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Published in:
Proceedings of the royal society of edinburgh section a-Mathematics

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2000

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

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Dissipative eigenvalue problems for a Sturm–Liouville operator with a singular potential

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(MS received 19 February 1999; accepted 11 May 1999)

In this paper we consider the Sturm–Liouville operator \( d^2/dx^2 - 1/x \) on the interval \([a, b]\), \( a < 0 < b \), with Dirichlet boundary conditions at \( a \) and \( b \), for which \( x = 0 \) is a singular point. In the two components \( L^2(a, 0) \) and \( L^2(0, b) \) of the space \( L^2(a, b) = L^2(a, 0) \oplus L^2(0, b) \) we define minimal symmetric operators and describe all the maximal dissipative and self-adjoint extensions of their orthogonal sum in \( L^2(a, b) \) by interface conditions at \( x = 0 \). We prove that the maximal dissipative extensions whose domain contains only continuous functions \( f \) are characterized by the interface condition \( \lim_{x \to 0^+} \left( f'(x) - f'(-x) \right) = \gamma f(0) \) with \( \gamma \in \mathbb{C}^+ \cup \mathbb{R} \) or by the Dirichlet condition \( f(0^+) = f(0^-) = 0 \). We also show that the corresponding operators can be obtained by norm resolvent approximation from operators where the potential \( 1/x \) is replaced by a continuous function, and that their eigen and associated functions can be chosen to form a Bari basis in \( L^2(a, b) \).

1. Introduction

In this paper we consider the differential expression

\[
l[f](x) := -f''(x) - \frac{f(x)}{x}
\]

and the corresponding differential equation

\[
-f''(x) - \frac{f(x)}{x} - \lambda f(x) = 0
\]

*Dedicated to Professor Boele Braaksma on the occasion of his 65th birthday, in friendship.
on the interval \([a, b]\), where \(a < 0 < b\), with the boundary conditions
\[
f(a) = f(b) = 0.
\]

(1.3)

Since the potential is not summable at \(x = 0\), it is not a classical Sturm–Liouville problem. We associate with this boundary eigenvalue problem two minimal operators in the spaces \(L^2([a, 0])\) and \(L^2((0, b])\). Since these operators are in the limit case at \(x = 0\), they are not self-adjoint and their direct sum operator \(S\) in the space \(L^2([a, b])\) is symmetric with defect index \((2, 2)\). It is the aim of this paper to describe all self-adjoint and maximal dissipative extensions of \(S\) in \(L^2([a, b])\). Recall that an operator \(A\) in some Hilbert space \(\mathcal{H}\) is called dissipative if \(\text{Im}(Af, f) \geq 0\) for all \(f \in \mathcal{H}\) and maximal dissipative if it does not have a proper dissipative extension. In particular, we also describe those extensions among them for which the domain consists only of continuous functions. This set turns out to be a one-parameter family of operators \(T_\gamma, \gamma \in \mathbb{C}^+ \cup \{\infty\}\), which are defined by the interface condition
\[
\lim_{x \to 0^+} (f'(x) - f'(-x)) = \gamma f(0) \quad \text{if } \gamma \in \mathbb{C},
\]
and by
\[
f(0^+) = f(0^-) = 0 \quad \text{if } \gamma = \infty.
\]

The problem (1.1) has been studied by several authors \([4, 8, 12]\). In \([4]\) the potential \(-x^{-1}\) is replaced by the regular potential \(-x(i\varepsilon)^{-1}\) and the resulting operator for \(\varepsilon \to 0\) is considered. This operator is the extension \(T_\gamma\) with \(\gamma = -i\pi\) (see Remark 5.2). In \([8]\) the operator \(T_\infty\) is studied: it is the direct sum of two self-adjoint operators on \([a, 0]\) and \((0, b]\), respectively, with Dirichlet boundary conditions. Gunson treats the operators \(T_{-i\pi}\) [12, theorem 2.6 and eqn (2.13)] and \(T_\infty\) [12, theorem 2.2 and eqn (2.1)] as well as \(T_0\), where the potential \(-x^{-1}\) is considered in the distributional sense as the Cauchy principal value [12, theorem 2.4 and eqn (2.9)]. This self-adjoint operator is also studied in \([1]\) from the viewpoint of quasi-derivatives. We mention that the operators \(T_\gamma\) considered here have discrete spectrum. The case where the interval \([a, b]\) is replaced by the real axis is also considered in \([12]\). In this case the corresponding operators \(T_{i\theta}\) with \(0 < \theta < \pi\) also have an absolutely continuous spectrum and \(T_{i\pi}\) has only absolutely continuous spectrum. For a more recent discussion about the potential \(-x^{-1}\) in the physics literature, we refer to \([14, 17, 18, 20]\), and the references therein.

In § 2 we introduce the symmetric operator \(S\). In § 3 all self-adjoint and maximal dissipative extensions of \(S\) are described by an interface condition at \(0\). Here we use essentially the fact that all these extensions are contained in \(S^*\). There also exist extensions of \(S\) in \(L^2([a, b])\) with a non-empty resolvent set which are not contained in \(S^*\) [3]. The extensions \(T_\gamma, \gamma \in \mathbb{C} \cup \{\infty\}\), are described in § 4. By a method already used in \([12]\) it is shown that the extensions \(T_\gamma\) for \(\gamma \in \mathbb{C}\) can be obtained as norm resolvent limits of operators generated by regular potentials. An analogous result for the case \(\gamma = \infty\) can be found in [3]. In § 5 we express the solutions of equation (1.2) by Whittaker functions in order to get information about the characteristic determinant and the asymptotics of the eigenvalues. This is used in § 6, where we prove that the system of root vectors of the operator \(T_\gamma\) forms a Bari basis in \(L^2([a, b])\). Finally, the Fourier coefficients of the corresponding expansions
are expressed by inner products in $L^2([a,b])$ with the complex conjugate functions of the root functions (which are the root functions of the adjoint operator).

2. The symmetric operator $S$

Let $a < 0 < b$. We consider the differential expression $l[f]$ from (1.1) on the intervals $I := [a,b]$, $I_- := [a,0)$ and $I_+ := (0,b]$; at the endpoints $a$ and $b$ we always impose the Dirichlet boundary conditions (1.3). In the space $L^2(I_\pm)$ a minimal operator $L_\pm$ and a maximal operator $L^*_\pm$, which is the adjoint of the minimal operator in $L^2(I_\pm)$, are associated with the differential expression $l$. The domain of the maximal operator $L^*_\pm$ is

$$D(L^*_\pm) := \{ f \in L^2(I_\pm) : f, f' \in AC_{loc}(I_\pm), f(b) = 0, l[f] \in L^2(I_\pm) \}$$

and $L^*_\pm f = l[f]$ if $f \in D(L^*_\pm)$. Here, for example, $AC_{loc}(I_\pm)$ is the set of locally absolutely continuous functions on $I_\pm$. The set $D(L^*_\pm)$ and the operator $L^*_\pm$ are defined correspondingly. To describe the domains of the minimal operators $L_\pm$, we introduce for $f, g \in D(L^*_\pm)$ and $x, x_1, x_2 \in I_\pm$ the sesquilinear forms

$$[f, g]_{x_1} := f(x)g'(x) - f'(x)g(x), \quad [f, g]_{x_1}^2 := [f, g]_{x_2} - [f, g]_{x_1}. \quad (2.1)$$

Then Green’s formula becomes

$$[f, g]_{x_1}^2 = \int_{x_1}^{x_2} (l[f](x)g(x) - f(x)l[g](x)) \, dx. \quad (2.2)$$

It implies that the limits $\lim_{x \to 0^\pm} [f, g]_x = [f, g]_0^\pm$ exist and are finite and that the sesquilinear forms $[\cdot, \cdot]_{x_1}^2$ are continuous on $D(L^*_\pm)$ with respect to the $L^*_\pm$-graph norms. The domains of the minimal operators can be described as follows [7, theorem 2.3]:

$$D(L_-) = \{ f \in D(L^*_\pm) : [f, g]_a^0 = 0 \text{ for all } g \in D(L^*_\pm) \}, \quad (2.3)$$

$$D(L_+) = \{ f \in D(L^*_\pm) : [f, g]_b^0 = 0 \text{ for all } g \in D(L^*_\pm) \}, \quad (2.4)$$

and Green’s formula (2.2) implies that the operators $L_\pm$ are symmetric.

Consider on the interval $[a,b]$ the functions

$$u(x) = x \quad \text{and} \quad v(x) = 1 - x \ln |x|. \quad \text{We choose numbers } \varepsilon_1, \varepsilon_2 : 0 < \varepsilon_1 < \varepsilon_2 < \min \{ -a, b \} \text{ and twice continuously differentiable functions } u_\pm \text{ on } I_\pm \text{ with the properties}$$

$$u_+(x) := \begin{cases} u(x) & \text{if } 0 < x < \varepsilon_1, \\ 0 & \text{if } \varepsilon_2 < x < b, \end{cases} \quad u_-(x) := \begin{cases} 0 & \text{if } a < x < -\varepsilon_2, \\ u(x) & \text{if } -\varepsilon_1 < x < 0, \end{cases}$$

and, analogously, functions $v_\pm$. For $x$ in a neighbourhood of 0,

$$l[u_\pm](x) = -1, \quad l[v_\pm](x) = \ln |x|,$$

hence $l[u_\pm], l[v_\pm] \in L^2(I_\pm)$ and $u_\pm, v_\pm \in D(L^*_\pm)$. Further,

$$[v_-, v_-]_a^0 = \lim_{x \to 0^-} (v_-(x)v'_-(x) - v'_-(x)v_-(x)) = 0, \quad (2.5)$$

$$[u_-, v_-]_b^0 = \lim_{x \to 0^-} (u_-(x)v'_-(x) - u'_-(x)v_-(x)) = -1, \quad (2.6)$$
and, analogously,
\[
\begin{align*}
[u_-, u_+]_{a}^0 &= [v_+, v_+]_{0+}^b = [u_+, u_+]_{0+}^b = 0, \\
[v_-, u_+]_{a}^0 &= [v_+, u_+]_{0+}^b = [u_+, v_+]_{0+}^b = 1.
\end{align*}
\] (2.7)

The sesqilinear forms $[,]_{a}^0$ and $[,]_{0+}^b$ vanish on $D(L_-)$ and $D(L_+)$, respectively; see equations (2.3) and (2.4). Therefore, the functions $u_\pm$ and $v_\pm$ are linearly independent modulo $D(L_\pm)$. Since $l$ is a second-order differential operator and boundary conditions at $a$ and $b$ have been fixed, the dimension of the factor space $D(L_\pm^*)/D(L_\pm)$ is at most 2, and we find
\[
D(L_-^*) = D(L_-) + \text{span}\{u_-, v_\}, \quad D(L_+^*) = D(L_+) + \text{span}\{u_+, v_+\}. \tag{2.8}
\]

Now we consider in the Hilbert space
\[
\mathcal{L}^2(I) = \mathcal{L}^2(I_-) \oplus \mathcal{L}^2(I_+)
\] (2.9)

the operator $S := L_- \oplus L_+$. Evidently, $S^* = L_+^* \oplus L_-^*$ and on $D(S^*)$ we define the sesqilinear form
\[
[f, g] := [f_-, g_-]_{a}^0 + [f_+, g_+]_{0+}^b, \quad f, g \in D(S^*), \tag{2.10}
\]
where $f = f_- + f_+$ and $g = g_- + g_+$ are the decompositions of the elements $f$ and $g$ with respect to (2.9). Relation (2.2) implies the Green’s formula
\[
[f, g] = (S^* f, g) - (f, S^* g), \quad f, g \in D(S^*), \tag{2.11}
\]
and the sesqilinear form on the left-hand side is again continuous in the $S^*$-graph norm on $D(S^*)$.

We extend the functions $u_\pm$ and $v_\pm$ to the whole interval $[a, b]$ as follows:
\[
\tilde{u}_-(x) := \begin{cases} u_-(x) & \text{if } x \in [a, 0), \\ 0 & \text{if } x \in (0, b], \end{cases} \quad \tilde{u}_+(x) := \begin{cases} 0 & \text{if } x \in [a, 0), \\ u_+(x) & \text{if } x \in (0, b], \end{cases}
\]
and $\tilde{v}_\pm$ are defined analogously. All these extended functions belong to $D(S^*)$. On $f \in D(S^*)$ the following functionals $u_\pm, v_\pm$ are defined:
\[
u_\pm f := [f, \tilde{u}_\pm], \quad u_+ f := [f, \tilde{u}_+], \quad v_- f := [f, \tilde{v}_-], \quad v_+ f := [f, \tilde{v}_+]. \tag{2.12}
\]
From (2.3) and (2.4) it follows that the functionals $u_\pm, v_\pm$ vanish on $D(S)$, and the definition of the functions $\tilde{u}_\pm, \tilde{v}_\pm$ yields for $f \in D(S^*)$ the relations
\[
u_\pm f = \pm f(0 \pm), \quad v_\pm f = \pm \lim_{x \to 0 \pm} (f'(x) + f(x)(1 + \ln |x|)), \tag{2.13}
\]
where we have used that the functions $f \in D(S^*)$ satisfy the relation
\[
f'(x) = O(\ln |x|) \quad \text{for } x \to 0; \tag{2.14}
\]
see [8, lemma 2.2]. Since the operators $L_\pm$ are symmetric, also $S$ is a symmetric operator and we have
\[
D(S) = \{ f \in D(S^*) : u_- f = u_+ f = v_- f = v_+ f = 0 \}. \tag{2.15}
\]
and
\[ \mathcal{D}(S^*) = \mathcal{D}(S) + \text{span}\{\tilde{u}_-, \tilde{u}_+, \tilde{v}_-, \tilde{v}_+\} \tag{2.16} \]

Therefore, the defect index of the operator \( S \) is \((2, 2)\).

**Lemma 2.1.** If \( f \in \mathcal{D}(S) \), it holds that
\[ f(x) = o(x), \quad f'(x) = o(1) \quad \text{for} \quad x \to 0, \tag{2.17} \]
and
\[ \mathcal{D}(S) = \{ f \in \mathcal{D}(S^*) : f, f' \text{ are continuous in } 0 \text{ and } f(0) = f'(0) = 0 \}. \tag{2.18} \]

**Proof.** If \( f \in \mathcal{D}(S) \), then (2.15) and the first relation in (2.13) imply, for \( x \to 0 \),
\[ f(x) = o(1). \tag{2.19} \]
Now relation (2.14) yields the sharper estimate
\[ f(x) = \int_0^x f'(t) \, dt = O(x \ln |x|), \tag{2.20} \]
and if we observe that \( v_{\pm} f = 0 \), it follows by (2.15) and the second relation in (2.13) that
\[ f'(x) = -(1 + \ln |x|)O(x \ln |x|) + o(1) = o(1) \]
and finally
\[ f(x) = \int_0^x f'(t) \, dt = o(x). \]
Thus the relations (2.17) and the inclusion
\[ \mathcal{D}(S) \subset \{ f \in \mathcal{D}(S^*) : f, f' \text{ are continuous in } 0 \text{ and } f(0) = f'(0) = 0 \} \]
are proved. The equality sign in (2.18) follows now from (2.16) and the fact that no linear combination \( f \) of the functions \( u_{\pm}, v_{\pm} \), except the trivial one, has the property that \( f \) and \( f' \) are continuous and fulfill \( f(0) = f'(0) = 0 \).

\[ \square \]

**3. The self-adjoint and the maximal dissipative extensions of \( S \)**

The symmetric operator \( S \) in \( \mathcal{L}^2(I) \) with defect index \((2, 2)\), which was associated with the differential expression \( l \) from (1.1) and the Dirichlet boundary conditions (1.3), has self-adjoint and maximal dissipative canonical extensions; here *canonical* means that these extensions act in the originally given space \( \mathcal{L}^2(I) \). We shall characterize these extensions by interface conditions at \( 0 \).

To this end, we first observe that all symmetric and dissipative canonical extensions of \( S \) are restrictions of the adjoint \( S^* \) (see [11, theorem 3.1.3] and [15, theorem 1.3.7]). Relation (2.15) implies that such an extension is determined by a linear relation between the functionals \( u_{\pm}, v_{\pm} \), which are defined on \( \mathcal{D}(S^*) \). Let \( b : \mathcal{D}(S^*) \to \mathbb{C}^4 \) be the mapping
\[ b := \begin{pmatrix} u_- \quad v_- \quad u_+ \quad v_+ \end{pmatrix}^T, \tag{3.1} \]
by $J_0$ we denote the $2 \times 2$ matrix

$$J_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and by $J$ the $4 \times 4$ matrix

$$J = \begin{pmatrix} J_0 & 0 \\ 0 & -J_0 \end{pmatrix}.$$ 

**Proposition 3.1.** The linear mapping $b$ from (3.1) has these properties:

(i) $\mathcal{R}(b) = \mathbb{C}^4$,
(ii) $\ker b = \mathcal{D}(S)$,
(iii) $\left( S^* f, g \right) - \left( f, S^* g \right) \overset{\text{i}}{=} (b^*)_f J^b f$, $f, g \in \mathcal{D}(S^*)$.

**Proof.** The definitions (2.12) and the relations (2.5), (2.6) and (2.7) imply

$$b_{\tilde{u}_-} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad b_{\tilde{v}_-} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad b_{\tilde{u}_+} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad b_{\tilde{v}_+} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad (3.3)$$

and (i) follows. Statement (ii) is a consequence of (2.15).

In order to prove (iii), we observe that, according to (2.16), each $f \in \mathcal{D}(S^*)$ is a linear combination of an element $f_0 \in \mathcal{D}(S)$ and $\tilde{u}_+, \tilde{v}_+$. Relations (2.5), (2.6) and (2.7) imply that $f = f_0 + f_1$ with $f_0 \in \mathcal{D}(S)$ and

$$f_1 := (\tilde{u}_- f) \tilde{v}_- - (\tilde{v}_- f) \tilde{u}_- - (\tilde{u}_+ f) \tilde{v}_+ + (\tilde{v}_+ f) \tilde{u}_+.$$ 

With an analogous decomposition of $g \in \mathcal{D}(S^*)$ it follows from (2.11), (2.3) and (2.4) that

$$\left( S^* f, g \right) - \left( f, S^* g \right) \overset{\text{i}}{=} [f_1, g_1]_i.$$

By means of (2.11), (2.5), (2.6) and (2.7) we find for the expression on the right-hand side the form

$$(b^*)_f J^b f,$$

and relation (iii) is proved. \hfill $\square$

We equip the space $\mathbb{C}^4$ with the inner product generated by $J$: $(Jx, y) := y^* Jx$. Then a subspace $\mathcal{U}$ of $\mathbb{C}^4$ is called $J$-non-negative ($J$-neutral, respectively) if $(Jx, x) \geq 0$ ($= 0$, respectively) for all $x \in \mathcal{U}$.

**Corollary 3.2.** The operator $T$ is a (maximal) dissipative canonical extension of $S$ if and only if $\mathcal{U} = \{ b^* f : f \in \mathcal{D}(T) \}$ is a (maximal) $J$-non-negative subspaces of $\mathbb{C}^4$, and $T$ is a (maximal) symmetric canonical extension of $S$ if and only if this subspace is (maximal) $J$-neutral.
Indeed, it follows from statement (iii) of proposition 3.1 that the operator $T \subset S^*$ is, for example, dissipative if and only if, for all $f \in \mathcal{D}(T)$, it holds that

$$0 \leq 2 \text{Im}(Tf, f) = \Re \left( \frac{(Tf, f) - (f, Tf)}{i} \right) = \Re \left( \frac{(S^*f, f) - (f, S^*f)}{i} \right) = (b_f)^* J b_f.$$ 

The other claims follow in the same way.

In the sequel, $B$ denotes a complex $2 \times 4$ matrix, which we write also as a block matrix

$$B = (C \quad D)$$

with two $2 \times 2$ matrices $C$ and $D$; $J_0$ is the matrix defined in (3.2). Since the eigenvalues of the matrix $J$ are $\pm 1$, each of multiplicity 2, the maximal $J$-non-negative subspaces of $\mathbb{C}^4$ are of dimension 2.

**Theorem 3.3.** The operator $T$ is a maximal dissipative canonical extension of $S$ if and only if

$$\mathcal{D}(T) = \{ f \in \mathcal{D}(S^*): B b f = 0 \},$$

where the $2 \times 4$ matrix $B = (C \quad D)$ is such that its rank is 2 and the inequality

$$C J_0 C^* \leq D J_0 D^*$$

holds; $T$ is a self-adjoint canonical extension of $S$ if and only if the rank of the matrix $B$ in (3.4) is 2 and the relation

$$C J_0 C^* = D J_0 D^*$$

holds.

**Proof.** By corollary 3.2, $T$ is maximal dissipative if and only if $\mathcal{U} = \{ b f : f \in \mathcal{D}(T) \} = \ker B$ is maximal $J$-non-negative. This is the case if and only if $\mathcal{U}^\perp = \mathcal{R}(B^*)$ is maximal $J$-nonpositive, which is equivalent to (3.5) and rank $B = 2$. The proof of the second statement of the theorem is similar. \[\square\]

If we write the matrices $C$ and $D$ in the form

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

the interface condition $B b f = 0$ in (3.4) becomes

$$c_{11} f(0-) - c_{12} \lim_{x \to 0-} \frac{f'(x) + (1 + \ln |x|) f(x)}{x} = 0,$$

$$- d_{11} f(0+) + d_{12} \lim_{x \to 0+} \frac{f'(x) + (1 + \ln |x|) f(x)}{x} = 0,$$

$$c_{21} f(0-) - c_{22} \lim_{x \to 0-} \frac{f'(x) + (1 + \ln |x|) f(x)}{x} = 0,$$

$$- d_{21} f(0+) + d_{22} \lim_{x \to 0+} \frac{f'(x) + (1 + \ln |x|) f(x)}{x} = 0.$$ 

(3.7)
4. Continuity at the origin

In this section we consider those maximal dissipative canonical extensions $T$ of the symmetric operator $S$ for which the functions $f \in \mathcal{D}(T)$ are continuous at zero. Continuity of $f$ at zero means that $f(0-) = f(0+)$, which according to (2.13) is equivalent to $u_- f + u_+ f = 0$. Therefore, these extensions are described by a matrix $B$ with the property

$$c_{11} = d_{11} \neq 0, \quad c_{12} = d_{12} = 0,$$

and we can assume that

$$C = \begin{pmatrix} 1 & 0 \\ c_{21} & c_{22} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ d_{21} & d_{22} \end{pmatrix}.$$ 

Condition (3.5) is equivalent to

$$c_{22} = d_{22} \quad \text{and} \quad \frac{c_{21} c_{22} - c_{22} c_{21}}{i} \leq \frac{d_{21} d_{22} - d_{22} d_{21}}{i}. \quad (4.1)$$

If $c_{22} = d_{22} = 0$, matrix $B$ can be supposed to have the form

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$ 

If $c_{22} = d_{22} \neq 0$ we can assume that this number is 1, and inequality (4.1) becomes $\text{Im} \ c_{21} \leq \text{Im} \ d_{21}$. By subtracting a multiple of the first row of $B$ from the second row, we arrive at the following result.

**Theorem 4.1.** The functions in the domain of the maximal dissipative canonical extension $T$ of $S$ are continuous in $0$ if and only if the matrix $B$ in (3.4) can be chosen as

$$B_\gamma = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \gamma & 1 \end{pmatrix} \quad \text{with} \quad \text{Im} \ \gamma \geq 0, \quad (4.2)$$

or as

$$B_\infty = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$ 

This extension $T$ is self-adjoint if and only if in (4.2) $\text{Im} \ \gamma = 0$ or if $B$ is of the form (4.3).

The extension $T$ of $S$ having the form (3.4) with $B = B_\gamma$ is denoted by $T_\gamma, \gamma \in \mathbb{C}^+ \cup \{\infty\}$. It is easy to see that also for $\gamma \in \mathbb{C}^-$ an extension $T_\gamma$ is defined by the same interface conditions; then the operator $-T_\gamma$ is maximal dissipative.

In order to write the boundary conditions for the extension $T_\gamma$ in a more explicit form than (3.7), we need a lemma.

**Lemma 4.2.** If $f \in \mathcal{D}(S^*)$ and $f(0+) = f(0-)$, then

$$\lim_{x \to 0^+} (f(x) - f(-x))(1 + \ln |x|) = 0. \quad (4.4)$$
Proof. If \( f \in \mathcal{D}(S) \), the claim follows from (2.17). So it remains to consider linear combinations
\[
    f = \alpha_- \tilde{u}_- + \beta_- \tilde{v}_- + \alpha_+ \tilde{u}_+ + \beta_+ \tilde{v}_+,
\]
for which, because of the continuity of \( f \) at 0, also \( \beta_- = \beta_+ =: \beta \). Hence \( f \) has the form
\[
    f = \alpha_- \tilde{u}_- + \alpha_+ \tilde{u}_+ + \beta v,
\]
and relation (4.4) follows easily from the definition of functions \( \tilde{u}_\pm \) and \( v \). \( \square \)

**Theorem 4.3.** The extension \( T_\gamma, \gamma \in \mathbb{C} \cup \{ \infty \} \), of \( S \) is given by interface conditions of the form
\[
    f(0-) = f(0+), \quad \lim_{x \to 0^+} (f'(x) - f'(-x)) = \gamma f(0) \quad \text{if } \gamma \in \mathbb{C},
\]
and by the Dirichlet interface conditions
\[
    f(0+) = f(0-) = 0 \quad \text{if } \gamma = \infty.
\]

\( T_\gamma \) is self-adjoint if and only if \( \gamma \in \mathbb{R} \cup \{ \infty \} \).

**Proof.** If the matrix \( B = B_\gamma \) given by (4.2), then the interface conditions at 0 for \( f \in \mathcal{D}(T) \subset \mathcal{D}(S^*) \) are \( f(0-) = f(0+) \) and
\[
    - \lim_{x \to 0^-} (f'(x) + (1 + \ln |x|) f(x)) + \lim_{x \to 0^+} (f'(x) + (1 + \ln |x|) f(x)) = \gamma f(0+).
\]

By lemma 4.2, relation (4.7) is equivalent to relation (4.5). If the matrix \( B = B_\infty \) given by (4.3), we obtain the Dirichlet interface conditions. \( \square \)

For the canonical extensions of \( S \) which were considered in [12], it was shown there that they are norm resolvent limits of Sturm–Liouville operators with a regular potential. We show by the same method as in [12] that this is true for all the operators \( T_\gamma, \gamma \in \mathbb{C} \). To this end, we define for \( \gamma \in \mathbb{C} \) and \( \varepsilon > 0 \) the Sturm–Liouville operators \( T_{\gamma,\varepsilon} \) as follows:
\[
    \mathcal{D}(T_{\gamma,\varepsilon}) := \{ f \in L^2(I) : f, f' \in AC_{\text{loc}}(I), \ f'' \in L^2(I), \ f(a) = f(b) = 0 \},
\]
\[
    (T_{\gamma,\varepsilon} f)(x) := -f''(x) - \frac{1}{\varepsilon} \left( \frac{1 + \gamma/i\pi}{x + i\varepsilon} + \frac{1 - \gamma/i\pi}{x - i\varepsilon} \right) f(x).
\]

**Theorem 4.4.** For \( \gamma \in \mathbb{C} \), the operator \( T_\gamma \) is the norm resolvent limit of the operators \( T_{\gamma,\varepsilon} \) if \( \varepsilon \to 0^+ \).

**Proof.** On the set
\[
    \mathcal{D} := \{ f \in AC_{\text{loc}}(I) : f' \in L^2(I), \ f(a) = f(b) = 0 \}
\]
we consider the following sesquilinear forms:

\[
\begin{align*}
\ell_0^0[f, g] &:= \int_a^b f'(x)g'(x) \, dx, \\
q_\varepsilon[f, g] &:= -\int_a^b \frac{f(x)g(x)}{x+i\varepsilon} \, dx, \quad \text{if } \varepsilon \neq 0, \quad q_0[f, g] := -P \int_a^b \frac{f(x)\overline{g(x)}}{x} \, dx, \\
b[f, g] &:= f(0)g(0),
\end{align*}
\]

where \( P \) denotes the Cauchy principal value. The form \( \ell_0^0 \) is closed and non-negative; the forms \( q_0 \) and \( b \) are symmetric and \( \ell_0^0 \)-bounded with relative bound zero \([12, \text{lemmas } 2.3 \text{ and } 2.5]\). Hence, according to \([13, \text{theorem } VI.1.33]\),

\[
t_\gamma := \ell_0^0 + q_0 + \gamma b
\]
is a closed sectorial form on \( \mathcal{D} \). By the second representation theorem \([13, \text{ theorem } VI.2.1]\), there exists an \( m \)-sectorial operator \( T_{t_\gamma} \) such that

1. \( \mathcal{D}(T_{t_\gamma}) \subset \mathcal{D} \);
2. \( t_\gamma[f, g] = (T_{t_\gamma} f, g) \), \( f \in \mathcal{D}(T_{t_\gamma}) \), \( g \in \mathcal{D} \);
3. \( \mathcal{D}(T_{t_\gamma}) \) is a core of \( t_\gamma \);
4. if \( f \in \mathcal{D} \), \( y \in \mathcal{L}^2(I) \) such that the equality \( t_\gamma[f, g] = (y, g) \) holds for all \( g \) in a core of \( t_\gamma \), then \( f \in \mathcal{D}(T_{t_\gamma}) \) and \( T_{t_\gamma} f = y \).

We shall show that \( T_{t_\gamma} = T_\gamma \). Theorem 4.3 implies \( \mathcal{D}(T_\gamma) \subset \mathcal{D} \), and for \( f \in \mathcal{D}(T_\gamma) \) and \( g \in \mathcal{D} \) it holds that

\[
(T_\gamma f, g) = \left( \int_a^0 + \int_0^b \right) \left( -f''(x) - \frac{f(x)}{x} \right) g(x) \, dx
= \lim_{\varepsilon \to 0^+} \left( \int_a^{-\varepsilon} + \int_0^b \right) \left( -f''(x) - \frac{f(x)}{x} \right) g(x) \, dx
= P \int_a^b \left( f'(x)g'(x) - \frac{f(x)g(x)}{x} \right) \, dx + \lim_{\varepsilon \to 0^+} (f'(\varepsilon)g(\varepsilon) - f'(-\varepsilon)g(-\varepsilon))
= \ell_0^0[f, g] + q_0[f, g] + \lim_{\varepsilon \to 0^+} (f'(\varepsilon)g(\varepsilon) - f'(-\varepsilon)g(-\varepsilon)) - \gamma f(0)g(0).
\]

(4.8)

If \( g \in \mathcal{D} \), we have

\[
|g(x) - g(0)| \leq \int_0^x |g'(s)| \, ds \leq \sqrt{x} \|g'(s)\|.
\]

Therefore, relation (2.14) yields, for \( f \in \mathcal{D}(T_\gamma) \),

\[
\lim_{x \to 0^+} (f'(x)g(x) - f'(-x)g(-x)) - \gamma f(0)g(0)
= \lim_{x \to 0^+} (f'(x) - f'(-x) - \gamma f(0))g(0) = 0.
\]
Hence (4.8) becomes
\[(T_\gamma f, g) = t_\gamma [f, g], \quad f \in \mathcal{D}(T_\gamma), \quad g \in \mathcal{D},\]
which implies \(T_\gamma \subset T_{\gamma,e}\). Since, on the other hand, \(T_\gamma\) or \(-T_\gamma\) is a maximal dissipative operator, in this inclusion the equality sign must prevail.

The differential operator \(T_{\gamma,e}\) is associated with the sesquilinear form
\[t_{\gamma,e} = t^0 + \frac{\pi i + \gamma}{2\pi i} q_\varepsilon + \frac{\pi i - \gamma}{2\pi i} q_{-\varepsilon},\]
which is also defined on \(\mathcal{D}\). As in the proof of [12, theorem 3.3], for \(f, g \in \mathcal{D}\) it follows that
\[|q_{\pm \varepsilon}[f, g] - (q_0[f, g] \mp \pi i b[f, g])| = o(1)t^0[f, g] + o(1)(f, g), \quad \varepsilon \to 0+,\]
and we get
\[t_{\gamma,e}[f, g] - t_\gamma[f, g] = o(1)t^0[f, g] + o(1)(f, g), \quad \varepsilon \to 0+.\]

Now the resolvent convergence of the operators \(T_{\gamma,e}\) to \(T_\gamma\) follows from [13, theorem VI.3.6].

5. Representation of the solutions by Whittaker functions

In this section we express the resolvents of the extensions \(T_\gamma\) from § 4 by means of Whittaker functions. We first recall Whittaker’s differential equation [2, 5, 16, 21]:
\[
\frac{d^2 f(z)}{dz^2} + \left( -\frac{1}{4} + \frac{\kappa}{z} + \frac{1 - \mu^2}{4z^2} \right) f(z) = 0.
\]

Two linearly independent solutions of this differential equation are the Whittaker functions
\[
M_{\kappa,\mu}(z/2) = z^{(1+\mu)/2}e^{-z/2}\Phi\left(\frac{1}{2}(1 + \mu) - \kappa, 1 + \mu, z\right),
\]
\[
W_{\kappa,\mu}(z/2) = z^{(1+\mu)/2}e^{-z/2}\Psi\left(\frac{1}{2}(1 + \mu) - \kappa, 1 + \mu, z\right),
\]
where \(\Phi\) is the confluent hypergeometric function. In the following we use the function \(\psi(z) := \Gamma'(z)/\Gamma(z)\), and for complex numbers \(\alpha\) and \(\beta\) and an integer \(k\) the symbols
\[(\alpha)_k := \alpha(\alpha + 1) \cdots (\alpha + k - 1), \quad d_k(\alpha, \beta) := \psi(\alpha + k) - \psi(1 + k) - \psi(\beta + k).\]

Then the function \(\Phi\) is given by the relation
\[\Phi(\alpha, \beta, z) = \sum_{k=0}^{\infty} \frac{d_k(\alpha, \beta) z^k}{(\beta)_k k!}.\]
and in the case that $\beta$ is a positive integer, $\Psi(\alpha, \beta, z)$ admits the following representation [2, § 6.1, § 6.7, formula (13)]:

$$
\Psi(\alpha, \beta, z) = \frac{(-1)^\beta}{\Gamma(\beta)\Gamma(\alpha - \beta + 1)} \left( \Phi(\alpha, \beta, z) \ln z + \sum_{k=0}^{\infty} \frac{(\alpha)_k d_k(\alpha, \beta) z^k}{(\beta)_k k!} \right) + \frac{\Gamma(\beta - 1)}{\Gamma(\alpha)} \sum_{k=0}^{\beta-2} \frac{(\alpha - \beta + 1)_k z^{k-\beta+1}}{(2 - \beta)_k k!}.
$$

(5.2)

If we make the substitution

$$
\mu = 1, \quad \kappa = \frac{i}{2\sqrt{\lambda}}, \quad z = \frac{x}{\kappa} = -2i\sqrt{\lambda} x,
$$

equation (5.1) becomes equation (1.2): $l[f] - \lambda f = 0$. Therefore, two linearly independent solutions of (1.2) are the functions

$$
\begin{align*}
M(x, \lambda) &= M_{i/2\sqrt{\lambda}, 1/2}(-2i\sqrt{\lambda} x), \\
W(x, \lambda) &= \Gamma(1 - i/2\sqrt{\lambda}) W_{i/2\sqrt{\lambda}, 1/2}(-2i\sqrt{\lambda} x);
\end{align*}
$$

(5.3)

see also [4, 8]. The function $M$ is entire in $x$, whereas $W$ has a logarithmic branch point at $x = 0$. The function $W$ is understood as the principal branch, which is obtained from the principal branch of the logarithm in (5.2).

With the functions $M(x, \lambda)$ and $W(x, \lambda)$ we form for $\lambda \neq 0$ the solutions

$$
\begin{align*}
f_-(x, \lambda) &:= \left\{ \begin{array}{ll}
\frac{M(a, \lambda) W(x, \lambda) - W'(a, \lambda) M(x, \lambda)}{M(a, \lambda) W'(a, \lambda) - W'(a, \lambda) M'(a, \lambda)} & \text{if } x < 0, \\
0 & \text{if } x > 0,
\end{array} \right. \\
f_+(x, \lambda) &:= \left\{ \begin{array}{ll}
0 & \text{if } x < 0, \\
\frac{M(b, \lambda) W(x, \lambda) - W(b, \lambda) M(x, \lambda)}{M(b, \lambda) W'(b, \lambda) - W'(b, \lambda) M'(b, \lambda)} & \text{if } x > 0.
\end{array} \right.
\end{align*}
$$

(5.4)

(5.5)

They satisfy for $x \neq 0$ the differential equation $l[f] - \lambda f = 0$ and the boundary conditions

$$
\begin{align*}
f_-(a, \lambda) &= 0, & f_-'(a, \lambda) &= 1, \\
f_+(b, \lambda) &= 0, & f_+'(b, \lambda) &= 1.
\end{align*}
$$

If $x \neq 0$ is fixed, $f_\pm(x, \lambda)$ are entire functions in $\lambda$. Further, we introduce the kernel

$$
K(x, \xi; \lambda) := \left\{ \begin{array}{ll}
\frac{M(\xi, \lambda) W(x, \lambda) - W(\xi, \lambda) M(x, \lambda)}{M(\xi, \lambda) W'(\xi, \lambda) - W'(\xi, \lambda) M'(\xi, \lambda)} & \text{if } \xi \leq x < 0, \\
\frac{M(\xi, \lambda) W(x, \lambda) - W(\xi, \lambda) M(x, \lambda)}{M(\xi, \lambda) W'(\xi, \lambda) - W'(\xi, \lambda) M'(\xi, \lambda)} & \text{if } 0 < x \leq \xi, \\
0 & \text{otherwise},
\end{array} \right.
$$

(5.6)

It satisfies for $x \neq 0$ and $x \neq \xi$ the differential equation

$$
- \frac{\partial^2 K}{\partial x^2}(x, \xi; \lambda) - \frac{K(x, \xi; \lambda)}{x} = \lambda K(x, \xi; \lambda)
$$

(5.7)
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and the boundary conditions

\[ \frac{\partial K}{\partial x}(\xi^+, \xi; \lambda) = 1 \text{ if } \xi < 0, \quad \frac{\partial K}{\partial x}(\xi^-, \xi; \lambda) = -1 \text{ if } \xi > 0. \]

We introduce the following operators \( K_\lambda, \lambda \in \mathbb{C} \), in \( L^2(I) \):

\[(K_\lambda f)(x) := \int_a^b K(x, \xi; \lambda) f(\xi) \, d\xi, \quad f \in L^2(I).\]

Then \( K_\lambda f \in \mathcal{D}(S^*) \) and \((S^* - \lambda)K_\lambda f = f\) for arbitrary \( f \in L^2(I) \). This implies for functions \( f \in \mathcal{D}(S^*) \) that \( K_\lambda(S^* - \lambda)f = f + g \) with \( g \in \ker(S^* - \lambda) \). If \( f \) vanishes identically near \( a \) and \( b \), then also \( K_\lambda(S^* - \lambda)f \) does. In this case \( g = 0 \), and \( K_\lambda(S^* - \lambda)f = f \), which yields \( \hat{u}_\pm, \hat{v}_\pm \in \mathcal{R}(K_\lambda) \) and further

\[ \mathcal{R}(bK_\lambda) = \mathbb{C}^4. \quad (5.6) \]

The functions \( f_-(:, \lambda) \) and \( f_+(:, \lambda) \) span the kernel \( \ker(S^* - \lambda) \). For given \( f \in L^2(I) \) the equation

\[(T_\gamma - \lambda)f = y \quad (5.7)\]

is satisfied if and only if \( f = c_- f_- + c_+ f_+ + K_\lambda y \) with numbers \( c_- \) and \( c_+ \) such that \( B_\gamma b(c_- f_- + c_+ f_+ + K_\lambda y) = 0 \). Relation \((5.6)\) implies that the latter equation has a unique solution for arbitrary \( y \in L^2(I) \) if and only if the \( 2 \times 2 \) matrix

\[ M_\gamma(\lambda) := (B_\gamma b(f_-(:, \lambda); B_\gamma b(f_+(:, \lambda))) \quad (5.8) \]

is invertible, and the solution of equation \((5.7)\) can be written as

\[ f(x) = (K_\lambda y)(x) - (f_-((x, \lambda); f_+((x, \lambda))) M_\gamma(\lambda)^{-1} B_\gamma b(K_\lambda y). \quad (5.9) \]

For the following theorem see \[19, \text{I} \S 2\].

**Theorem 5.1.** Suppose \( \gamma \in \mathbb{C} \) and let \( M_\gamma(\lambda) \) be the matrix function from \((5.8)\). Then \( \lambda \in \rho(T_\gamma) \) if and only if \( \det M_\gamma(\lambda) \neq 0 \), and in this case the resolvent \((T_\gamma - \lambda)^{-1}\) is given by \((5.9)\): \( (T_\gamma - \lambda)^{-1}y = f \). The eigenvalues of \( T_\gamma \) are geometrically simple, and the length of the Jordan chain of \( T_\gamma \) at an eigenvalue \( \lambda \) equals the order of the zero \( \zeta = \lambda \) of the function \( \det M_\gamma(\zeta) \).

**Proof.** If \( \det M_\gamma(\lambda) \neq 0 \), the resolvent \((T_\gamma - \lambda)^{-1}\) exists and is given by \((5.9)\). Now suppose that \( \det M_\gamma(\lambda) = 0 \). Then the non-zero 2-vector \((c_-, c_+)^T \) belongs to \( \ker M_\gamma(\lambda) \) if and only if the function \( f(x) := c_- f_-((x, \lambda) + c_+ f_+(x, \lambda) fulfills the interface condition \( B_\gamma b f = 0 \) and hence is an eigenfunction of \( T_\gamma \) at \( \lambda \). Since all eigenfunctions of \( T_\gamma \) at \( \lambda \) are of this form and the matrix \( M_\gamma(\lambda) \) is not the zero matrix, the geometric multiplicity of the eigenvalue \( \lambda \) equals one.

Suppose now that \( \lambda \) is a zero of order \( m \) of the function \( \det M_\gamma(\zeta) \). Then \((5.9)\) implies that the length of the Jordan chain of \( T_\gamma \) at \( \lambda \) is at most \( m \). A chain of length \( m \) can be obtained as follows. Since

\[ M_\gamma(\zeta) = \begin{pmatrix} m_{\gamma,11}(\zeta) & m_{\gamma,12}(\zeta) \\ m_{\gamma,21}(\zeta) & m_{\gamma,22}(\zeta) \end{pmatrix} \]
is not the zero matrix, at least one entry does not vanish. Suppose, for example, that this is $m_{\gamma,11}(\lambda)$; the other cases can be treated similarly. With the matrices

$$E(\zeta) = \begin{pmatrix} 1 & 0 \\ -m_{\gamma,21}(\zeta) & m_{\gamma,11}(\zeta) \end{pmatrix}, \quad F(\zeta) = \begin{pmatrix} 1 & -m_{\gamma,12}(\zeta) \\ 0 & m_{\gamma,11}(\zeta) \end{pmatrix}$$

we get

$$E(\zeta)M(\zeta)F(\zeta) = \begin{pmatrix} m_{\gamma,11}(\zeta) & 0 \\ 0 & \det M(\zeta) \end{pmatrix}.$$ 

Therefore, the analytic family of vectors

$$\begin{pmatrix} c_-(\zeta) \\ c_+(\zeta) \end{pmatrix} = F(\zeta) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

fulfills for $\zeta \to \lambda$ the relations

$$\begin{pmatrix} c_-(\zeta) \\ c_+(\zeta) \end{pmatrix} \neq 0, \quad \begin{pmatrix} d_-(\zeta) \\ d_+(\zeta) \end{pmatrix} = M(\zeta) \begin{pmatrix} c_-(\zeta) \\ c_+(\zeta) \end{pmatrix} = O((\zeta - \lambda)^m).$$

Then (3.3) and (5.8) give

$$f(\cdot, \zeta) = c_-(\zeta)f_-(\cdot, \zeta) + c_+(\zeta)f_+(\cdot, \zeta) - d_-(\zeta)v_-(\cdot) - d_+(\zeta)v_+(\cdot) \in \mathcal{D}(T_{\gamma}),$$

and the relation $(T_{\gamma} - \zeta)f(\cdot, \zeta) = O((\zeta - \lambda)^m)$ implies that the functions

$$f_i(\cdot, \lambda) := \frac{\partial f(\cdot, \lambda)}{\partial \lambda^i}, \quad i = 0, 1, \ldots, m - 1,$$

form a Jordan chain at $\lambda$.

In the following we need some asymptotic properties of the eigenvalues of the operators $T_{\gamma}$. To this end, we study the asymptotic behaviour of the functions $f_M$ and $f_W$. The relations (5.3) imply the following asymptotics. If $\lambda \in \mathbb{C} \setminus \{0\}$ is fixed, then for $x \to 0$,

$$f_M(x, \lambda) = -2i\sqrt{\lambda}x + O(x^2), \quad (5.10)$$

$$f_W(x, \lambda) = e^{-z/2} - \kappa_ze^{-z/2}((1 + O(z) \ln z + d_0(1 - \kappa, 2) + O(z))$$

$$= 1 + i\sqrt{\lambda}x - \ln z - d_0(1 - \kappa, 2)x + O(x^2 \ln x)$$

$$= 1 - x \ln |x| + c_\lambda(x)x + O(x^2 \ln x), \quad (5.11)$$

where

$$c_\lambda(x) := i\sqrt{\lambda} - d_0\left(1 - \frac{i}{2\sqrt{\lambda}}, 2\right) + \ln |x| - \ln(-2i\sqrt{\lambda}x). \quad (5.12)$$

Note that $c_\lambda(x)$ does not depend on $|x|$, hence it is bounded if $x \to \pm 0$. Further, it holds that

$$c_\lambda(1) - c_\lambda(-1) = \ln(2i\sqrt{\lambda}) - \ln(-2i\sqrt{\lambda}) = i\pi. \quad (5.13)$$
Relations (5.10) and (5.11) imply
\[ f_W(0-, \lambda) = f_W(0+, \lambda) = 1, \quad f_M(0-, \lambda) = f_M(0+, \lambda) = 0 \] (5.14)
and
\[
\begin{align*}
\lim_{x \to 0^-} (f'_M(x, \lambda) + (1 + \ln |x|)f_M(x, \lambda)) &= -2i\sqrt{\lambda}, \\
\lim_{x \to 0^+} (f'_W(x, \lambda) + (1 + \ln |x|)f_W(x, \lambda)) &= c\lambda(-1), \\
\lim_{x \to 0^+} (f'_W(x, \lambda) + (1 + \ln |x|)f_W(x, \lambda)) &= c\lambda(+1),
\end{align*}
\] (5.15)
where \( c\lambda(x) \) is given by (5.12).

Remark 5.2. Boyd [4] considered the boundary value problem (1.1) with boundary conditions (1.3), replacing the potential \(-x^{-1}\) first by \(-(x - i\varepsilon)^{-1}\) with \(\varepsilon > 0\) and letting \(\varepsilon \to 0\). He required the eigenfunctions to admit an analytic continuation onto the lower half-plane. This requirement specifies an interface condition in \(x = 0\), which, however, turns out not to be self-adjoint. Indeed, the solutions of (1.1) which admit an analytic continuation onto the lower half-plane are linear combinations of the functions \(f_M(x, \lambda)\) and \(\tilde{f}_W(x, \lambda)\), where \(\tilde{f}_W(x, \lambda)\) equals the function \(f_W(x, \lambda)\) for positive real \(x\), and with the branch cut at \(\arg x = \pi/2\). This corresponds to a branch cut in the logarithm in the definition of the function \(\Psi\) in (5.2) at \(\arg z = \arg \sqrt{\lambda}\). For real \(x\) and \(-\pi < \arg \lambda \leq \pi\), this means
\[
\tilde{f}_W(x, \lambda) = \begin{cases} 
 f_W(x, \lambda) & \text{if } x > 0, \\
 f_W(x, \lambda) - \frac{\pi}{\sqrt{\lambda}} f_M(x, \lambda) & \text{if } x < 0.
\end{cases}
\]
Now it follows from (5.13), (5.14) and (5.15) that
\[
b_{f_M}(\cdot, \lambda) = \begin{pmatrix} 0 \\
2i\sqrt{\lambda} \\
0 \end{pmatrix}, \quad b_{\tilde{f}_W}(\cdot, \lambda) = \begin{pmatrix} 1 \\
-c\lambda(1) - i\pi \\
-1 \end{pmatrix}.
\]
These vectors span the kernel of the \(2 \times 4\) matrix \(B_{-i\pi}\). Therefore, the operator which was considered in [4] is (up to its sign) \(T_{-i\pi}\).

In order to study the asymptotic behaviour of the functions \(f_M\) and \(f_W\) for \(\lambda \to \infty\), we use the following relations [2, 6.13(1) and (2)] [16, 4.7(2)-(4)]:
\[
\Phi(\alpha, \beta, z) = \frac{\Gamma(\beta)e^{\alpha\pi\sgn \ln z}}{\Gamma(\beta - \alpha)z^{-\alpha}} + \frac{\Gamma(\beta)}{\Gamma(\alpha)}e^{z^2z^{\alpha-\beta}}
\]
\[+ O(z^{-\alpha-1}) + O(e^{z^2z^{\alpha-\beta-1}}), \quad \text{if } z \to \infty. \]
\[\Psi(\alpha, \beta, z) = z^{-\alpha} + O(z^{-\alpha-1}) \quad \text{if } z \to -\infty. \]
if \(z \to \infty\). The expansion (5.16) holds in the sector \(-\pi < \arg z < \pi\), the expansion (5.17) in the sector \(-3\pi/2 < \arg z < 3\pi/2\). If \(x \in \mathbb{R}\setminus\{0\}\) is fixed, then for \(\kappa \to 0\),
\[
\Gamma(1 \pm \kappa) = 1 + O(\kappa), \quad e^{\pm i\pi(1-\kappa)} = -1 + O(\kappa), \quad z^\kappa = e^{\kappa(\ln x - \ln \kappa)} = 1 + O(\kappa \ln \kappa),
\]
and we find the following asymptotics for $\lambda \to \infty$ in the sector $-\pi < \arg \lambda < \pi$:

$$f_M(x, \lambda) = e^{-i\sqrt{\lambda}x} - e^{i\sqrt{\lambda}x} + O\left(\frac{\ln \lambda}{\sqrt{\lambda}} e^{i|x\ln \sqrt{\lambda}|}\right), \quad (5.18)$$

$$f_W(x, \lambda) = e^{i\sqrt{\lambda}x} + O\left(\frac{\ln \lambda}{\sqrt{\lambda}} e^{i\sqrt{\lambda}x}\right). \quad (5.19)$$

**Theorem 5.3.** If $\gamma \in \mathbb{C}$, then the spectrum $\sigma(T_\gamma)$ consists of isolated normal eigenvalues $\lambda_n$, $n \in \mathbb{N}$, of geometric multiplicity one, and all but finitely many of them are simple. If they are numbered according to non-decreasing absolute value, then the following asymptotic formula holds:

$$\lambda_n = \frac{\pi^2 n^2}{(b - a)^2} + O(\ln n) \quad \text{for} \ n \to \infty. \quad (5.20)$$

In the proof of the theorem we use the following lemma.

**Lemma 5.4.** An entire function $F(z)$ of the form

$$F(z) = \sin z + O\left(\frac{\ln |z|}{z} \exp |\Im z|\right) \quad \text{for} \ |z| \to \infty$$

has infinitely many zeros and all but finitely many of them are simple. For $n \in \mathbb{Z}$ with $|n|$ sufficiently large, there is a disc of radius

$$\rho_n = O\left(\frac{\ln |n|}{n}\right) \quad \text{for} \ |n| \to \infty,$$

around the point $n\pi$ which contains exactly one zero of $F(z)$; outside these discs lie only finitely many zeros of $F(z)$.

**Proof.** Since $F(z)$ is entire and does not vanish identically, its zeros are countable and have no accumulation point in $\mathbb{C}$. We consider the zeros only in the right half-plane; the zeros in the left half-plane can be treated similarly. There exist positive real numbers $r, C_1, C_2$ such that for the zeros $\zeta = s + it$, with $|\zeta| > r$,

$$|\sinh t| \leq |\sin \zeta| \leq C_1 \left|\frac{\ln \zeta}{\zeta}\right| \exp |t| \leq 2C_1 \left|\frac{\ln \zeta}{\zeta}\right| (|\sinh t| + 1).$$

Hence

$$1 + \frac{1}{|\sinh t|} \geq \frac{1}{C_2} \left|\frac{\zeta}{\ln \zeta}\right|, \quad |\zeta| > r,$$

which implies that all zeros $\zeta$ lie in a strip $|t| \leq C$ with $C > 0$, and with $C_3 = C_1 \exp C$,

$$|\sin \zeta| \leq C_3 \left|\frac{\ln \zeta}{\zeta}\right|, \quad |\zeta| > r.$$

Denote by $R_n$, $n \in \mathbb{N}$, the rectangle

$$n\pi - \pi/2 \leq \Re z \leq n\pi + \pi/2, \quad -C \leq \Im z \leq C.$$

Then

$$m := \min_{z \in \partial R_n} |\sin z| > 0$$
If we put
\[ |F(z) - \sin z| < m \leq |\sin z|, \quad z \in \partial R_n. \]
Rouché’s theorem implies that \( F(z) \), like \( \sin z \), has exactly one zero in \( R_n \) for \( n \geq n_0 \) and that this zero is simple. We now claim that for \( n \) sufficiently large, the zero of \( F(z) \) in \( R_n \) lies in a circle of radius \( \rho_n = O(n^{-1}\ln n) \) around the zero \( z = n\pi \) of \( \sin z \). To prove the claim, first choose \( \rho > 0 \) such that the inequality
\[ |\sin z| \geq \frac{1}{2}|z - \pi n| \]
holds for all \( n \in \mathbb{N} \) and \( |z - \pi n| \leq \rho \). Then choose \( C_4 \) such that
\[ |F(z) - \sin z| < \frac{C_4}{2} \left( \frac{\ln n}{n} \right), \quad z \in R_n, \ n \geq 2. \]
Finally, choose \( n_1 \geq \max(2, n_0) \) so large that \( \rho_n := C_4(n^{-1}\ln n) < \rho \) for all \( n \geq n_1 \). Then, for \( n \geq n_1 \) and \( |z - \pi n| = \rho_n \),
\[ |F(z) - \sin z| < \frac{1}{2}\rho_n = \frac{1}{2}|z - \pi n| \leq |\sin z|. \]
The claim now follows again from Rouché’s theorem. \( \square \)

**Proof of theorem 5.3.** We consider the matrix \( B_\gamma \) from (4.2) and the corresponding \( 2 \times 2 \) matrix function \( M_\gamma \) defined by (5.8). Since \( M_\gamma(\lambda), \ \lambda \in \mathbb{C} \), is never the zero matrix, the geometric multiplicity of the eigenvalues of \( T_\gamma \) is one; see theorem 5.1. A straightforward calculation shows that, up to a non-zero factor, the determinant \( \det M_\gamma(\lambda) \) equals
\[
\begin{vmatrix}
    f_M(a, \lambda) & f_M(b, \lambda) \\
   -2i\sqrt{\chi} f_W(a, \lambda) + (i\pi - c_\lambda(1)) f_M(a, \lambda) & -2i\sqrt{\chi} f_W(b, \lambda) + (\gamma - c_\lambda(1)) f_M(b, \lambda)
\end{vmatrix}
= \begin{vmatrix}
    f_M(a, \lambda) & f_M(b, \lambda) \\
   -2i\sqrt{\chi} f_W(a, \lambda) + i\pi f_M(a, \lambda) & -2i\sqrt{\chi} f_W(b, \lambda) + \gamma f_M(b, \lambda)
\end{vmatrix}
\]
Now, relations (5.18) and (5.19) imply that this determinant for \( \lambda \to \infty \) asymptotically behaves like
\[
-2i\sqrt{\chi} \left| \begin{array}{cc}
    f_M(a, \lambda) & f_M(b, \lambda) \\
   f_W(a, \lambda) & f_W(b, \lambda)
\end{array} \right| + O(e^{|(b-a)\ln \chi|})
= -2i\sqrt{\chi} \left| \begin{array}{cc}
    e^{-i\sqrt{\chi}a} - e^{i\sqrt{\chi}a} & e^{-i\sqrt{\chi}b} - e^{i\sqrt{\chi}b} \\
    e^{-i\sqrt{\chi}a} & e^{-i\sqrt{\chi}b}
\end{array} \right| + O(e^{|(b-a)\ln \chi|} \ln \lambda)
= 4\sqrt{\chi}(b - a)\sqrt{\chi} + O(e^{|(b-a)\ln \chi|} \ln \lambda).
\]
If we put \( \zeta = (b-a)\sqrt{\chi} \), apply lemma 5.4, and observe again theorem 5.1, then the claim follows. \( \square \)
6. Basis properties of the root vectors of $T_\gamma$

Recall that a sequence $(f_n), n \in \mathbb{N}$, of elements of a separable Hilbert space $\mathcal{H}$ is called a basis of $\mathcal{H}$ if each $y \in \mathcal{H}$ has a unique representation

$$y = \sum_{n=1}^{\infty} c_n f_n, \quad \text{with } c_n \in \mathbb{C}, \ n \in \mathbb{N},$$

where the sum converges in the norm of $\mathcal{H}$. The basis $(f_n), n \in \mathbb{N}$, of $\mathcal{H}$ is called a Bari basis if it is quadratically close to an orthonormal basis $\{e_n; n \in \mathbb{N}\}$ of $\mathcal{H}$, which means that

$$\sum_{n=1}^{\infty} \|f_n - e_n\|^2 < \infty.$$ 

For this notion and its properties, see, for example, [9, ch. VI]. We use the following criterion about the existence of a Bari basis [9, theorem VI.4.1]:

**Criterion.** Let $T$ be a bounded dissipative operator in a Hilbert space such that $T - T^*$ is compact. Denote by $\mu_n, n \in \mathbb{N}$, the mutually different eigenvalues of $T$ and by $l_n$ the geometric multiplicity of $\mu_n$, and suppose that

$$\sum_{n \neq m} \min(l_n, l_m) \frac{\text{Im}\mu_n \text{Im}\mu_m}{|\mu_n - \mu_m|^2} < \infty,$$

where the sum runs over all $n, m \in \mathbb{N}$ such that $n \neq m$ and $\text{Im}\mu_n \neq 0$, $\text{Im}\mu_m \neq 0$. If we choose in each eigenspace of $T$ an orthonormal basis, then the sequence of all these basis elements forms a Bari basis in its closed linear hull.

We also use the well-known result of Lidski\textsuperscript{\textregistered} [9, theorem V.2.3]:

**Result.** A dissipative trace class operator has a complete system of root vectors.

If $\gamma$ is real or $\infty$, then the operator $T_\gamma$ is self-adjoint. By an argument as in the proof of the following theorem, it follows that its resolvent is a trace class operator. Hence $T_\gamma, \gamma \in \mathbb{R} \cup \{\infty\}$, has an orthonormal basis of eigenfunctions. The main result of this section is the following theorem.

**Theorem 6.1.** If $\gamma \in \mathbb{C}^+ \cup \mathbb{C}^-$, then the root vectors of $T_\gamma$ can be chosen to form a Bari basis of $\mathcal{L}^2(I)$.

**Proof.** Let $l \in \rho(T_\gamma) \cap \rho(T_0)$ be a real number. The spectral mapping theorem and theorem 5.3 imply that the eigenvalues $\eta_n, n \in \mathbb{N}$, of $(T_\gamma - l)^{-1}$ satisfy the relation

$$\eta_n = \frac{1}{cn^2 + O(\ln n)} = \frac{1}{cn^2} + O\left(\frac{\ln n}{n^4}\right) \quad \text{for } n \to \infty$$

(6.2)

with $c := \pi^2(b-a)^{-2}$. By theorem 4.3, $T_0$ is self-adjoint, hence also $(T_0 - l)^{-1}$ is self-adjoint, and since its eigenvalues satisfy relation (6.2), it is a trace class operator. If $\gamma \neq 0$, the difference $(T_\gamma - l)^{-1} - (T_0 - l)^{-1}$ is one-dimensional and therefore also $(T_\gamma - 1)^{-1}$ is a trace class operator.

In order to prove that the root vectors of $T_\gamma$ form a Bari basis, we suppose that $\gamma \in \mathbb{C}^+$; the case $\gamma \in \mathbb{C}^-$ can be treated analogously. The operator $-(T_\gamma - l)^{-1}$ is...
dissipative and a trace class operator. Therefore, the closed linear span of its root vectors is the whole space \( L^2(I) \). Next we verify that the eigenvalues of \((T_\gamma - l)^{-1}\), which we denote by \( \eta_n \), satisfy condition (6.1). Since the algebraic multiplicity of all but finitely many eigenvalues is one by theorem 5.3, this condition simplifies to

\[
\sum_{1 \leq m < n} \frac{\text{Im} \eta_m \text{Im} \eta_n}{|\eta_m - \overline{\eta_n}|^2} < \infty. \tag{6.3}
\]

Relation (6.2) implies for \( 1 \leq m < n \) and suitable constants \( C_1, C_2, C_3 \) that

\[
\frac{\text{Im} \eta_m \text{Im} \eta_n}{|\eta_m - \overline{\eta_n}|^2} \leq C_1 \frac{\ln m \ln n}{|(n - m)(n + m) - C_1 (\ln m + \ln n)|^2} \leq C_3 (\frac{\ln(m + n)}{2})^2; \tag{6.4}
\]

here we have used the inequalities \( \ln n, \ln m \leq \ln(n + m) \) and the fact that \((n - m)^{-1}(n + m)^{-1}(\ln m + \ln n) \to 0 \) if \( m < n, n \to \infty \).

Since for sufficiently large \( x \) the function \( x^{-1} \ln x \) is decreasing, then with \( k = n - m \) and some constant \( C_4 \), we finally obtain

\[
\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{(\ln(2m + k))^2}{k^2(2m + k)^2} \leq C_4 \sum_{n=1}^{\infty} \frac{(\ln 2m)^2}{(2n)^2} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.
\]

\[\square\]

If \( \gamma \in \mathbb{C}^+ \cup \mathbb{C}^- \), then \( T_\gamma \) or \( -T_\gamma \) is dissipative and it is easy to see that the relation

\[T^*_\gamma = T_\gamma^*\]

holds. Denote by \( (\lambda)_n, n \in \mathbb{N} \), the sequence of (mutually different) eigenvalues of \( T_\gamma \), and denote by

\[g_{n,1}, g_{n,2}, \ldots, g_{n,m_n}\]

a basis of the root subspace of \( T_\gamma \) corresponding to \( \lambda_n \), such that the system of all elements \( g_{n,k}, k = 1, 2, \ldots, m_n, n \in \mathbb{N} \), is a Bari basis of \( L^2(I) \). Then the complex conjugate functions

\[\overline{g}_{n,1}, \overline{g}_{n,2}, \ldots, \overline{g}_{n,m_n}\]

form a basis of the root subspace of \( T^*_\gamma = T_\gamma^* \) corresponding to \( \overline{\lambda}_n \). We introduce for \( n \in \mathbb{N} \) the \( m_n \times m_n \) matrix

\[G_n := \begin{pmatrix}
(g_{n,1}, \overline{g}_{n,1}) & \cdots & (g_{n,m_n}, \overline{g}_{n,1}) \\
\vdots & \ddots & \vdots \\
(g_{n,1}, \overline{g}_{n,m_n}) & \cdots & (g_{n,m_n}, \overline{g}_{n,m_n})
\end{pmatrix}.
\]

The root subspaces of \( T_\gamma \) at \( \lambda_n \) and of \( T^*_\gamma \) at \( \overline{\lambda}_m \) are orthogonal if \( m \neq n \), and are in duality if \( m = n \). Hence the matrix \( G_n \) is invertible. For \( y \in L^2(I) \) we define numbers \( c_{n,k}, k = 1, 2, \ldots, m_n, n \in \mathbb{N} \), by the relation

\[
\begin{pmatrix}
c_{n,1}(y) \\
\vdots \\
c_{n,m_n}(y)
\end{pmatrix} := G_n^{-1} \begin{pmatrix}
(y, \overline{g}_{n,1}) \\
\vdots \\
(y, \overline{g}_{n,m_n})
\end{pmatrix} \quad \text{for } n \in \mathbb{N}. \tag{6.4}
\]
**Theorem 6.2.** If $\gamma \in \mathbb{C} \setminus \mathbb{R}$, then each element $y \in \mathcal{L}^2(I)$ admits the following unique expansion,

$$y = \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} c_{n,k}(y) g_{n,k},$$  \hspace{1cm} (6.5)

where the left sum converges in the norm of $\mathcal{L}^2(I)$.

**Proof.** For $y = g_{n_0,l}$ with $1 \leq l \leq m_{n_0}$, the expansion (6.5) follows from the definitions of the matrix $G_n$ and of the coefficients $c_{n,k}(y)$ and from the fact that $c_{n,k}(g_{n_0,l}) = 0$ if $n \neq n_0$. For arbitrary $y \in \mathcal{L}^2(I)$ it is now a consequence of the properties of a Bari basis. \hfill \Box

If the elements $g_{n,k}$, $k = 1, 2, \ldots, m_k$, which span the root subspace of $T_{\gamma}$ at $\lambda_n$ are chosen to form a Jordan chain:

$$(T_{\gamma} - \lambda_n)g_{n,1} = 0, \quad (T_{\gamma} - \lambda_n)g_{n,2} = g_{n,1}, \quad (T_{\gamma} - \lambda_n)g_{n,m_n} = g_{n,m_n-1},$$

then the elements $\overline{g}_{n,k}$, $k = 1, 2, \ldots, m_n$, form a Jordan chain of $T^*_\gamma$ at $\overline{\lambda}_n$ and we get

$$(g_{n,k}, \overline{g}_{n,l}) = \left( (T_{\gamma} - \lambda_n)^{m_n-k} g_{n,m_n}, (T^*_\gamma - \overline{\lambda}_n)^{m_n-l} \overline{g}_{n,m_n} \right)$$

$$= \left( (T_{\gamma} - \lambda_n)^{2m_n-(k+l)} g_{n,m_n}, \overline{g}_{n,m_n} \right).$$

Therefore, the matrix $G_n$ is now a Hankel matrix and right lower triangular. Since $G_n$ is invertible, the numbers $(g_{n,k}, \overline{g}_{n,l})$ with $k + l = m_n$ are not zero. Now it is easy to see that the Jordan chain $g_{n,k}$, $k = 1, 2, \ldots, m_n$, can be modified such that the matrix $G_n$ becomes $(g_{n,1}, \overline{g}_{n,m_n})$ times the $m_n$-sip matrix $(\delta_{k,m_n-l+1})_{k,l=1}^{m_n}$ [10, theorem I.3.3]. Indeed, replace the Jordan chain $g_{n,k}$ by a Jordan chain $g'_{n,k}$, the last element of which has the form $g'_{n,k} = \sum_{k=1}^{m_n} \alpha_k g_{n,k}$, and determine the $\alpha_k$ such that $(g'_{n,k}, \overline{g}_{n,l}) = \delta_{k,m_n-l+1}$, $k, l = 1, 2, \ldots, m_n$. With this choice of the Jordan chains at all the eigenvalues $\lambda_n$ of $T_{\gamma}$, expansion (6.5) simplifies to

$$y = \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} (y, \overline{g}_{n,m_n-k+1}) g_{n,k}.$$

**Acknowledgments**

B.B. was supported by the Fonds zur Förderung der Wissenschaftlichen Forschung (Project P 12176 MAT).

**References**

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(Issued 15 December 2000)