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Sensitivity Analysis of the Greedy Heuristic for Binary Knapsack Problems

Diptesh Ghosh* Nilotpal Chakravarti† Gerard Sierksma‡

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Abstract

Greedy heuristics are a popular choice of heuristics when we have to solve a large variety of $\mathcal{NP}$-hard combinatorial problems. In particular for binary knapsack problems, these heuristics generate good results. If some uncertainty exists beforehand regarding the value of any one element in the problem data, sensitivity analysis procedures can be used to know the tolerance limits within which the value may vary will not cause changes in the output. In this paper we provide a polynomial time characterization of such limits for greedy heuristics on two classes of binary knapsack problems, namely the 0-1 knapsack problem and the subset sum problem.

We also study the relation between algorithms to solve knapsack problems and algorithms to solve their sensitivity analysis problems, the conditions under which the sensitivity analysis of the heuristic generates bounds for the tolerance limits for the optimal solutions, and the empirical behavior of the greedy output when there is a change in the problem data.

Keywords: Sensitivity Analysis, Heuristics, Knapsack Problems
AMS Subject Classification: 90C31

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1. Introduction

Binary knapsack problems are among some of the most widely studied problems in combinatorial optimization (see, for example, Martello and Toth [6]). Algorithms to solve a large variety of combinatorial problems, such as capital budgeting, cargo loading, and vehicle routing, either reduce to solving knapsack problems, or solve a large number of such problems en route to their solutions. Since the optimization versions of these problems are \(NP\)-hard, practical solution techniques do not ask for optimality, but are heuristics that generate feasible, suboptimal solutions. \(\epsilon\)-optimal heuristics form a special class of heuristics for which, in case of maximization problems, the objective function values of the solutions output are greater than \((1-\epsilon)\) times those of the optimal solution.

Heuristics are usually compared on their execution speeds and quality of solutions (either empirically, or probabilistically, or using their worst case performance results), but recently some work has been done on the effect of perturbations in problem data on the performance of the heuristics (see, for example, Chakravarti and Wagelmans [2], and Kolen et al. [5]). This body of work is referred to as the sensitivity analysis of heuristics. Although sensitivity analysis of optimal solutions to combinatorial optimization problems is an established problem (see, for example, Gal and Greenberg [3]), sensitivity analysis of \(\epsilon\)-optimal heuristics is not. In Wagelmans [8] the following definition for sensitivity analysis of heuristics is suggested.

**Problem SA-Wagelmans**

**Input** An instance \(I\) of a combinatorial optimization problem, an \(\epsilon\)-optimal heuristic \(H\) and a solution \(S\) to \(I\) under \(H\).

**Output** For each problem parameter \(p\) the following values

- \(\beta_p^W = \sup (\delta | S \text{ remains } \epsilon\text{-optimal under } H \text{ when } p \rightarrow p + \delta)\)
- \(\alpha_p^W = \sup (\delta | S \text{ remains } \epsilon\text{-optimal under } H \text{ when } p \rightarrow p - \delta)\)

In Wagelmans [8] it is shown that this problem is \(NP\)-hard when the underlying combinatorial optimization problem is \(NP\)-hard, even when the heuristic is of polynomial complexity. In this paper, we study the following related problem.

**Problem SA-Heuristic**

**Input** An instance \(I\) of a combinatorial optimization problem, a heuristic \(H\) and a solution \(S\) to \(I\) under \(H\).

**Output** For each problem parameter \(p\) the following values

- \(\beta_p^H = \sup (\delta | S \text{ remains the heuristic output when } p \rightarrow p + \delta)\)
- \(\alpha_p^H = \sup (\delta | S \text{ remains the heuristic output when } p \rightarrow p - \delta)\)

We refer to \(\beta_p\) and \(\alpha_p\) respectively as the upper and lower tolerance limits of the parameter \(p\). We agree with Wagelmans [8] that this is indeed not the sensitivity analysis
of the heuristic solution, but rather of the heuristic method. However, it is easy to see that $\beta_p^W \geq \beta_p^H$ and $\alpha_p^W \geq \alpha_p^H$.

In this paper we consider two types of binary knapsack problems, the 0-1 knapsack problem and the subset sum problem.

The 0-1 knapsack problem can be described as follows.

**Problem KP($[e_j]_j$)**

**Instance:** An integer $n \geq 2$, a set $E$ of $n$ elements $e_j = \langle p_j, w_j \rangle$, $j = 1, \ldots, n$ where $p_j, w_j$ are positive integers. A positive integer $c$.

**Output:** A subset $E_1$ of $E$ such that $\sum_{j \in E_1} w_j \leq c$, and $\forall E_2 \subseteq E$ with $\sum_{j \in E_2} w_j \leq c$, $\sum_{j \in E_1} p_j \geq \sum_{j \in E_2} p_j$.

An integer programming formulation of the problem is the following.

Maximize $\sum_{j=1}^n p_j x_j$ subject to $\sum_{j=1}^n w_j x_j \leq c$, $x_j \in \{0, 1\}$ for $j = 1, \ldots, n$. (KP)

We assume that all $p_j$'s, $w_j$'s and $c$ are positive integers, $w_j \leq c$ for $j = 1, \ldots, n$; $\sum_{j=1}^n w_j \geq c$; and that $\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \cdots \geq \frac{p_n}{w_n}$.

An interesting fact may be noted here. SS is a special case of KP, and any algorithm for solving KP instances can be easily modified to solve SS instances (by providing $w_j$ values instead of the $p_j$ values as input). However the sensitivity analysis problems for KP and SS are not similarly related, and it is not easy to modify an algorithm for calculating the tolerance limits for KP instances to output the tolerance limits for SA instances (or vice versa). The reason for this is the following. When a particular $w_j$ value changes in a SS instance, it affects both the constraint and the objective function. In case of KP instances, such a change would only affect the constraint. Again, when
a particular $p_j$ value changes in a KP instance, only the objective function is affected. This has no parallel in SS instances. Suppose we have an algorithm designed to output tolerance limits for KP instances. For this algorithm, calculating tolerance limits for $w_j$ values in SS instances would amount to handling simultaneous changes in two problem parameters (the $w_j$ value and the corresponding $p_j$ value) — something the algorithm is not designed to do. If we have an algorithm for calculating tolerances for SS instances, and if we want to use it to calculate the tolerance limit for a $p_j$ value in a KP instance, the algorithm would be expected to handle a situation where the objective function changes but not the constraint. An algorithm to calculate tolerances for SS instances is not designed to handle such cases.

In this paper, we will concentrate on the greedy heuristic. The greedy heuristic works in the following manner. It assumes a prior ordering of the elements in $E$. (We will refer to this ordering as the input ordering in the remainder of the paper.) The heuristic considers the elements (e_j's) one at a time and decides whether it can be added to $E_1$ without violating the constraint. If it can, the heuristic adds $e_j$ to $E_1$. The heuristic stops when there are no more elements to consider. We will refer to the output of the greedy heuristic as the greedy solution. In this paper we will assume the input ordering of the elements of $E$ to be a non-increasing order of $p_j$/$w_j$ ratios for KP instances and a non-increasing order of $w_j$ values for SS instances.

We will now explain some notations that we use in this paper. $X = \{x_j\}$ denotes a solution to an instance of KP or SS. $X^H = \{x^H_j\}$ refers to the greedy solution and $X^* = \{x^*_j\}$ to the optimal solution for the instance. The objective function value of a solution $X$ is denoted by $z_X$. We use $Z^H$ and $Z^*$ to denote the objective functions of the greedy heuristic and any exact algorithm. Notice that $z_X$, $Z^H$, and $Z^*$ are all functions of the problem parameters ($p_j$, $w_j$, and $c$ for KP instances, and $w_j$ and $c$ for SS instances).

We also need to differentiate between $z^{X^H}$ and $z^{X^*}$ on one hand and $Z^H$ and $Z^*$ on the other. $z^{X^H}$ and $z^{X^*}$ are in a sense, tied to the solutions $X^H$ and $X^*$ respectively. If the data in the instance changes sufficiently, $X^H$ may cease to remain the solution output by the greedy heuristic and the previously optimal $X^*$ may become sub-optimal but $z^{X^H}$ and $z^{X^*}$ will still reflect the objective function values of these solutions. $Z^H$ and $Z^*$ however will change and store the objective function values of the new greedy solution, and the new optimal solution respectively. We will use the superscript $\ast$ to denote the tolerance limits of optimal solutions. For instance $\beta_Z$ denotes the maximum amount by which the parameter $c$ can vary without affecting the optimality of $X^*$.

The remainder of the paper is organized as follows. In Section 2 we characterize the tolerance limits output by SA-Heuristic for KP and SS, i.e. the $\beta^H$ and $\alpha^H$ values. In Section 3 we analyze the characteristics of $Z^H$ and $Z^*$, and illustrate their behaviour with as a function of changes in the values of certain problem parameters. It may appear
that in certain situations, $\beta^H$ and $\alpha^H$ values can serve as bounds for the corresponding $\beta^*$ and $\alpha^*$ values. We characterize these situations in Section 4. In Section 5 we report the results of limited computational tests on the empirical behavior of $X^H$, and conclude the paper with Section 6 on discussions and conclusions.

2. Characterization of tolerance limits

In this section, we characterize the $\beta^H$ and $\alpha^H$ values for KP (Subsection 2.1) and SS (Subsection 2.2). Recall that these are limits within which each problem parameter may vary while the other parameters remain unchanged, for $X^H$ to remain the output for the greedy heuristic for the instance. The formal proofs for the characterizations are available in Ghosh [4].

2.1 0-1 Knapsack Problem

Consider a KP instance $[(e_j)|c]$, where $e_j = (p_j, w_j)$ denotes element $j$. We define $p_j = \frac{p_j}{w_j}$, $\rho_0 = \infty$, $r_k = c - \sum_{j=1}^{k} w_j x^H_j$, $r_0 = c$, and $s_k = \begin{cases} w_k - r_{k-1} & \text{if } x_k^H = 0, \\ \infty & \text{otherwise} \end{cases}$ for $k = 1, \ldots, n$.

This means that $r_k$ is the portion of $c$ that is available after the heuristic has considered $e_j$, while $s_k$ denotes the amount by which $w_k$ exceeds $r_{k-1}$ for any element $e_k$ that was not included in the greedy solution. $r_k$ is non-increasing in $k$.

If $c$ is increased, $X^H$ remains the greedy solution until the increase is large enough to admit some element which could not be admitted before. It is easy to see therefore, that the greedy solution remains unchanged until $c$ increases by $\min_{1 \leq j \leq n} (s_j)$. Similarly, $X^H$ would remain the greedy solution until $c$ decreases sufficiently so that the greedy heuristic refuses to admit an element that it admitted before (i.e. decreases by $r_n$).

The greedy heuristic expects its input to be arranged in non-increasing order of $p_j$ values. When the $p_j$ value or the $w_j$ value of an element $e_j$ changes, so does its $p_j$ value, which may cause the greedy solution to change. Let us first consider the case where the problem data changes so as to increase $p_j$. This could be due to an increase in $p_j$ or a decrease in $w_j$. Let us assume that the change is due to an increase in $p_j$. The value of $p_j$ does not affect the constraint in any way, so the only way in which this change could affect the greedy solution is by bringing about a change in the input ordering. The nature of this change is also easy to determine. It would cause $e_j$ to be selected by the greedy heuristic, when it had not been selected earlier. We next consider an increase in $p_j$ due to a decrease in $w_j$. The effects of such a decrease is more complicated. It can cause a
change in the input ordering. Even if it does not, it will increase $r_k$ values for all $k \geq j$ and decrease $s_k$ values for all $k \geq j$ if $x_j^H = 1$. If $x_j^H = 0$, only $s_j$ will decrease. These changes may affect the greedy solution in a variety of ways. If $x_j^H = 1$, the decrease could cause the $s_k$ value of some element $e_k$, $k \geq j$ to reduce to 0, thus including it in the greedy solution. If $x_j^H = 0$, the decrease in $s_j$ or the new position of $e_j$ in the input ordering could set $x_j^H$ to 1 after the change.

Next let us consider a decrease in $p_j$ due to a decrease in $p_j$ or an increase in $w_j$. Since a change in the $p_j$ value cannot affect the feasibility of any solution, the only way in which such a change can affect the greedy solution is by causing a change the input ordering so that the greedy heuristic, which may have set $x_j^H = 1$ before the change would set it to 0 after the change. If the decrease in $p_j$ is due to an increase in $w_j$, then again we have several effects. If $x_j^H = 1$ originally, then this change would cause $r_k$ to decrease for all $k \geq j$ and $s_k$ to increase for all $k > j$ with $x_k^H = 0$. If $x_j^H$ was initially set to 0, then a change in $w_j$ would cause $s_j$ to increase by a corresponding amount. If the increase is sufficient, $X_j^H$ could cease to be the output of the greedy heuristic. This could be due to $r_n$ decreasing to 0, or due to a change in the input ordering.

These arguments lead to the following characterization of $\beta_j^H$ and $\alpha_j^H$ values for parameters of KP.

- $c$
  \begin{align*}
  \beta_c^H &= \min_{1 \leq j \leq n}(s_j), \\
  \alpha_c^H &= r_n.
  \end{align*}

- $p_j$
  \begin{align*}
  \text{If } x_j^H &= 1, \beta_{p_j}^H = \infty. \\
  \text{If } x_j^H &= 0, \beta_{p_j}^H &= [\rho_m w_j - p_j] \text{ where } m = \max_{1 \leq k \leq j} \{k : x_k^H = 1, r_k < w_j \leq r_{k-1}\}. \\
  \text{If } x_j^H &= 1, \alpha_{p_j}^H &= \begin{cases} [p_j - \rho_m w_j] & \text{where } \exists m \text{ such that } \\
  & m = \min_{j < k \leq n} \{k : x_k^H = 0, s_k \leq w_j\}, \\
  & \text{otherwise.} \end{cases} \\
  \text{If } x_j^H &= 0, \alpha_{p_j}^H &= p_j.
  \end{align*}

- $w_j$
  \begin{align*}
  \text{If } x_j^H &= 1, \beta_{w_j}^H &= \begin{cases} \min(r_n, \lceil \frac{p_j}{\rho_m} - w_j \rceil) & \text{where } \exists m \text{ such that } \\
  & m = \min_{j < k \leq n} \{k : x_k^H = 0, s_k \leq w_j\}, \\
  & \text{otherwise.} \end{cases} \\
  \text{If } x_j^H &= 0, \beta_{w_j}^H = c - w_j, \\
  \text{If } x_j^H &= 1, \alpha_{w_j}^H = \min(w_j, \min_{j < k \leq n}(s_k)). \\
  \text{If } x_j^H &= 0 \text{ then } \alpha_{w_j}^H &= \min(s_j, \lceil w_j - \frac{p_j}{\rho_m} \rceil) \text{ where } m = \max_{j < k \leq n} \{k : r_{k-1} \geq \frac{p_j}{\rho_m} \}. 
  \end{align*}
Let us illustrate this characterization with an example.

**Example 1.** Consider the following instance of KP in which the elements are arranged in non-increasing order of $\rho$ values.

<table>
<thead>
<tr>
<th>$e_j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_j$</td>
<td>81</td>
<td>64</td>
<td>24</td>
<td>83</td>
<td>98</td>
<td>75</td>
<td>35</td>
<td>77</td>
<td>58</td>
<td>21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$w_j$</td>
<td>7</td>
<td>8</td>
<td>11</td>
<td>78</td>
<td>99</td>
<td>79</td>
<td>38</td>
<td>98</td>
<td>79</td>
<td>57</td>
<td>277</td>
<td></td>
</tr>
<tr>
<td>$x_j^H$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_j$</td>
<td>$\infty$</td>
<td>11.57</td>
<td>8.00</td>
<td>2.18</td>
<td>1.06</td>
<td>0.99</td>
<td>0.95</td>
<td>0.92</td>
<td>0.79</td>
<td>0.73</td>
<td>0.37</td>
<td></td>
</tr>
<tr>
<td>$r_j$</td>
<td>277</td>
<td>270</td>
<td>262</td>
<td>251</td>
<td>173</td>
<td>74</td>
<td>74</td>
<td>36</td>
<td>36</td>
<td>36</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>$s_j$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>5</td>
<td>$\infty$</td>
<td>62</td>
<td>43</td>
<td>21</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For this instance $\min_{1\leq j\leq n} |s_j|$ is 5 (for $e_6$). If $c$ increases by more than 5, the greedy heuristic outputs $\{1, 1, 1, 1, 0, 0, 0, 0, 0\}$ i.e. sets $x^H_6 = 1$, thereby changing the greedy solution. Similarly, the greedy solution remains unaltered until $c$ reduces by more than 36 (i.e. $r_n$) after which $x^H_7$ is set to 0 (the greedy heuristic outputs $\{1, 1, 1, 1, 0, 0, 0, 0, 0\}$).

We next illustrate our characterization of the $\beta^H$ and $\alpha^H$ values for elements $e_j$ with $x^H_j = 1$.

**Example 2.** Consider the problem in Example 1. It is clear that an increase in the $p_j$ value for an element with $x^H_j = 1$ will not affect the greedy solution.

If the $p_j$ value decreases, then there may be two situations. Consider the element $e_7$. If $p_7$ decreases, $\rho_7$ will decrease, so that the input ordering will eventually change (for example, the ordering becomes $<e_1, \ldots, e_6, e_8, e_7, e_9, e_{10}>$ when $p_7$ decreases by 5). In this case, such changes will not affect the greedy solution, and $p_7$ could decrease to 0 (i.e. $\beta^H_{p_7} = p_7$). If however, $p_5$ decreased, then a decrease of 4 (i.e. $|p_5 - p_6 w_5|$) would cause the input ordering would change to $<e_1, \ldots, e_4, e_6, e_5, e_7, \ldots, e_{10}>$, and the greedy output to change to $\{1, 1, 1, 0, 1, 0, 0, 0\}$.

Next assume that $w_7$ increases. This causes $\rho_7$, $r_7$, $r_8$, $r_9$, and $r_{10}$ to decrease, and $s_8$, $s_9$, and $s_{10}$ to increase. If $w_7$ increases by more than 36 (i.e. $r_{10}$), then the input ordering will change to $<e_1, \ldots, e_6, e_8, e_9, e_7, e_{10}>$, and the greedy output to $\{1, 1, 1, 1, 0, 1, 1, 0, 0, 1\}$. If $w_5$ increased however, the greedy solution would change as soon as the increase exceeded 5 (i.e. $|p_5 - w_5|$). Such a change would cause the input ordering to change to $<e_1, \ldots, e_4, e_6, e_5, e_7, \ldots, e_{10}>$ and the greedy solution to $\{1, 1, 1, 1, 0, 1, 0, 0, 0\}$. 
Next we consider decreases in \( w_j \) values. If \( w_7 \) decreases, then \( p_7 \) increases and can cause the input ordering to change. If the decrease is by 21 (i.e. \( s_{10} \)), then the greedy solution changes, since this change causes \( x_{10}^H \) to be set to 1. However it is possible for \( w_j \) to decrease to 0 without causing any change in the greedy solution (for e.g. for \( w_3 \)).

Finally we illustrate our characterization of the the \( H^1 \) and \( H^1 \) values for elements \( e_j \) with \( x_j^H = 0 \).

**Example 3.** Refer to Example 1. We first consider increases in \( p_j \) values. Assume an increase in \( p_6 \). This increases \( s_6 \) and can potentially change the input ordering. If \( p_6 \) increases by 4 (i.e. \( p_5 w_6 - p_6 \)) the greedy heuristic will accept \( e_6 \) before \( e_5 \) and set \( x_6^H = 1 \) thereby altering the solution.

It is easy to see that a decrease in \( p_6 \) will not affect the greedy solution. Hence \( p_6 \) could reduce to 0 without changing the greedy solution.

Next we consider changes in \( w_j \) values. Assume that \( w_6 \) increases. This causes \( s_6 \) to decrease, and \( s_6 \) to increase, and will not affect the solution. Hence \( w_6 \) can increase until it reaches the value \( c_6 \).

If \( w_j \) decreases, it cause \( s_j \) to decrease and \( p_j \) to increase. This can change the output of the greedy heuristic in two ways. For example, if \( w_{10} \) decreases by 21 (\( s_{10} \)) the input ordering will not change but the greedy heuristic would set \( x_{10}^H = 1 \). But if \( w_6 \) decreases by 4 (i.e. \( w_6 - \frac{d}{p_5} \)) the input ordering changes to \( e_1 \cdots e_4 e_6 e_5 e_7 \cdots e_{10} > \) and the greedy heuristic would set \( x_6^H = 1 \).

### 2.2 The Subset Sum Problem

We define \( r_k = c - \sum_{j=1}^{k} w_j x_j^H \) and \( s_k = \begin{cases} w_k - r_{k-1} & \text{if } x_k^H = 0, \\ \infty & \text{otherwise.} \end{cases} \) The interpretations of \( r_k \) and \( s_k \) are identical to that in KP; \( r_k \) is non-increasing in \( k \) and a plot of the finite values of \( s_k \) with \( k \) yields a saw-tooth curve.

Recall that for SS, the input ordering is a non-increasing ordering of \( w_j \) values. The dependence of the greedy solution on \( c \) is very similar to that in the case of KP — the greedy solution remains unchanged until \( c \) either decreases by \( r_n \) or increases by \( \min_{1 \leq j \leq n}(s_j) \).

Let us consider an increase in \( w_j \). If \( x_j^H = 1 \), then this increase causes \( r_k \) values to decrease by an equal amount for all \( k \geq j \) and \( s_k \) values to increase for all \( k > j \) with \( x_k^H = 0 \). If \( x_j^H = 0 \), then the increase causes \( s_j \) to increase by the same amount. Additionally, such an increase may cause a change in the input ordering. In case \( x_j^H \) was originally set to 1, the increase would affect the greedy solution if \( w_j \) increased by an
amount greater than \( r_n \). If \( x_{Hj}^I \) was originally set to 0, the greedy solution can change only if the input ordering becomes such that \( w_j \) is input to the greedy heuristic early enough for \( x_{Hj}^I \) to be set to 1.

If \( w_j \) decreases, the effects are the reverse of the effects mentioned above, i.e. if \( x_{Hj}^I = 1 \) originally, then \( r_k \) values will increase by an equal amount for all \( k \geq j \) and \( s_k \) values decrease for all \( k > j \) with \( x_{Hk}^I = 0 \). If \( x_{Hj}^I = 0 \), then \( s_j \) registers a corresponding decrease. If \( x_{Hj}^I = 0 \), then \( s_j \) would become 0 thus changing the greedy solution. If originally \( x_{Hj}^I = 1 \), the greedy solution changes when the value of \( w_j \) decreases till \( s_p \) for some \( p > j \) decreases to 0, or the input ordering changes appropriately.

This discussion leads to the following characterization of \( \beta^H \) and \( \alpha^H \) values for parameters of SS.

\[ \begin{align*}
\beta_{c}^H &= \min_{1 \leq i \leq n} (s_i), \\
\alpha_{c}^H &= r_n, \\
\beta_{w_j}^H &= r_n. \\
\beta_{w_j}^H &= \max_{1 \leq k < j} (w_k : x_{Hk}^I = 1, r_k > 0) - w_j. \\
\alpha_{w_j}^H &= \begin{cases} 
\min(s_p, w_j - w_t) & \text{if } \exists t, \text{ and } k > j \text{ such that } x_{Hk}^I = 0, \\
 s_p & \text{if } \exists k > j \text{ such that } x_{Hk}^I = 0, \\
 w_j & \text{otherwise.} 
\end{cases} \\
\text{where } p = \min_{j \leq m \leq n} (m : s_m = \min_{j \leq k \leq n} (s_k)), \text{ and } t = \min_{j \leq m \leq n} (m : x_m = 0). \\
\text{If } x_{Hj}^I = 0, \ \alpha_{w_j}^H &= s_j. 
\end{align*} \]

As in the case of KP, let us illustrate these tolerance limits with an example.

**Example 4.** Consider the following instance of SS in which the elements are arranged in non-increasing order of \( w_j \) values.

<table>
<thead>
<tr>
<th>j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_j )</td>
<td>99</td>
<td>98</td>
<td>83</td>
<td>81</td>
<td>79</td>
<td>57</td>
<td>38</td>
<td>11</td>
<td>8</td>
<td>7</td>
<td>274</td>
</tr>
<tr>
<td>( x_{Hj}^I )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( r_j )</td>
<td>175</td>
<td>77</td>
<td>77</td>
<td>77</td>
<td>77</td>
<td>20</td>
<td>20</td>
<td>9</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( s_j )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>( \infty )</td>
<td>18</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>
If \( c \) increases by 2 (i.e. \( s_5 \)), then the greedy heuristic sets \( x_5^H = 1 \), thus changing the greedy solution. Again, if \( c \) decreases by more than 1 (i.e. \( r_{10} \)), the greedy heuristic sets \( x_j^H = 0 \), thereby changing the greedy solution.

We now examine the effect of changes in \( w_j \) on \( \beta^H \) and \( \alpha^H \) values for elements \( e_j \) with \( x_j^H = 1 \).

**Example 5.** Consider \( e_8 \) in Example 4. If \( w_8 \) increases, then \( r_8, r_9, \) and \( r_{10} \) decreases and \( s_{10} \) increases. If \( w_8 \) increases by more than 1 (i.e. \( r_{10} \)), it causes the greedy heuristic to set \( x_j^H = 0 \), thereby changing the greedy solution. Notice that we do not consider changes in input ordering since, if there exists \( e_j, e_{j+1} \) such that \( w_j > w_{j+1}, x_j^H = 0, x_{j+1}^H = 1 \), then \( w_j - w_{j+1} > r_{j+1} \geq r_n \).

If \( w_8 \) decreases, then \( r_8, r_9, \) and \( r_{10} \) increases and \( s_{10} \) decreases. If the decrease is by more than 4 (i.e. \( w_8 - w_{10}, \) since in this case \( t = 10 \)), the initial ordering changes to \( <e_1, \cdots, e_7, e_9, e_{10}, e_8> \). The greedy heuristic now sets \( x_{10}^H = 1 \), i.e. the greedy solution changes. If there was no element \( e_j \) such that \( w_j \leq w_8 \) and \( x_j^H = 0 \), then a decrease in \( w_8 \) would never affect the greedy solution, and so \( w_8 \) could reduce to 0.

We finally examine the effect of changes in \( w_j \) on \( \beta^H \) and \( \alpha^H \) values for elements \( e_j \) with \( x_j^H = 0 \).

**Example 6.** Consider \( e_7 \) in Example 4. If \( w_7 \) increases, \( s_7 \) increases by an equal amount. If the increase is by more than 19 (i.e. \( w_6 - w_7 \)), then the initial ordering changes to \( <e_1, \cdots, e_5, e_7, e_6, e_8, \cdots, e_{10}> \) and the greedy heuristic sets \( x_{10}^H = 1 \).

If \( w_7 \) decreases, then \( s_7 \) decreases by the same amount. If \( w_7 \) decreases by more than 18 (i.e. \( s_7 \)), it causes the greedy heuristic to out put a different solution with \( x_j^H = 1 \).

Note that here also we do not consider changes in input ordering since, if there exists \( e_j, e_{j+1} \) such that \( w_j > w_{j+1}, x_j^H = 0, x_{j+1}^H = 1 \), then \( w_j - w_{j+1} \geq s_j \).

One more interesting result is clear from the characterizations presented in Subsections 2.1 and 2.2. Notice that the tolerance limits of all the problem parameters for KP are polynomial functions of \( p_k, r_k \) and \( s_k \) values while those of SS are polynomial functions of \( r_k \) and \( s_k \) values. These values can be calculated by careful book-keeping when the greedy heuristic is generating the greedy solution. Hence we have the following theorem.

**Theorem 1**  The complexity of calculating \( \beta^H \) and \( \alpha^H \) values of any parameter of KP or SS is polynomial in the size of the problem.
3. The objective functions $Z^H$ and $Z^*$

In this section, we study $Z^H$ and $Z^*$ as functions of the problem parameters. We will assume that the value of exactly one parameter of the instance at hand varies. If this parameter is $c$ for either a KP instance or a SS instance, or $w_j$ for a KP instance, $Z^H$ (and $Z^*$) are piecewise linear discontinuous functions with the linear sections having a slope of 0. This is because neither $c$ for KP or SS instances, nor $w_j$ for KP instances appear in the objective function, and hence a change in their values cannot cause a linear change in the objective function. If the parameter is $p_j$ for a KP instance or $w_1$ for a SS instance, then $Z^H$ (and $Z^*$) are piecewise linear functions with the linear sections having slopes of $+1$ or 0 depending on whether or not $e_i \in E_1$ in the greedy (and optimal) solution. We will now illustrate this with examples.

Example 7. Let us consider the KP instance where $E = \{<65,50>, <6,5>, <25,25>\}$ and $c = 65$. We first assume that $c$ changes from 0 to $\sum_{j=1}^{j=3} w_j$ (i.e. 80). When $0 \leq c < 5$, both $Z^H$ and $Z^*$ equal 0. As soon as $c = 5$, both $Z^H$ and $Z^*$ assume a value of 6. When $c$ increases to 25, a new optimal solution is reached and $Z^*$ equals 25. $Z^H$ however remains at 6. Both $Z^H$ and $Z^*$ assume values of 31 when $30 \leq c < 50$, of 65 when $50 \leq c < 55$, and of 71 when $55 \leq c < 75$. When $c$ reaches 75, $Z^*$ assumes a value of 90 ($Z^H$ remains at 71) and when it touches 80, both $Z^H$ and $Z^*$ assume values of 96. Figure 3a shows the functions $Z^H(c)$ and $Z^*(c)$. Notice that both are piecewise linear and are constant over intervals. $Z^H(c)$ is discontinuous at $c = 5, 30, 50, 55, 80$, while $Z^*(c)$ is discontinuous at $c = 5, 25, 30, 50, 55, 75, 80$.

Next we assume that $p_2$ changes in the problem instance. When $0 \leq p_2 < 5$, the input ordering for the greedy heuristic is $<e_1, e_3, e_2>$. Both the greedy and the optimal solution to this instance is $\{1, 0, 1\}$ and hence both $Z^H$ and $Z^*$ equal 80. When $p_2$ crosses 5 the input ordering for the greedy heuristic changes to $<e_1, e_2, e_3>$ which causes the greedy solution to become $\{1, 1, 0\}$ and the $Z^H$ value decreases to 70. When $p_2 \geq 7$ the input ordering for the greedy heuristic again changes (this time to $<e_2, e_1, e_3>$) but the greedy solution remains $\{1, 1, 0\}$. So when $p_2 \geq 5$, $Z^H$ increases linearly with $p_2$ and the function has a slope of $+1$. $Z^*$ remains constant at 80 when $5 \leq p_2 \leq 15$ but when $p_2 > 15$ the optimal solution becomes $\{1, 1, 0\}$, $Z^*$ increases linearly with $p_2$. Figure 3b shows the functions $Z^H(p_2)$ and $Z^*(p_2)$. Notice that both the functions are piecewise linear, and have regions where the slope is $+1$ as well as regions where the functions are constant. $Z^H$ has a discontinuity at $p_2 = 5$ while $Z^*$ is continuous.

Next we assume that $w_2$ changes. Note that $w_2$ can only vary in the interval $[0, c]$. The input ordering for the greedy heuristic is $<e_2, e_1, e_3>$ when $0 < w_2 < 5$, $<e_1, e_2, e_3>$ when $5 \leq w_2 \leq 6$, and $<e_1, e_3, e_2>$ when $w_2 > 6$. Hence the greedy solution is $\{1, 1, 0\}$ for $0 < w_2 \leq 15$ and changes to $\{1, 0, 0\}$ when $w_2 > 15$. The opti-
Figure 1: The $Z^H$ and $Z^*$ functions for KP

mal solution mimics the greedy solution in this case. $Z^H$ and $Z^*$ therefore both remains constant at 71 when $0 < w_2 \leq 15$ and at 65 when $w_2 > 15$. Figure 3c shows the functions $Z^H(w_2)$ and $Z^*(w_2)$. Notice that both the functions are piecewise linear and are constant over intervals. Both $Z^H$ and $Z^*$ are discontinuous at $w_2 = 15$.

Example 8. Consider an instance of SS with $E = \{14, 9, 7\}$ and $c = 18$. We first assume that $c$ changes from 0 to $\sum_{j=1}^{3} w_j$ (i.e. 30). When $0 \leq c < 7$, both $Z^H$ and $Z^*$ equal 0. As soon as $c = 7$, both $Z^H$ and $Z^*$ assume a value of 7. The values of both $Z^H$ and $Z^*$ increase to 9 when $9 \leq c < 14$ and to 14 when $14 \leq c < 16$. When
c increases to 16, a new optimal solution is reached (\{0, 1, 1\}), and \(Z^*\) equals 16. \(Z^H\) however remains at 14. Both \(Z^H\) and \(Z^*\) assume values of 21 when \(21 \leq c < 23\), of 23 when \(23 \leq c < 30\), and of 30 when \(c = 30\). Figure 3a shows the functions \(Z^H(c)\) and \(Z^*(c)\). Notice that both the functions are piecewise linear, continuous and have regions where the slope is +1 as well as regions where the functions are constant.

Finally we assume that \(w_2\) changes. As in case of the KP instance, \(w_2\) can vary in the interval \([0, c]\). When \(0 < w_2 \leq 7\), the input ordering for the greedy heuristic is \(< e_1, e_3, e_2 >\). Both the greedy output and the optimal solution are \(\{1, 0, 1\}\) when \(0 < w_2 \leq 4\) and \(\{1, 0, 0\}\) when \(4 < w_2 \leq 7\). So both \(Z^H\) and \(Z^*\) remain constant at 18 when \(0 < w_2 \leq 4\) and at 14 when \(4 < w_2 \leq 7\). When \(7 < w_2 \leq 14\), the input ordering for the greedy heuristic changes to \(< e_1, e_2, e_3 >\). In this whole range, the greedy solution remains \(\{1, 0, 0\}\) and \(Z^H\) remains constant at 14. The optimal solution however becomes \(\{0, 1, 1\}\) when \(7 < c \leq 11\) and \(Z^*\) increases linearly with \(w_2\) in this range. When \(11 < c \leq 14\), the optimal solution again becomes \(\{1, 0, 0\}\) and \(Z^*\) remains constant at 14. The input ordering for the greedy heuristic again changes when \(w_2 > 14\) and becomes \(< e_2, e_1, e_3 >\). So when \(14 < c \leq 18\), the greedy heuristic outputs \(\{0, 1, 0\}\) and \(Z^H\) increases linearly with \(w_2\). The optimal solution is also \(\{0, 1, 0\}\) when \(w_2\) is in this interval and so \(Z^*\) also increases linearly with \(w_2\). Figure 3b shows the functions \(Z^H(w_2)\) and \(Z^*(w_2)\). Notice that here too, both the functions are piecewise linear and have regions where the slope is +1 as well as regions where the slope is 0. \(Z^H\) is discontinuous at \(w_2 = 4\) while \(Z^*\) is discontinuous at \(w_2 = 4\) and 11.

\[\beta^H \text{ and } \alpha^H \text{ as bounds for } \beta^* \text{ and } \alpha^*\]

The literature (see for example, Ramaswamy [7], Wagelmans [8]) points out that it is hard to calculate \(\beta^*\) and \(\alpha^*\) values for parameters in \(\mathcal{NP}\)-hard combinatorial optimization problems. We see however from the characterizations in Section 2, that calculating \(\beta^H\) and \(\alpha^H\) values is easy for binary knapsack problems. This raises the question — “Can we use \(\beta^H\) and \(\alpha^H\) values as bounds for the corresponding \(\beta^*\) and \(\alpha^*\) values for binary knapsack problems?” To answer this question, we will consider separately the case where \(z^{X^H} = z^X\), and the case where \(z^{X^H} < z^X\). We will consider the case where the objective function remains constant until a tolerance limit is reached and then varies linearly. Other cases can be dealt with in a very similar manner.

Let \(z^{X^H} = z^X\). We first assume that the value of a parameter \(v\) increases. When the increase exceeds \(\beta^*_v\), let us assume that \(Z^*(v)\) increases. We also assume that the \(Z^H(v)\) function mimics the \(Z^*(v)\) function, in the sense that \(Z^H(v)\) increases after \(v\) increases by more than its \(\beta^*_v\) value. This situation is shown in Figure 4a. It is clear that \(Z^H_0\) can
never be a correct representation of the $Z^H$ function (because, for a range of values of $v$, $Z^H_a(v) > Z^*(v)$). In such cases $\beta^H$ (or $\alpha^H$) has to be an upper bound to the $\beta^*$. If $Z^H(v)$ function does not mimic the $Z^*(v)$ function (shown in Figure 4b), $\beta^H$ cannot be used as a bound for the corresponding $\beta^*$ value, since in such cases, both $Z^H_a$ and $Z^H_b$ are possible representations of the $Z^H$ function. We can also show that if $Z^*(v)$ decreases after $v$ increases by more than $\beta^*(v)$ and $Z^H(v)$ mimics $Z^*(v)$, then $\beta^H_v$ would be a lower bound to the corresponding $\beta^*_v$ value (see Figure 4c). It is interesting to note that if $Z^*(v)$ decreases after a tolerance limit is reached, $Z^H(v)$ has to mimic $Z^*(v)$ (see Figure 4d).

Next let $z^{X^H} < z^{X^*}$. We again assume that the value of a parameter $v$ increases, and when the increase exceeds $\beta^*_v$, $Z^*(v)$ increases. This situation is shown in Figure 4a. Due to the existence of a gap between the values of the $Z^*$ and $Z^H$ functions at the present value of $v$, all of the functions $Z^H_a$ through $Z^H_d$ are possible representations of the $Z^H$ function. This implies that we cannot use $\beta^H_v$ as a bound on $\beta^*_v$. From Figure 4b, we see that $\beta^H_v$ cannot be used as a bound on $\beta^*_v$ even when $Z^*(v)$ decreases after $v$ increases by more than $\beta^*_v$.

Exactly similar arguments hold for the relation between $\alpha^H$ and $\alpha^*$ values.

In the remainder of this section, we therefore concentrate on instances of KP and SS where $z^{X^H} = z^{X^*}$.
Let us first consider instances of KP. Assume that $c$ increases. The set of feasible solutions increases in size and hence, when $c \rightarrow c + \beta_c^*,$ $Z^*$ cannot deteriorate. By a similar argument, $Z^*$ cannot improve when $c \rightarrow c - \alpha_c^*.$ Notice that changes in $c$ cannot alter the input ordering. So when $c$ exceeds $c + \beta_c^H,$ $x_j$ changes from 0 to 1, and $x_k$ is set to 0 for all $k > j$ (where $j$ is the lowest index affected by the increase). Since $p_j > \sum_{k=j+1}^n p_k x_k^H$ before the change, this change will never cause a deterioration in $Z_c^H.$ Following a similar argument, we can show that $Z_c^H$ never improves when $c \rightarrow c - \alpha_c^H.$ Hence we see that $Z_c^H$ mimics $Z^*$ when $c$ changes, which implies that $\beta_c^H$ can be used as an upper bound for $\beta_c^*$ and $\alpha_c^H$ can be used as a lower bound for $\alpha_c^*$ when $X^H = X^*.$
Unfortunately, the nature of changes in \( Z^H \) can be quite different from the nature of changes in \( Z^* \) when \( p_j \) and \( w_j \) values reach their tolerance limits. We illustrate this with the following example.

**Example 9.** Consider an instance of KP with \( E = \{ < 150, 100 >, < 25, 25 >, < 4, 5 > \} \) and \( c = 127 \). It is easy to see that \( X^H = X^* = \{ 1, 1, 0 \} \), and \( Z^H = Z^* = 175 \).

Consider an increase in \( p_3 \). \( \beta^*_{p_3} = 21 \). When \( p_3 \rightarrow p_3 + \beta^*_{p_3} \), the optimal solution changes to \( \{ 1, 0, 1 \} \) and \( Z^* \) increases. However, \( \beta^*_{p_3} = 1 \), and if \( p_3 \) increases by more than this amount, the input ordering changes. The greedy solution becomes \( \{ 1, 0, 1 \} \) in this case too, but at this stage, it causes a decrease in \( Z^H \).

Next consider a decrease in \( w_3 \). \( \alpha^*_{w_3} = 3 \). If \( w_3 \) decreases by more than this amount, the optimal solution becomes \( \{ 1, 1, 1 \} \) and \( Z^* \) increases to 179. However, \( \alpha^H_{w_3} = 3 \) and any further decrease in \( w_3 \) causes a change in the input ordering. The greedy solution then becomes \( \{ 1, 0, 1 \} \) and \( Z^H \) decreases to 154.

This implies that \( \beta^H_{p_j}, \alpha^H_{w_j}, \beta^H_{w_j}, \) and \( \alpha^H_{w_j} \) cannot, in general be used as bounds for the corresponding \( \beta^*_p, \alpha^*_p, \beta^*_w, \) and \( \alpha^*_w \) values.

Let us next consider instances of SS. Note that the objective function values for these problems are bounded above by the value of \( c \). Due to reasons identical to those in the case of KP, \( Z^* \) either remains the same or increases when \( c \rightarrow c + \beta^*_c \), and either remains the same or decreases when \( c \rightarrow c - \alpha^*_c \). Changes in \( c \) cannot alter the the input ordering in SS instances. So when \( c \rightarrow c + \beta^H_c, x^H_j \) changes from 0 to 1, and
Consider changes in \( w_j \) for elements \( e_j \) with \( x_j^H = x_j^* = 1 \). Assume that \( w_j \) increases. \( X^* \) remains the optimal solution until \( z^{X^*} \) reaches \( c \), after which this solution becomes infeasible. Hence when \( w_j \rightarrow w_j + \beta^H_{w_j}, \) \( X^* \) cannot improve. In case of the greedy heuristic, \( Z^H \) increases until \( w_j \rightarrow w_j + \beta^H_{w_j} \). \( \beta^H_{w_j} \) is reached due to one of two reasons. Either \( z^{X^H} \) reaches a value of \( c \) (which causes a further increase in \( w_j \) to render the \( X^H \) infeasible), or any further increase in \( w_j \) causes a change in the input ordering, and moves \( e_j \) to a position just ahead of \( e_k \) such that \( x_{j-1}^H = \cdots = x_{k+1}^H = 1 \) and \( x_k^H = 0 \) before the change. In the former situation, \( Z^H \) clearly cannot improve. The latter situation is impossible to reach. Next assume that \( w_j \) decreases. This causes \( z^{X^H} \) of all solutions \( X \) with \( x_j = 1 \) to decrease. Hence \( X^* \) ceases to be the optimal solution when \( z^{X^{new}} < z^{X^{new}} \) for some other solution \( X^{new} \) with \( x_j^{new} = 0 \). So when \( w_j \rightarrow w_j - \alpha^H_{w_j} \), \( Z^* \) does not decrease. \( \alpha^H_{w_j} \) can be reached in several ways. It is possible that the decrease in \( w_j \) would cause some \( x_k^H, k > j \) to be changed from 0 to 1. This would cause an increase in \( Z^H \). It is also possible that the decrease in \( w_j \) would change the input ordering and place \( e_j \) after some \( e_k \) such that \( x_{j-1}^H = \cdots = x_{k+1}^H = 1 \) and \( x_k^H = 0 \) before the change. This would cause the greedy heuristic to set \( x_k^H = 1 \) which will keep \( Z^H \) from reducing further. A third possibility is that \( w_j \) would reduce to 0. Clearly, \( Z^H \) cannot decrease at this stage. From the nature of changes in the objective function values, we can conclude that in such cases, \( \beta^H_{w_j} \) is a lower bound on \( \beta^*_{c} \) and \( \alpha^H_{c} \) is an upper bound for \( \alpha^*_{c} \).

Finally we consider changes in \( w_j \) for elements \( e_j \) with \( x_j^H = x_j^* = 0 \). Assume that \( w_j \) increases. \( Z^* \) remains equal to \( z^{X^*} \) until \( z^{X^*} \) for some other solution \( X \) with \( x_j = 1 \) exceeds this value. Hence \( Z^* \) increases when \( w_j \rightarrow w_j + \beta^H_{w_j} \). In case of the greedy heuristic, \( Z^H \) does not change until the increase in \( w_j \) causes a change in the input ordering, and sets \( e_j \) in a position just ahead of \( e_k \) such that \( x_{j-1}^H = \cdots = x_{k+1}^H = 0 \) and \( x_k^H = 1 \) before the change. If \( w_j \) increases further, it is clear that \( Z^H \) would increase. Next assume that \( w_j \) decreases. \( z^{X^H} \) remains unchanged, and \( z^{X^H} \) values of all solutions \( X \) (feasible and otherwise) with \( x_j = 1 \) decrease. When the decrease in \( w_j \) is large enough, a previously infeasible solution becomes feasible and \( Z^* = c \). Hence \( Z^* \) does not deteriorate when \( w_j \rightarrow w_j - \alpha^H_{w_j} \). Consider the greedy heuristic. If \( w_j \) decreases,
remains unchanged until $w_j$ decreases enough to be able to be incorporated into the current greedy solution thus setting $Z^H$ to $c$, i.e. it does not deteriorate. Hence in these cases $\beta^H_{w_j}$ and $\alpha^H_{w_j}$ act as upper bounds on $\beta^*_w$ and $\alpha^*_w$ respectively.

Hence, if $z^{X^H} < z^{X^*}$, in general we cannot use $\beta^H$ and $\alpha^H$ as bounds for the corresponding $\beta^*$ and $\alpha^*$ values. Even when $X^H = X^*$, $\beta^H$ and $\alpha^H$ values can respectively be used as upper and lower bounds for the corresponding $\beta^*$ and $\alpha^*$ values for variations in $w_j$ and $c$ in SS instances and in $c$ for KP instances, but not for variations in $p_j$ and $w_j$ values in KP instances.

5. **Empirical behaviour of $x^H$**

In this section we report the results of certain preliminary computations we carried out to test the empirical behavior of $x^H$. The computations were carried out with an aim to answer the following two questions.

1. How does the performance ratio of the greedy heuristic vary when a particular problem parameter is altered?
2. How often are the $\beta^W$ and $\alpha^W$ values reached in randomly generated problem instances?

The second question is interesting for binary knapsack problems since the empirical performance of the greedy heuristic is very different from its worst-case performance.

We used the KP instance of Example 1 to see how the performance ratio of the greedy heuristic varied when various problem parameters are changed. In particular we studied the variation of the performance ratio with changes in $c$, $p_3$, $w_3$, $p_6$, and $w_6$. (Recall that $x^H_3 = 1$ and $x^H_6 = 0$ for that instance). The performance ratio is the ratio of the objective function value of $z^{X^H}$ to $z^{*}$.

Due to our assumptions that none of the $w_j$ values can exceed $c$ and that $c = \sum_{j=1}^n w_j$, and the fact that the original greedy solution has to be feasible in the changed problem instance, most of the problem parameters can only vary within certain ranges. For example, the value of $c$ can vary between 241 and 554, that of $w_3$ between 0 and 32 and that of $w_6$ between 0 and 100. $p_3$ and $p_6$ can take on any non-negative value. Figure 5 shows the variation of $z^{x^H}$ with the various problem parameters. It is interesting to see that for this instance, the performance ratio of 0.5 is reached only in the case where $p_6$ increases to 443.

We used the SS instance $E = \{16, 13, 9, 6, 3\}$, $c = 27$ to see the variation of the performance ratio of the greedy heuristic varied when various problem parameters are
Figure 5: Variation of $\frac{x^H}{z^H}$ with problem parameters in KP instance

changed. Again due to our assumptions there are bounds on the values that $c$ and $w_j$'s can assume. In particular we studied the variation of the performance ratio with changes in $c$, $w_3$, and $w_2$. ($x_3^H = 1$ and $x_2^H = 0$ for this instance). Figure 5 shows the variation of $\frac{x^H}{z^H}$ with the various problem parameters.

Next we report computations to see how often the $\beta^W$ and $\alpha^W$ values are reached in randomly generated instances of KP and SS. Note that the worst-case performance ratio for both KP and SS is 0.5 under very weak assumptions (see, for example, Martello
and Toth [6]). So for these problems, $\beta^W$ and $\alpha^W$ values will be reached only if the performance ratios drop to 0.5.

In case of KP instances, it is easy to see that if $x_j^H = 0$ for any element $e_j$, then the performance ratio of $X^H$ would drop to 0.5 (i.e. $\beta^W_{p_j}$ would be reached) if $p_j$ was raised sufficiently. However we cannot obtain such general results for the other problem parameters. We generated six sets with 10 problem instances in each set for our empirical tests. The particulars of these instances are given in Table 1.

We observed that the performance ratios dropped sufficiently only for problem instances in the problem set KP-1. $\beta^W_{w_3}$ was reached in eight of the problem instances, and $\alpha^W_{w_2}$ in only one of the instances.

For SS instances, we do not even need to perform tests to answer this question. For these instances $Z^H$ and $Z^*$ are both bounded above by the value of $c$. Therefore $\beta^W$ or $\alpha^W$ values would be reached only if $z^{X_H} < 0.5c$. However, the empirical performance of the greedy heuristic for SS is far superior (see, for example, Ghosh [4], or Martello and Toth [6]), and hence $\beta^W$ or $\alpha^W$ values would rarely be reached in practice.
Table 1: Problem sets for testing $\beta^W$ and $\alpha^W$

| Problem Set | $|E|$ | Numbers drawn from | $c$ as a fraction of $\sum_j w_j$ |
|-------------|------|-------------------|---------------------------------|
| KP-1        | 10   | $\mathcal{U}[1,100]$ | 0.2                             |
| KP-2        | 10   | $\mathcal{U}[1,100]$ | 0.5                             |
| KP-3        | 10   | $\mathcal{U}[1,100]$ | 0.8                             |
| KP-4        | 50   | $\mathcal{U}[1,1000]$ | 0.2                             |
| KP-5        | 50   | $\mathcal{U}[1,1000]$ | 0.5                             |
| KP-6        | 50   | $\mathcal{U}[1,1000]$ | 0.8                             |

6. Discussion and Conclusion

In this paper we have studied the sensitivity analysis problem when the greedy heuristic is applied to two common binary knapsack problems — the 0-1 knapsack problem and the subset sum problem. The sensitivity analysis problem as proposed in Wagelmans [8] is $\mathcal{NP}$-hard for these problems, and so we concentrated on a related sensitivity analysis problem that we defined in the introductory section. We noted the interesting fact that even though the 0-1 knapsack problem and the subset sum problem were closely related, their sensitivity analysis problems were not. In particular, even though any algorithm that can solve instances of 0-1 knapsack problems can solve instances of subset sum problems, we cannot design an algorithm for sensitivity analysis of any one of these problems that can be easily modified to perform sensitivity analysis of the other.

The paper contains a characterization of the tolerance limits for the two knapsack problems. We illustrated these limits with the help of examples. We also illustrated the behavior of the objective function values of the optimal as well as the greedy solution when the data in the instance is perturbed. We also derived the cases in which the tolerance limits output by the sensitivity analysis of the greedy heuristic (our definition) can serve as bounds to the tolerance limits of the optimal solution.

The paper also contains some empirical tests to evaluate heuristic tolerance limits. One interesting observation in this section was that the tolerance limits proposed in Wagelmans [8] are very rarely reached in the case of binary knapsack problems. This is because the tolerance limits in Wagelmans [8] are based on the worst-case performances of heuristics, which cannot be seen as representative of the empirical performance for many heuristics for combinatorial problems.
References


