a.$$

The wave operators in the proposition intertwine the absolutely continuous parts of the operators $H, H', H^{-1}$ and $A_{0}$.

**Proposition 5.1.** Assume $0 < v < 1$.

(a) If in the model $A_{0}$ is multiplication by the spectral parameter $\lambda$ in $L^2(\mathbb{R}^+)$ and $\varphi(\lambda) = \lambda^{v/2}/\sqrt{2}$, then the kernels of the wave operators

$$(W^\pm f)(x) = \int_0^{+\infty} \hat{\varphi}_{\pm}(x, \lambda) f(\lambda) \, d\lambda.$$
are given by
\[ e^*_{\pm}(x, \lambda) = 6(x - \lambda) - \frac{x^{2v2} \lambda^{v2}}{2(\lambda^{\pm} + \lambda^{\pm})} \cdot \]
\[ Q_{\pm}(z) = \frac{\pi (-z)^v}{2 \sin \pi v}. \]

(b) In the setting where \( A_0 \) is the Bessel operator, the kernels of the wave operators
\[ (W^{\pm} f)(y) = \int_0^{+\infty} e^*_{\pm}(y, \lambda) f(\lambda) \, d\lambda \]
are given by (0.2),
\[ e^*_{\pm}(y, \lambda) = c^*_{\pm}(\lambda) \sqrt{\lambda}\cos xJ_{\lambda}(y \sqrt{\lambda}) + \sin \alpha J_{\lambda}(y \sqrt{\lambda}), \quad y, \lambda \in \mathbb{R}^+, \]
with
\[ c^*_{\pm}(\lambda) = - \frac{1}{\sqrt{2}} \left( \frac{\pi \lambda v^{1/v} + \pi}{2 \sin \pi v} \right)^{-1} \left( t^2 + \frac{\pi^2}{4 \sin^2 \pi v} \right)^{1/2} \]
and \( \alpha \) defined by
\[ \cos \alpha = - t \left( t^2 + \frac{\pi^2}{4 \sin^2 \pi v} \right)^{-1/2}, \quad \sin \alpha = - \frac{\pi}{2 \sin \pi v} \left( t^2 + \frac{\pi^2}{4 \sin^2 \pi v} \right)^{-1/2}. \]
The eigenfunction expansion in terms of the functions \( e^*_{\pm}(y, \lambda) \) and \( e(y, -s) \) is given by
\[ f(x) = \int_0^{+\infty} e^*_{\pm}(x, \lambda) \left( \int_0^{+\infty} e^*_{\pm}(y, \lambda) f(y) \, dy \right) \, d\lambda + \frac{f^*(x, c(-s))}{c(-s)} e(x, -s), \]
with the understanding that the last summand is omitted if \( t \geq 0 \).

Proof. In the model \( A_0 \) is multiplication by \( \lambda \) in \( L^2(\mathbb{R}^+) \), \( \chi(\lambda) = \frac{\lambda^v}{\sqrt{2}} \cdot \]
\[ S_0 = A_0 | \text{dom } S_0, \quad \text{dom } S_0 = \{ u \in \text{dom } A_0 | \langle u, \chi \rangle = 0 \}, \]
\[ \chi_{\pm}(\lambda) = R_0(\mu_0) \chi(\lambda) = \frac{\lambda^{v2}}{\sqrt{2}(\lambda - \mu_0)}, \]
\[ \varphi(\lambda, z) = \varphi_{\pm}(\lambda) = (I + (z - \mu_0) R_0(z)) \chi_{\pm}(\lambda) = \frac{\lambda^{v2}}{\sqrt{2}(\lambda - z)}, \]
and

\[ Q_0(z) = Q_0(\mu_0) + (z - \mu_0)\langle \varphi(z), \varphi(\mu_0) \rangle_0 = -\frac{\pi (-z)^\gamma}{2 \sin \pi \mu}, \]

which corresponds to formula (3.3) with \( m = 0 \). Finally, from Krein’s formula

\[(H' - z)^{-1} = (A_0 - z)^{-1} - \frac{\langle \cdot, \varphi(z) \rangle_0}{Q_0(z) - t} \varphi(z) \]

we obtain that all canonical selfadjoint extensions of \( S_0 \) are given by

\[ H' = \left\{ u + c_0, A_0 u + c_0 \varphi(\mu_0)^\gamma \right\} | u \in \text{dom} A_0, c = \frac{\langle u, \varphi(\mu_0)^\gamma \rangle_0}{t - Q_0(\mu)} \}, \]

\[ t \in \mathbb{R} \cup \{ \infty \}. \]

Krein’s formula implies that in the model representation \( R'(z) \) is an integral operator

\[(R'(z)) f(x) = \int_{0}^{\infty} R'(x, y; z) f(y) \, dy \]

with kernel

\[ R'(x, y; z) = \frac{\delta(x - y)}{y - z} - \frac{y^{\gamma/2}}{y - z} \frac{1}{2(\sqrt{Q_0(z)} - t)} \frac{x^{\gamma/2}}{x - z}. \]

Using that \( P_\infty = I \), the identity operator on \( L^2(\mathbb{R}^+) \), and that \( E_0(\mathcal{A}) \) is the operator in \( L^2(\mathbb{R}^+) \) of multiplication by the characteristic function of the interval \( \mathcal{A} \), we get

\[(W^\pm f)(x) = s - \lim_{\sigma \downarrow 0} \pm i\eta \int_{\mathbb{R}} R'(x, \lambda; \lambda) f(\lambda) \, d\lambda. \]

Since

\[ \pm i\eta R'(x, \lambda; \lambda) = \delta(x - \lambda) - \frac{\lambda^{\gamma/2}x^{\gamma/2}}{2(\sqrt{Q_0(\lambda)} + i\eta) - t(x - \lambda \pm i\eta)}, \]

the formula for the kernels \( \hat{e}_\pm(x, \lambda) \) can now easily be verified.
We now calculate the kernel $e_-(y, \lambda) = H_{\lambda}^{-1} \hat{e}_-(\lambda, \lambda)(y)$ in the setting of the Bessel operator. From the second formula in (0.4), the fact that the Hankel transform is unitary, and (3.1), it follows that

$$\frac{1}{2} \int_0^\infty \sqrt{y} J_\lambda(y \sqrt{x}) \frac{x^{\nu/2}}{x - \lambda - i0} dx = (-\lambda - i0)^{\nu/2} \sqrt{y} K_\lambda(y \sqrt{-\lambda - i0})$$

$$= i \frac{\pi}{2} \lambda^{\nu/2} \sqrt{y} H^{(1)}_\lambda(y \sqrt{\lambda}),$$

where $(-\lambda - i0)^{\nu/2} = e^{-i\nu \lambda^{\nu/2}}$. Expressing $H^{(1)}_\lambda(y)$ in terms of $J_\lambda(y)$ and $J_{\lambda}(-y)$ (see [E, p. 4]),

$$H^{(1)}_\lambda(y) = (i \sin \pi \nu)^{-1} (J_\lambda(y) - J_{\lambda}(y) e^{-i\pi}) \quad (5.1)$$

we obtain

$$e_-(y, \lambda) = \frac{1}{\sqrt{2}} \int_0^\infty \sqrt{y} J_\lambda(y \sqrt{x}) \hat{e}(x, \lambda) \, dx$$

$$= \frac{1}{\sqrt{2}} c_1(\lambda) \sqrt{y} J_\lambda(y \sqrt{\lambda}) + \frac{1}{\sqrt{2}} c_2(\lambda) \sqrt{y} J_{\lambda}(-y \sqrt{\lambda}),$$

where

$$c_1(\lambda) = 1 + \frac{\pi}{2 \sin \pi \nu (Q_0(\lambda + i0) - t)} , \quad c_2(\lambda) = -\frac{\pi}{2 \sin \pi \nu (Q_0(\lambda + i0) - t)} \lambda^\nu .$$

Using the definition of $x$ and that

$$Q_0(\lambda + i0) = -\frac{\pi \lambda^\nu e^{-i\nu \pi}}{2 \sin \pi \nu} ,$$

we see that the expressions for $e_-(y, \lambda)$, the one here and the one in the proposition, coincide. The formula for the other kernel follows from $e_+(y, \lambda) = e_-(y, \lambda)^*$. The eigenfunction expansion formula follows from the remark preceding the proposition.  

In the remainder of this section we sketch a similar picture for the case Hypotheses 3.1 hold. We consider three cases $t \neq 0, \infty, t = 0$, and $t = \infty$.

The case $t \neq 0, \infty$. In this case $\sigma(H^t) \cap \mathbb{R}^+ = \emptyset$. The operators $H^t$ and $A_0$ act in different spaces in the nonnegative model from Section 2 as well as in the setting of the Bessel operator. The definition of the wave operator should therefore be adapted to a two space version; see for instance [RS]. Recall from Section 4 that $H_m = L_m^+ \oplus L_m^-$ is the spectral decomposition of
\( \Pi_m \) with respect to the spectra \( \sigma_d(H') \) and \( \sigma_w(H') \). By Theorem 4.2, \( \mathscr{L}_1 \) is a Hilbert subspace of \( \Pi_m \). Let \( \Pi_m \) be the projection in \( \Pi_m \) onto \( \mathscr{L}_1 \) and \( J \) be the inclusion map: \( \mathscr{H}_0 \hookrightarrow \Pi_m = \mathscr{H}_0 \oplus \mathbb{C}^m \oplus \mathbb{C}^m \). Then the wave operators \( W^\pm(H', A_0) \) are defined by

\[
W^\pm = W^\pm(H', A_0) = s - \lim_{\tau \to \pm \infty} e^{i H' \tau} Q_{ac} \mathcal{J} e^{-i A_0 \tau} P_{ac}.
\]

These operators are well defined, continuous and complete because the rank of the difference of the resolvents is one; see [DT]. Moreover, they define isometries from \( \mathscr{H}_0 \) to \( \mathscr{L}_1 \). \( W^\pm \) can be calculated explicitly through the time-independent representation

\[
W^\pm = \lim_{\eta \downarrow 0} \left[ e^{i H' \eta} Q_{ac} \mathcal{J} e^{-i A_0 \eta} P_{ac} \right] = \lim_{\eta \downarrow 0} \left[ e^{i H' \eta} Q_{ac} \mathcal{J} e^{-i A_0 \eta} P_{ac} \right].
\] (5.2)

We first consider the model from Section 2. \( A_0 \) is multiplication by \( \lambda \) in \( L^2(\mathbb{R}^+) \) and \( \gamma(\lambda) = \lambda^{1/2}/\sqrt{2} \). From (3.9) it is clear that in this model

\[
(R'(z) J f)(x) = \int_0^\infty R'(x, y; z) f(y) \, dy
\]

with kernel

\[
R'(x, y; z) = \begin{pmatrix} \delta(x - y) & \frac{\sqrt{2} \eta^{1/2 - m}}{\sqrt{2} (x - z)} \left( \sum_{j=1}^{2m} \frac{x^{m_j} - y^{m_j}}{(m_j + j - 1)/(m_j + m_j - j)} \right) \right) \\
\frac{\sqrt{2} \eta^{1/2 - m}}{\sqrt{2} (x - z)} \left( \sum_{j=1}^{2m} \frac{x^{m_j} - y^{m_j}}{(m_j + j - 1)/(m_j + m_j - j)} \right) \right) \\
0 & 0
\end{pmatrix}
\]

As in the proof of Proposition 5.1, \( P_{ac} = I \), the identity on \( \mathscr{H}_0 \). Moreover in this case we may also replace in (5.2) the projection \( Q_{ac} \) by the identity on \( \Pi_m \), because \( I_{\Pi_m} - Q_{ac} \) is the projection onto the spectral subspace \( \mathscr{L}_1 \) corresponding to the point spectrum of \( H' \) which lies on the circle \( |\lambda + i\eta| = s > 0 \) and has a positive distance from \( \mathbb{R}^+ \) (see Theorem 4.2) and hence

\[
s - \lim_{\eta \downarrow 0} \left[ e^{i H' \eta} Q_{ac} \mathcal{J} e^{-i A_0 \eta} P_{ac} \right] = 0.
\]

It follows that

\[
W^\pm f(x) = s - \lim_{\eta \downarrow 0} \left[ e^{i H' \eta} Q_{ac} \mathcal{J} e^{-i A_0 \eta} P_{ac} \right] f(\lambda) \, d\lambda.
\]
So the wave operators in the model are integral operators

\[ W^\pm f(x) = \int_0^\infty \hat{e}_\pm(x, \lambda) f(\lambda) d\lambda, \]

whose kernels

\[ \hat{e}_\pm(x, \lambda) = \text{column}((\hat{e}_d(x, \lambda)), (\hat{e}_0(\lambda)))_{j=1}^m, (\hat{e}_j(\lambda)))_{j=1}^m \]  

have components given by

\[ \hat{e}_d(x, \lambda) = \frac{\lambda^{v/2-m} \lambda^{v/2+m}}{Q(\lambda \mp i0, t)(x - \lambda \pm i0)} \]

and

\[ \hat{E}_0(\lambda) = -t \frac{\lambda^{v/2-j}}{\sqrt{2} Q(\lambda \mp i0, t)}, \quad \hat{E}_j(\lambda) = -\frac{\lambda^{v/2+j-1}}{\sqrt{2} Q(\lambda \mp i0, t)}. \]

Here

\[ Q(\lambda \mp i0, t) = \frac{\pi \lambda e^{\pm i\pi v}}{2 \sin \pi v} - t. \]

The eigenfunctions in the initial setting where \( A_0 \) is the Bessel operator, are obtained using the inverse Hankel transform.

**Theorem 5.2.** The wave operators \( W^\pm = W^\pm(H', A_0), \ t \neq 0, \infty, \) are integral operators with kernels given by

\[ e_\pm(y, \lambda) = \left( \frac{\lambda^{v/2-m} \lambda^{v/2+m}}{2 \sin \pi v \ sin x \lambda^{v/2+j-1}} \right) \]

Here \( c_\pm(\lambda) \) are the same as in Proposition 5.1 and \( [J_\pm(\lambda)] \) stands for the regular part of the function \( J_\pm(\lambda) \), which is obtained from \( J_\pm(\lambda) \) by removing the first \( m \) singular terms in the standard expansion of \( J_\pm(\lambda) \) in powers of its argument \( y \), that is, if \( J_\pm(\lambda) = \sum_{m=0}^\infty c_j \lambda^{v/2+j} \), then \( [J_\pm(\lambda)] = \sum_{m=0}^\infty c_j \lambda^{v/2+2j} \).

**Proof.** We explain briefly how formula (5.5) is obtained. The calculations are similar to the ones in the proof of Proposition 5.1 and make use...
of Proposition 4.3. The inverse Hankel transform of the component $\hat{e}(\lambda)$ in (5.3) is given by

$$(H^{-1}_r \hat{e}(\lambda))(y) = \frac{1}{\sqrt{2}} \sqrt{y} J_0(y \sqrt{x}) \frac{x^{\nu/2 + m}}{\sqrt{2} Q(\lambda + i0, \ell)} H^{-1}_r \left( \frac{x^{\nu/2 - m}}{\sqrt{2} (x - \lambda \pm i0)} \right).$$

In the proof of Proposition 4.3 we showed that

$$(H^{-1}_r \hat{e}(\lambda))(y) = \frac{1}{2} \int_0^\infty \sqrt{y} J_0(y \sqrt{x}) \frac{x^{\nu/2 - m} dx}{x - (\lambda + i0)} \left[ K_0(y \sqrt{x - \lambda - i0}) \right].$$

If (with the help of (5.1) and (0.4)) $K_0(z)$ is expressed as a linear combination of the modified Bessel functions $I_\nu(z)$:

$$K_0(z) = a I_\nu(z) + b I_{-\nu}(z),$$

then

$$(H^{-1}_r \hat{e}(\lambda))(y) = \frac{1}{\sqrt{2}} c_1(\lambda) \sqrt{y} J_0(y \sqrt{x}) + \frac{1}{\sqrt{2}} c_2(\lambda) \sqrt{y} [J_{-\nu}(y \sqrt{x})]$$

with the same coefficient functions $c_1(\lambda)$ and $c_2(\lambda)$ as in Proposition 5.1. This proves the formula for the first entry of $e^0(y, \lambda)$ in (5.5). The other entries of $e^0(y, \lambda)$ are obtained from the corresponding entries of $e^0(x, \lambda)$ and the definition of $\pi$ in Proposition 5.1. 

The wave operators define an expansion in generalized eigenfunctions which can be identified with the kernels of $W^{\pm}$: For all $f \in \mathcal{F}$, we have

$$f = W^{\pm} \tilde{f}, \quad \text{where} \quad \tilde{f} = W^{\pm} f.$$ 

We shall write down the expansion formula more explicitly. Recall that $\varphi(\lambda) = \Gamma_1 1$; hence if $\lambda \in \sigma(H^t)$, then $\varphi(\lambda)$ is the corresponding eigenvector and

$$c(\lambda) := \langle \varphi(\lambda), \varphi(\lambda^*) \rangle = \frac{\partial}{\partial z} Q(z) \bigg|_{z = \lambda}.$$

By Theorem 4.2, the eigenvalues of $H^t$ lie on the circle $|z| = s$ in the complex $z$ plane; $H^t$ has a real eigenvalue, namely $z = -s$, only if $\text{sign} \ell = (-1)^{\nu+1}$. Denote by $\lambda_j, j = 1, \ldots, k(t)$, the eigenvalues of $H^t$ in the upper half plane $C^+$. (Note $k(t) \leq m$.)
Corollary 5.3. With the above notation the expansion of an arbitrary 
$f \in \Pi_{ac}$ in terms of the generalized eigenfunctions of $H'$, $t \neq 0, \infty$, takes the 
form

$$f(y) = \int_{-\infty}^{+\infty} e_{\pm}(y, \lambda) \langle f, e_{\pm}(\lambda) \rangle \, d\lambda + \frac{\langle f, \varphi(-s) \rangle}{c(-s)} \varphi(y, -s) + \sum_{j=1}^{k(t)} \left( \frac{\langle f, \varphi(\lambda_j^+) \rangle}{c(\lambda_j^+)} - \frac{\langle f, \varphi(\lambda_j^-) \rangle}{c(\lambda_j^-)} \right) \varphi(y, \lambda_j)$$

with the understanding that the middle term on the right should be deleted 
when $\text{sign } t \neq (-1)^{[v]+1}$. In particular, the kernels of the wave operators 
$W^\pm(H', A_0)$ form a complete system in the Hilbert space $\Pi_{ac} = \mathcal{D}'$, 
$t \neq 0, \infty$: for $f \in \Pi_{ac}$ we have

$$f(y) = \int_{-\infty}^{+\infty} e_{\pm}(y, \lambda) \langle f, e_{\pm}(\lambda) \rangle \, d\lambda,$$

where the integral converges in the space $\Pi_{ac}$.

Two cases remain; we first consider $t = 0$ and then $t = \infty$.

The Case $t = 0$. $\sigma_{\text{pf}}(H^0) = \{0\}$ and $\lambda = 0$ is the only singular critical point of $H^0$. The original definition of wave operators is inapplicable. However, $H^0$ induces the selfadjoint operator $A_0$ in $H^0$ by considering $H^0$ as the quotient 
space $\mathcal{D}' / \mathcal{D}_0$; see Section 4. The wave operators for the pair $A_0, A_0$ is well 
defined and their kernels which are the generalized eigenfunctions of $A_0$ given 
by

$$e_0^0(x, \lambda) = \delta(x - \lambda) - \frac{x^{1/2} - m^{1/2} + m}{2Q(x + i0, 0)},$$

$$e_\pm^0(y, \lambda) = \frac{1}{\sqrt{2}} e^{-iy\sqrt{\lambda}} \sqrt{y} \left[ J_\pm(y \sqrt{\lambda}) \right].$$

Here of course the first kernel corresponds to the model case, and the 
second one to the setting of the Bessel operator. Denote by $u_1, u_2, ..., u_m$ a 
Jordan chain of $H^0$ at $\lambda = 0$; the $u_i$’s span $\mathcal{D}_0$. Let $w_1, w_2, ..., w_m$ be a basis 
for $\mathcal{D}_0^\perp$ such that $\langle u_i, w_j \rangle = \delta_{ij}$. The expansion of an arbitrary $f \in \mathcal{D}_0^\perp$ in the “eigenfunctions” of $A_0$ and the root vectors of $H^0|_{\mathcal{D}_0^\perp}$ takes the form

$$f(y) = \int_{-\infty}^{+\infty} e_{\pm}^0(y, \lambda) \langle f, e_{\pm}^0(\lambda) \rangle \, d\lambda + \sum_{j=1}^{m} u_j(y) \langle f, w_j \rangle.$$  

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Under the condition
\[ \int_0^{+\infty} \lambda^{-\nu/2 + j - 1} \langle f, e_{\pm}(\lambda) \rangle_0 \, d\lambda < \infty, \quad j = 1, \ldots, m, \]
this expansion can be rewritten as an expansion in the “eigenfunctions” of \( H^0 \),
\[ f(y) = \int_0^{+\infty} \text{column} \left( e_{\pm}(y, \lambda), 0, \left( \frac{\sqrt{2} \sin \frac{\pi v}{\pi} e^{i\pi v} \lambda^{-\nu/2 + j - 1} \right)_{j=1}^m \right) \]
\[ \times \langle f, e_{\pm}(\lambda) \rangle_0 \, d\lambda + \sum_{j=1}^m u_j(y) \left( \langle f, w_j \rangle - \frac{\sqrt{2} \sin \frac{\pi v}{\pi} e^{i\pi v} \lambda^{-\nu/2 + j - 1} \langle f, e_{\pm}(\lambda) \rangle_0 \, d\lambda \right). \]

The column in the first integral is just the limit \( \lim_{t \to 0} e_{\pm}(y, \lambda) \) of the “eigenfunction” (5.5).

The Case \( t = \infty \). \( \sigma_A(H^\infty) = \{ \infty \} \) and \( \infty \) is the only singular critical point of \( H^\infty \). As explained in Section 4, \( H^\infty \) induces the initial operator \( A_0 \) in \( H^0 \), when \( H^0 \) is viewed as the quotient space \( \mathcal{D}_\infty / \mathcal{D}_0 \). Evidently, the wave operators for the pair \( A_0, A_0 \) equal the identity and the generalized eigenfunctions of \( A_0 \) are given by
\[ e_0(x, \lambda) = \delta(x - \lambda), \quad e_0(y, \lambda) = \frac{1}{\sqrt{2}} J_z(y \sqrt{\lambda}). \quad (5.8) \]

Remark 5.4. (1) The functions in the first component of the column representation (5.5) of \( e(\lambda) \) can be considered as the generalized eigenfunctions of the Strauss extension [DLS] \( T_j(\lambda + i0) \) of \( S_0 \). Abstractly these relations are defined by
\[ T_j(z) = \{ \{ P_0 f, P_0 g \} | \{ f, g \} \in H', g - zf \in \mathcal{W}_0 \}, \quad z \in \mathbb{C}, \]
where \( P_0 \) is the projection in \( \Pi_m \) onto its first summand \( \mathcal{W}_0 \).

2. Even though the original definition of a wave operator is inapplicable when \( t = 0 \) and \( t = \infty \), the generalized eigenfunctions \( e(\lambda) \), as can be seen from the formulas, exist and define isometric operators on a dense domain. But these operators are not bounded.
3. We can look at the eigenfunctions as ordinary singular functions of the type (0.2) using the embedding $\tau_\delta$

$$e(y, \lambda) = \tau_\delta \tilde{e}(y, \lambda),$$
$$\tilde{e}(\lambda, y) = e(\lambda) \sqrt{y} (\cos \alpha J_{\lambda}(y \sqrt{\lambda}) + \sin \alpha J_{\lambda,-}(y \sqrt{\lambda})).$$

It is a symbolic writing in terms of kernels of the operator equality

$$W^- f = \tau_\delta \tilde{W}^- f, \quad f \in \mathcal{H}_{m+1},$$

where

$$\tilde{W}^- : \mathcal{H}_{m+1} \to \mathcal{H}_m, \quad \tilde{W}^- f = \int_0^{+\infty} \tilde{e}(\lambda, y) f(\lambda) d\lambda.$$

6. SCATTERING MATRICES

In this section we calculate the scattering operators $S^i := W^+*(H^i, A_0)$ $W^-(H^i, A_0)$ and $S^0 := W^+*(A^0, A_0) W^-(A^0, A_0)$. The scattering operator $S^a$ corresponding to the pair $A_0$, $A_0$ is of course $I$. We only consider the spectral model in which $A_0$ is the multiplication operator in $\mathcal{H}_0 = L^2(\mathbb{R}^+)$. Our calculations here follow the same lines as in [F] (for the potential scattering case) [BYA] (where abstract stationary schemes are considered) and [AP] (where as in this note Krein’s formula is used but in a Hilbert space setting). As our starting point we use the structure formula (3.9).

**Theorem 6.1.** Assume Hypotheses 3.1. The scattering operator $S^i$ on $\mathcal{H}_0$ is the operator of multiplication by a unimodular function $s^i(\lambda)$ (the scattering matrix), $(S^i g)(\lambda) = s^i(\lambda) g(\lambda)$, where

$$s^i(\lambda) = \begin{cases} \frac{Q(\lambda - i0, t)}{Q(\lambda + i0, t)} = \frac{-i\lambda^* - \cot(\pi t) \lambda^* - (2/\pi) t}{i\lambda^* - \cot(\pi t) \lambda^* - (2/\pi) t}, & t \neq 0, \\ \frac{Q(\lambda - i0)}{Q(\lambda + i0)} = \frac{-i - \cot(\pi t)}{i - \cot(\pi t)} = \lim_{t \to 0} s^i(\lambda), & t = 0, \end{cases}$$

and $Q_0$ is given by (3.3). In particular, $S^i$ is unitary and continuous in $t$.

**Proof.** We identify $S^i$ with $s^i$. The main technical trick in proving the structural formula

$$S^i(\lambda) = \frac{Q(\lambda - i0, t)}{Q(\lambda + i0, t)}$$
is the same as in the cited works and consists of combining the resolvent identity with the formulas for $W^±$ derived in Section 5.

We rewrite formula (3.9) in the slightly different form

$$
(H' - z)^{-1} \begin{pmatrix} g' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{R}_0(z) g \\ 0 \\ 0 \end{pmatrix} - \bar{P}_z \frac{\mathcal{A}_0 \mathcal{Z}_{-1}}{Q(z, t)}, \tag{6.1}
$$

where

$$
\mathcal{Q}(z, t) := z^{-2m}Q(z, t) \quad \text{and} \quad \bar{P}_z 1 := \begin{pmatrix} \mathcal{R}_0(z) \mathcal{Z}_{-1} \\ (iz^{-m+j})^m \mathcal{X}_{j=1}^{m} \end{pmatrix}.
$$

Projecting the resolvent identity for $R_t(z)$ on $H_0$ and using the written formula for $R_t(z) P_0$ we obtain the relation

$$
\mathcal{Q}(z, t) - \mathcal{Q}(\zeta, t)^* = \mathcal{P}^* \mathcal{P}_z, \tag{6.2}
$$

In this new notation we have

$$
(W^±g) = -\int_0^{+\infty} \mathcal{P}_{±i0} \frac{\mathcal{Z}_{-1}(\mathcal{Z})^* g(\mathcal{Z}) d\mathcal{Z}}{Q(\mathcal{Z} ± i0, t)}
$$

and

$$
(W^±*f)(\mathcal{Z}) = (P_0 f)(\mathcal{Z}) - \frac{\mathcal{Z}_{-1}(\mathcal{Z})}{Q(\mathcal{Z} ± i0, t)} \mathcal{P}^*_{±i0} f.
$$

Inserting these expressions in the formula $S = W^+ W^-$ we get

$$
(S'g)(\mathcal{Z}) = g(\mathcal{Z}) - \int_0^{+\infty} \mathcal{P}_{±i0} \frac{\mathcal{Z}_{-1}(\mathcal{Z})^* g(\mathcal{Z}) d\mathcal{Z}}{Q(\mathcal{Z} ± i0, t)} - \frac{\mathcal{Z}_{-1}(\mathcal{Z})}{Q(\mathcal{Z} ± i0, t)} \mathcal{P}^*_{±i0} Jg \\
+ \frac{\mathcal{Z}_{-1}(\mathcal{Z})}{Q(\mathcal{Z} ± i0, t)} \int_0^{+\infty} \mathcal{P}^*_{±i0} \mathcal{P}_{±i0} 1 \frac{\mathcal{Z}_{-1}(\mathcal{Z})^* g(\mathcal{Z}) d\mathcal{Z}}{Q(\mathcal{Z} ± i0, t)} . \tag{6.3}
$$
We observe the simple relations

\[
(P_0 \tilde{F}^*_{x+i0} 1)(\lambda) = R_0(x+i0) Z_{-1}(\lambda) = \frac{\lambda^i}{\lambda - x - i0},
\]
\[
\tilde{F}^*_{\lambda - i0} J g = \int_0^{+\infty} \frac{Z_{-1}(x)^* g(x)}{x - \lambda + i0} \, dx.
\]

It follows that the second summand in (6.3) can be rewritten as

\[
-Z_{-1}(\lambda) \int_0^{+\infty} \left( \frac{1}{\lambda - x - i0} - \frac{1}{\lambda - x + i0} \right) \frac{Z_{-1}(x)^* g(x)}{Q(x+i0, t)} \, dx
\]
\[
-\frac{Z_{-1}(\lambda)}{Q(\lambda + i0, t)} \tilde{F}^*_{\lambda - i0} J g = \int_0^{+\infty} \frac{Z_{-1}(x)^* g(x)}{Q(x+i0, t)} \, dx.
\]

and the third one as

\[
\frac{Z_{-1}(\lambda)}{Q(\lambda + i0, t)} \tilde{F}^*_{\lambda - i0} J g = \int_0^{+\infty} \frac{Z_{-1}(x)^* g(x)}{Q(x+i0, t)} \, dx.
\]

Hence the sum of the second and the third summands in (6.3) can be represented as a sum of the two quantities

\[
-Z_{-1}(\lambda) \int_0^{+\infty} \frac{1}{\lambda - x - i0} \frac{Z_{-1}(x)^* g(x)}{Q(x+i0, t)} \, dx
\]
\[
= \frac{2\pi i |Z_{-1}(\lambda)|^2}{Q(\lambda + i0, t)} g(\lambda)
\]

(where we used that \(\frac{1}{\lambda - x - i0} - \frac{1}{\lambda - x + i0} = 2\pi i (\lambda - x)\)) and

\[
-Z_{-1}(\lambda) \int_0^{+\infty} \frac{1}{Q(x+i0, t)} \frac{Z_{-1}(x)^* g(x)}{Q(\lambda + i0, t)} \, dx
\]
\[
= -\frac{Z_{-1}(\lambda)}{Q(\lambda + i0, t)} \int_0^{+\infty} \frac{Z_{-1}(x)^* g(x)}{Q(x+i0, t)} \, dx.
\]

On account of (6.2), the last quantity here is \(-1 \times\) the last summand in (6.3). Hence

\[
(S'g)(\lambda) = g(\lambda) - 2\pi i \frac{|Z_{-1}(\lambda)|^2}{Q(\lambda + i0, t)} g(\lambda).
\]
From $$\tilde{Q}(\lambda + i0, t) - \tilde{Q}(\lambda - i0, t) = 2i \text{ Im } \tilde{Q}(\lambda + i0, t) = 2\pi i |\chi_{-1}(\lambda)|^2$$ we obtain the claimed formula for the scattering matrix,

$$(S^t g)(\lambda) = \frac{\tilde{Q}(\lambda - i0, t)}{\tilde{Q}(\lambda + i0, t)} = \frac{Q(\lambda - i0, t)}{Q(\lambda + i0, t)}.$$ By the formula (3.3) for $$Q(z, t)$$ we have that

$$Q(\lambda \pm i0, t) = \frac{\pi}{2} \frac{\lambda}{\sin \pi \nu} e^{it \nu} - t = \frac{\pi}{2} (\pm i\lambda' - \cot(\pi \nu) \lambda') - t.$$ This proves the formula for $$s'(\lambda)$$ in the case $$t \neq 0$$. The formula for $$s_0'(\lambda)$$ can be recovered from a formula in [AP] as now $$A_0^0, A_0$$ are different selfadjoint extensions of $$S_0$$ in the Hilbert space $$\mathcal{H}_0$$ which are related by Krein’s formula (4.4) with $$t = 0$$. It can also be derived as above by writing the wave operators as

$$W^z(A_0^0, A_0) g(x) = \int_{-\infty}^{\infty} \mathcal{E}^0_\pm(x, \lambda) g(\lambda) d\lambda,$$ where $$\mathcal{E}^0_\pm(x, \lambda)$$ is given by (compare with (5.4))

$$\mathcal{E}^0_\pm(x, \lambda) = \frac{x^{\nu/2 - m} \sqrt{2}(x - \lambda)}{2Q_d(x + i0, \lambda) \mp i0, 0}(x - \lambda \pm i0).$$ If we change in the calculations above $$\Gamma_1$$ by $$x^{\nu/2 - m/2}(x - z) = R_d(z) \chi_{-1}(z)$$ and $$\tilde{Q}(z)$$ by $$Q_d(z)$$ and take into account that the inclusion map $$J$$ is the identity operator on $$\mathcal{H}_0$$, we also obtain the desired formula for $$s_0'(\lambda)$$.}

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