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Group morphology

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Abstract

In its original form, mathematical morphology is a theory of binary image transformations which are invariant under the group of Euclidean translations. This paper surveys and extends constructions of morphological operators which are invariant under a more general group $\mathbb{T}$, such as the motion group, the affine group, or the projective group. We will follow a two-step approach: first we construct morphological operators on the space $\mathscr{P}(\mathbb{T})$ of subsets of the group $\mathbb{T}$ itself; next we use these results to construct morphological operators on the original object space, i.e. the Boolean algebra $\mathscr{P}(E^n)$ in the case of binary images, or the lattice Fun $(E^n, \mathscr{T})$ in the case of grey-value functions $F : E^n \to \mathscr{T}$, where E equals $\mathbb{R}$ or $\mathbb{Z}$, and $\mathscr{T}$ is the grey-value set. $\mathbb{T}$-invariant dilations, erosions, openings and closings are defined and several representation theorems are presented. Examples and applications are discussed. © 2000 Pattern Recognition Society. Published by Elsevier Science Ltd. All rights reserved.

Keywords: Mathematical morphology; Image processing; Boolean algebra; Complete lattice; Minkowski operations; Symmetry group; Dilation; Erosion; Opening; Closing; Adjunction; Invariance; Representation theorems

1. Introduction

Mathematical morphology in its original form is a set-theoretical approach to image analysis [1,2]. It studies image transformations with a simple geometrical interpretation and their algebraic decomposition and synthesis in terms of elementary set operations. Such an algebraic decomposition enables fast and efficient implementations on digital computers, which explains the practical importance of such decompositions, see e.g. Ref. [3]. In order to reveal the structure of binary images, small subsets, called *structuring elements*, of various forms and sizes are translated over the image plane to perform shape extraction. In this way, one obtains image transformations which are invariant under translations. The basic 'object space' is the Boolean algebra of subsets of the image plane.

In practice, it may be necessary to relax the restriction of translation invariance. For example, some images have radial instead of translation symmetry [2, p.17], requiring a polar group structure, see Example 2.8 below. In this case the size of the structuring element is proportional to the distance from the origin. The appropriate generalization of Euclidean morphology with arbitrary *abelian* symmetry groups was worked out by Heijmans [4], see also Ref. [5]. In the case of grey-level images a lattice formulation is required, see Refs. [6–9]. Again one may introduce a symmetry group, and a complete characterization of morphological operators for the case that this group is abelian was obtained by Heijmans and Ronse [10,11].

This paper extends Euclidean morphology on $\mathbb{R}^n$ by including invariance under more general transformations using the following general set-up. Take an arbitrary set E and a group $\mathbb{T}$ of transformations acting *transitively* on E, meaning that for every pair of elements $x, y \in E$ there is a transformation $g \in \mathbb{T}$ mapping x to y. One says that E is a *homogeneous space* under $\mathbb{T}$. Then $\mathbb{T}$-invariant morphological operators on the space $\mathscr{P}(E)$ of subsets of E can be constructed [12–14]. A further extension concerns non-Boolean lattices, such as the space of grey-scale functions on E. The basic assumption made in this paper is that the lattice has a sup-generating family ℓ and

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a group $\mathbb{T}$ of automorphisms which acts *transitively* on ℓ, thus generalizing the work of Heijmans and Ronse [10,11] who considered the abelian case.

The motivation for this approach derives from computer vision, where an important question is how to take the projective geometry of the imaging process into account. In many situations one does not want to distinguish between rotated versions of the same object. This is, for example, the basic assumption made in integral geometry in order to derive a complete characterization (Hadwiger's Theorem) of functionals of compact, convex sets in $\mathbb{R}^n$ [15]. Another example occurs in *invariant pattern recognition*, where the goal is to recognize patterns irrespective of their orientation or location [16]. In *image understanding*, one wants to derive information about three-dimensional (3D) scenes from projections on a planar (2D) image screen. In this case it is natural to require invariance of image operations under the 3D camera rotations [17]. So one may require invariance under increasingly larger groups, such as the Euclidean motion group, the similarity group, the affine group or the projective group, which are all non-commutative groups. For general questions of invariance in computer vision, see, for example, Ref. [18].

The purpose of this paper is to describe the mathematical structure of group morphology. For practical applications special algorithms are required, which extend the basic translation-invariant operations supported by standard image processing packages. An in-depth discussion of such algorithmical and computational issues is beyond the scope of this paper; however, some pertinent remarks can be found in the example presented in Section 4.6.2 below.

The organization of this paper is as follows. In Section 2 we summarise Euclidean morphology together with some general lattice concepts, and present some material on group actions. Section 3 reviews the construction developed in Refs. [12–14] of morphological operators on Boolean lattices, which are appropriate for binary image processing. The starting point is a group $\mathbb{T}$ acting transitively on a set E. First, $\mathbb{T}$-invariant morphological operators are defined on the lattice $\mathscr{P}(\mathbb{T})$ of subsets of $\mathbb{T}$ by generalizing the Minkowski operations to non-commutative groups. Next morphological operators are constructed on the actual object space of interest $\mathscr{P}(E)$ by (i) mapping the subsets of E to subsets of $\mathbb{T}$, (ii) using the results for the lattice $\mathscr{P}(\mathbb{T})$, and (iii) projecting back to the original space $\mathscr{P}(E)$. Graphical illustrations are given for the case where $\mathbb{T}$ equals the Euclidean motion group M generated by translations and rotations. Section 4 deals with non-Boolean lattices, and as a special case we discuss $\mathbb{T}$-invariant morphological operators for grey-scale functions. The material in this section is new. Section 5 contains a summary and discussion.

2. Preliminaries

In this section we review Euclidean morphology and introduce some general concepts concerning complete lattices and group actions.

2.1. Euclidean morphology

Let E be the Euclidean space $\mathbb{R}^n$ or the discrete grid $\mathbb{Z}^n$. By $\mathscr{P}(E)$ we denote the set of all subsets of E ordered by set-inclusion. A binary image can be represented as a subset X of E. Now E is a commutative group under vector addition: we write $x + y$ for the sum of two vectors x and y, and $-x$ for the inverse of x. The following two algebraic operations are fundamental in mathematical morphology:

$$\textit{Minkowski addition: } X \oplus A = \{x + a : x \in X, a \in A\} = \bigcup_{a \in A} X_a = \bigcup_{x \in X} A_x,$$

$$\textit{Minkowski subtraction: } X \ominus A = \bigcup_{a \in A} X_{-a},$$

where $X_a = \{x + a : x \in X\}$ is the translate of the set X along the vector a.

In preparation for later developments we introduce here the operator $\tau_a : \mathscr{P}(E) \to \mathscr{P}(E)$ by $\tau_a(X) = X_a$, referred to as 'translation by a'. Clearly, $\tau_a \tau_{a'} = \tau_{a+a'}$, $\tau_a^{-1} = \tau_{-a}$. Hence the collection $\mathsf{T}: = \{\tau_a : a \in E\}$ also forms a group, called the *translation group*, which is 'isomorphic' (as a group) to E, for to each point a there corresponds precisely one translation $\tau_a \in \mathsf{T}$, i.e. the one which maps the origin to a. Because of this 1–1 correspondence, one usually ignores the distinction in Euclidean morphology.

Let the *reflected* or *symmetric* set of A be denoted by $\check{A} = \{-a : a \in A\}$. The transformations δ_A^{T} and $\varepsilon_A^{\mathsf{T}}$ defined by

$$\delta_A^{\mathsf{T}}(X) := X \oplus A = \{h \in E : (\check{A})_h \cap X \neq \emptyset\}, \tag{1}$$

$$\varepsilon_A^{\mathsf{T}}(X) := X \ominus A = \{h \in E : A_h \subseteq X\}, \tag{2}$$

are called *dilation* and *erosion* by the structuring element A, respectively. To distinguish these translation-invariant operations from later generalizations, we explicitly indicate the dependence on the Euclidean translation group T and refer to them as T-dilations and T-erosions.

There exists a *duality relation* with respect to set-complementation (X^c denotes the complement of the set X): $X \oplus A = (X^c \ominus \check{A})^c$, i.e. dilating an image by A gives the same result as eroding the background by $\check{A}$. To any mapping $\psi : \mathscr{P}(E) \to \mathscr{P}(E)$ we associate the (Boolean) *dual* mapping $\psi' : \mathscr{P}(E) \to \mathscr{P}(E)$ by

$$\psi'(X) = \{\psi(X^c)\}^c. \tag{3}$$

Remark 2.1. Matheron and Serra define the Minkowski subtraction of X by A as follows: $X \ominus A = \bigcap_{a \in A} X_a$. The advantage of this definition is that the duality relation does not involve a reflection of the structuring element. But it complicates the expression of adjunctions (see Section 2.2.2), which is a notion persisting in lattices without complementation.

Minkowski addition and subtraction have many standard algebraic properties [15]. Two important properties are distributivity w.r.t. union or intersection,

$$\left(\bigcup_{i\in I} X_i\right)\oplus A = \bigcup_{i\in I} X_i \oplus A, \quad \left(\bigcap_{i\in I} X_i\right)\ominus A = \bigcap_{i\in I} X_i \ominus A,$$

and translation invariance: $(X \oplus A)_h = X_h \oplus A$, $(X \ominus A)_h = X_h \ominus A$. Dilation and erosion are *increasing* mappings, i.e. mappings such that for all $X, Y \in \mathscr{P}(E), X \subseteq Y$ implies that $\psi(X) \subseteq \psi(Y)$.

Other important increasing transformations are the *opening* α_A^{T} and *closing* ϕ_A^{T} by a structuring element A:

$$\alpha_A^{\mathsf{T}}(X) := (X \ominus A) \oplus A = \bigcup \{A_h : h \in E, A_h \subseteq X\},$$

$$\phi_A^{\mathsf{T}}(X) := (X \oplus A) \ominus A = \bigcap \{(\check{A}^c)_h : h \in E, (\check{A}^c)_h \supseteq X\}.$$

The opening of X is the union of all the translates of the structuring element which are included in X. The closing of X by A is the complement of the opening of X^c by $\check{A}$.

2.2. Lattice concepts

Here we summarize the main concepts from lattice theory needed in this paper, cf. Refs. [6,7]. For a general introduction to lattice theory, see Birkhoff [19].

Definition 2.2. A *complete lattice* $(\mathscr{L}, \leqslant)$ is a partially ordered set $\mathscr{L}$ with order relation $\leqslant$, a supremum or join operation written $\vee$ and an infimum or meet operation written $\wedge$, such that every (finite or infinite) subset of $\mathscr{L}$ has a supremum (smallest upper bound) and an infimum (greatest lower bound). In particular there exist two universal bounds, the least element written $O_{\mathscr{L}}$ and the greatest element $I_{\mathscr{L}}$.

In the case of the power lattice $\mathscr{P}(E)$ of all subsets of a set E, the order relation is set-inclusion $\subseteq$, the supremum is the union $\bigcup$ of sets, the infimum is the intersection $\bigcap$ of sets, the least element is the empty set $\emptyset$ and the greatest element is the set E itself.

An *atom* is an element X of a lattice $\mathscr{L}$ such that for any $Y \in \mathscr{L}$, $O_{\mathscr{L}} \leqslant Y \leqslant X$ implies that $Y = O_{\mathscr{L}}$ or $Y = X$. A complete lattice $\mathscr{L}$ is called *atomic* if every element of $\mathscr{L}$ is the supremum of the atoms less than or equal to it. It is called *Boolean* if (i) it satisfies the distributivity laws $X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$ and $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$ for all $X, Y, Z \in \mathscr{L}$, and (ii) every element X has a unique complement X^c, defined by $X \vee X^c = I_{\mathscr{L}}$, $X \wedge X^c = O_{\mathscr{L}}$. The power lattice $\mathscr{P}(E)$ is an atomic complete Boolean lattice, and conversely any atomic complete Boolean lattice has this form.

2.2.1. Mappings

The composition of two mappings ψ_1 and ψ_2 on a complete lattice $\mathscr{L}$ is written $\psi_1\psi_2$, and instead of $\psi\psi$ we also write ψ^2. An *automorphism* of $\mathscr{L}$ is a bijection $\psi : \mathscr{L} \to \mathscr{L}$ such that for any $X, Y \in \mathscr{L}, X \leqslant Y$ if and only if $\psi(X) \leqslant \psi(Y)$. If ψ_1 and ψ_2 are operators on $\mathscr{L}$, we write $\psi_1 \leqslant \psi_2$ to denote that $\psi_1(X) \leqslant \psi_2(X)$ for all $X \in \mathscr{L}$.

Definition 2.3. A mapping $\psi : \mathscr{L} \to \mathscr{L}$ is called

(a) *idempotent*, if $\psi^2 = \psi$;
(b) *extensive*, if for every $X \in \mathscr{L}$, $\psi(X) \geqslant X$;
(c) *anti-extensive*, if for every $X \in \mathscr{L}$, $\psi(X) \leqslant X$;
(d) *increasing (isotone, order-preserving)*, if $X \leqslant Y$ implies that $\psi(X) \leqslant \psi(Y)$ for all $X, Y \in \mathscr{L}$;
(e) a *closing*, if it is increasing, extensive and idempotent;
(f) an *opening*, if it is increasing, anti-extensive and idempotent.

Definition 2.4. Let $\mathscr{L}$ and $\tilde{\mathscr{L}}$ be complete lattices. A mapping $\psi : \mathscr{L} \to \tilde{\mathscr{L}}$ is called

(a) a *dilation*, if $\psi(\vee_{i\in I} X_i) = \vee_{i\in I}\psi(X_i)$;
(b) an *erosion*, if $\psi(\wedge_{i\in I} X_i) = \wedge_{i\in I}\psi(X_i)$.

When $\mathbb{T}$ is an automorphism group of two lattices $\mathscr{L}$ and $\tilde{\mathscr{L}}$, a mapping $\psi : \mathscr{L} \to \tilde{\mathscr{L}}$ is called $\mathbb{T}$-*invariant* or a $\mathbb{T}$-*mapping* if it commutes with all $\tau \in \mathbb{T}$, i.e., if $\psi(\tau(X)) = \tau(\psi(X))$ for all $X \in \mathscr{L}, \tau \in \mathbb{T}$. Accordingly, one speaks of $\mathbb{T}$-dilations, $\mathbb{T}$-erosions, etc. If no invariance under a group is required, one may set $\mathbb{T} = \{\mathsf{id}_{\mathscr{L}}\}$, where $\mathsf{id}_{\mathscr{L}}$ is the identity operator on $\mathscr{L}$.

2.2.2. Adjunctions

Definition 2.5. Let $\varepsilon : \mathscr{L} \to \tilde{\mathscr{L}}$ and $\delta : \tilde{\mathscr{L}} \to \mathscr{L}$ be two mappings, where $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are complete lattices. Then the pair (ε, δ) is called an *adjunction* between $\mathscr{L}$ and $\tilde{\mathscr{L}}$, if for every $X \in \tilde{\mathscr{L}}$ and $Y \in \mathscr{L}$, the following equivalence holds: $\delta(X) \leqslant Y \Leftrightarrow X \leqslant \varepsilon(Y)$. If $\tilde{\mathscr{L}}$ coincides with $\mathscr{L}$ we speak of an *adjunction* on $\mathscr{L}$.

It has been shown [10,11,20] that in an adjunction (ε, δ), ε is an erosion and δ a dilation. Also, for every dilation $\delta : \tilde{\mathscr{L}} \to \mathscr{L}$ there is a unique erosion $\varepsilon : \mathscr{L} \to \tilde{\mathscr{L}}$ such that (ε, δ) is an adjunction between $\mathscr{L}$ and $\tilde{\mathscr{L}}$; ε is given by $\varepsilon(Y) = \vee\{X \in \tilde{\mathscr{L}} : \delta(X) \leqslant Y\}$, and is called the *upper adjoint* of δ. Similarly, for every erosion $\varepsilon : \mathscr{L} \to \tilde{\mathscr{L}}$

there is a unique dilation $\delta : \tilde{\mathscr{L}} \to \mathscr{L}$ such that (ε, δ) is an adjunction between $\mathscr{L}$ and $\tilde{\mathscr{L}}$; δ is given by $\delta(X) = \wedge\{Y \in \mathscr{L} : X \leqslant \varepsilon(Y)\}$, and is called the *lower adjoint* of ε. Finally, for any adjunction (ε, δ), the mapping $\delta\varepsilon$ is an opening on $\mathscr{L}$ and $\varepsilon\delta$ is a closing on $\tilde{\mathscr{L}}$. In the case that $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are identical, one sometimes refers to such openings and closings as *morphological* or *adjunctional* [7].

2.2.3. Sup-generating families

Definition 2.6. A subset ℓ of a lattice $\mathscr{L}$ is called *sup-generating*[1] if every element of $\mathscr{L}$ can be written as a supremum of elements of ℓ.

Let $\mathscr{L}$ be a lattice with sup-generating subset ℓ. For every $X \in \mathscr{L}$, let $\ell(X) = \{x \in \ell : x \leqslant X\}$. The following properties hold [7,10,11]:

$$X = \bigvee \ell(X), \tag{4}$$

$$\ell\left(\bigwedge_{j\in J} X_j\right) = \bigcap_{j\in J} \ell(X_j), \tag{5}$$

$$\ell\left(\bigvee_{j\in J} X_j\right) \geqslant \bigcup_{j\in J} \ell(X_j), \tag{6}$$

$$\bigvee\left(\bigcup_{j\in J} \ell(X_j)\right) = \bigvee_{j\in J} (X_j), \tag{7}$$

Note also that the operators $\ell : X \to \ell(X)$ and $\vee : G \mapsto \vee G$ (i) are increasing, and (ii) form an adjunction between $\mathscr{L}$ and $\mathscr{P}(\ell)$:

$$\bigvee G \leqslant X \Leftrightarrow G \subseteq \ell(X).$$

This equation, together with Eq. (4), also implies the equivalence

$$X \leqslant Y \Leftrightarrow \ell(X) \subseteq \ell(Y).$$

Atoms of a lattice $\mathscr{L}$ are always members of a sup-generating subset. $\mathscr{L}$ is atomic if the set of its atoms is sup-generating. For example, given a set E, the set of singletons is sup-generating in the lattice $\mathscr{P}(E)$.

2.3. Group actions

Let E be a non-empty set, $\mathbb{T}$ a transformation group on E. Each element $g \in \mathbb{T}$ is a mapping $g : E \to E$, satisfying (i) $gh(x) = g(h(x))$, and (ii) $e(x) = x$, where e is the unit element of $\mathbb{T}$, and gh denotes the product of two group elements g and h. Instead of $g(x)$ we will usually write gx. We say that $\mathbb{T}$ is a *group action* on E [21,22]. $\mathbb{T}$ is called *transitive on* E if for each $x, y \in E$ there is a $g \in \mathbb{T}$ such that $gx = y$, and *simply transitive* when this element g is unique. A *homogeneous space* is a pair $(\mathbb{T}, E)$ where $\mathbb{T}$ is a group acting transitively on E. Any transitive abelian group $\mathbb{T}$ is simply transitive. The *stabilizer* of $X \in E$ is the subgroup $\mathbb{T}_x := \{g \in \mathbb{T} : gx = x\}$. Let ω be an arbitrary but fixed point of E, henceforth called the *origin*. The stabilizer $\mathbb{T}_\omega$ will be denoted by Σ from now on:

$$\Sigma := \mathbb{T}_\omega = \{g \in \mathbb{T} : g\omega = \omega\}.$$

The set $g_x\Sigma := \{g_x s : s \in \Sigma\}$ of group elements which map ω to a given point x is called a *left coset*. Here g_x is a representative (an arbitrary element) of this coset.

In the following we present some examples of homogeneous spaces. In each case $\mathbb{T}$ denotes the group and E the corresponding set.

Example 2.7 (*Euclidean group*). E = Euclidean space $\mathbb{R}^n$, $\mathbb{T}$ = the Euclidean translation group T. T is abelian, therefore it can be identified with E [14]. Elements of T can be parameterized by vectors $h \in \mathbb{R}^n$, with τ_h the translation over the vector $h : \tau_h x = x + h, x \in \mathbb{R}^n$.

Example 2.8 (*Polar group*). $E = \mathbb{R}^2 \setminus \{0\}$, $\mathbb{T}$ = the abelian group generated by rotations and scalar multiplication w.r.t. the origin. In this case points of E can be given in polar coordinates $(r, \theta), r > 0, 0 \leqslant \theta < 2\pi$. Again $\mathbb{T}$ can be identified with E and the group multiplication is $(r_1, \theta_1)*(r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$, cf. Ref. [5].

Example 2.9 (*Spherical group*). E = the sphere S^2, $\mathbb{T}$ = the non-abelian group $SO(3)$ of rotations in 3-space (see Ref. [23]). The subgroup leaving a point p fixed is the set of all rotations around an axis through p and the center of the sphere.

Example 2.10 (*Translation-rotation group*). E = Euclidean space $\mathbb{R}^2$, $\mathbb{T}$ = the Euclidean motion group M (proper Euclidean group, group of rigid motions) [24]. The subgroup leaving a point p fixed is the set of all rotations around p. M is not abelian. The collection of translations forms a subgroup, the translation group T. The stabilizer Σ equals the group R of rotations around the origin, which is abelian. A group element $\gamma_{h,\phi}$, $h \in \mathbb{R}^2$, $\phi \in [0, 2\pi)$, acts upon a point $x \in \mathbb{R}^2$ as follows:

$$\gamma_{h,\phi}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

Let τ_h denote the unique (Euclidean) translation by h (cf. Example 2.7), and let r_ϕ be the rotation around the origin over an angle ϕ. It is easy to verify that $\gamma_{h,\phi} = \tau_h r_\phi$. From the relations

$$\tau_h \tau_{h'} = \tau_{h+h'}, \quad r_\phi r_{\phi'} = r_{\phi+\phi'}, \quad r_\phi \tau_h = \tau_{r_\phi h} r_\phi, \tag{8}$$

[1] The dual concept is that of an inf-generating subset [7].

it is clear that we can represent any element of the motion group as the product of a single rotation around the origin followed by a single translation. The last equality in Eq. (8) expresses the fact that the motion group M is the *semi-direct product* of T and R [21,22].

We now introduce a graphical representation of the group elements. Define a *pointer* p to be a pair $(x, \mathbf{v})$, where x is a point in the plane and $\mathbf{v}$ a unit vector attached to x. We call x the *base-point* of p. Define the *base-pointer* b to be the pair $(\omega, \mathbf{e}_1)$, where $\mathbf{e}_1 = (1{,}0)$, i.e., b is a horizontal unit vector attached to the origin ω. Any pointer p represents a unique element of M: if $p = (x, \mathbf{v})$, where $\mathbf{v} = (\cos\phi, \sin\phi)$, then this element is precisely the motion $\gamma_{h,\phi}$ which maps b to p. The 2D rotation group R is represented by the set of unit vectors attached to the origin, and T is represented by the collection of horizontal unit vectors attached to points of $\mathbb{R}^2$. In the discrete case we will use a hexagonal grid, and M will denote the subgroup of all motions which leave the grid invariant. Also, T now becomes a discrete set of translations, and R is a finite group with six elements: rotations around the origin over $k \cdot 60$ deg, $k = 1, 2, \dots, 6$. The reader may refer to Fig. 1, where subsets of the grid are indicated by dots and subsets of M by dots with one or more unit vectors attached to them. Notice also that the coset $\tau_y\Sigma := \{\tau_y r : r \in R\}$ of all motions carrying the origin to a given point y is represented on the hexagonal grid by the six unit vectors attached to y.

Example 2.11 (*Affine group*). E = Euclidean space $\mathbb{R}^n (n \geqslant 2)$, $\mathbb{T}$ = the affine group. The subgroup Σ leaving the origin fixed is the linear group $GL(n, \mathbb{R})$, whose elements are $n \times n$ invertible matrices a. A group element acts upon a point $x \in E$ as follows:

$$\gamma_{h,a} x = ax + h, \quad a \in GL(n, \mathbb{R}), h \in \mathbb{R}^n.$$

Let $\rho_a : x \to ax$ denote the linear transformation by the matrix a. Then $\gamma_{h,a} = \tau_h \rho_a$. The relation $\rho_a \tau_h \rho_a^{-1} = \tau_{\rho_a h}$ again expresses the fact that the affine group is the semi-direct product of T and $GL(n, \mathbb{R})$ [21,22].

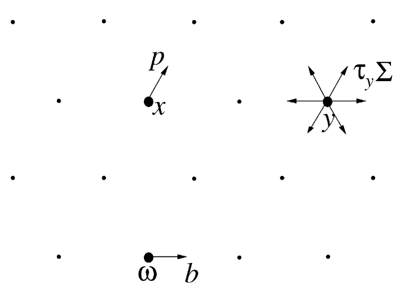


Fig. 1. Representation of elements of the Euclidean motion group on the hexagonal grid. b: base-pointer. p: pointer with base-point x. $\tau_y\Sigma$: the collection of group elements which map the origin ω to y. Each pointer represents a unique group element.

3. Group morphology for Boolean lattices

This section reviews the construction developed in Refs. [12–14] of morphological operators on Boolean lattices, appropriate for binary image processing, with a transitive group action. First we consider in Section 3.1 the case that E is a homogeneous space under a group $\mathbb{T}$ acting *simply transitively* on E. In this case there is a bijection between E and $\mathbb{T}$: let ω (the 'origin') be an arbitrary point of E, and associate to any $x \in E$ the unique element of $\mathbb{T}$ which maps ω to x. Hence in the simply transitive case is sufficient to study the power lattice $\mathscr{P}(\mathbb{T})$, i.e. the set of subsets of $\mathbb{T}$ ordered by set-inclusion.

The second case is that of a group $\mathbb{T}$ acting *transitively* on E. The object space of interest is again the Boolean lattice $\mathscr{P}(E)$ of all subsets of E. The general strategy is to make use of the results for the simply transitive case, by 'lifting' subsets of E to subsets of $\mathbb{T}$, applying morphological operators on $\mathscr{P}(\mathbb{T})$, and then 'projecting' the results back to the original space E.

The constructed operators are illustrated for the Euclidean motion group M acting on the hexagonal grid, using the representation by pointers introduced in Example 2.10.

3.1. Minkowski operators on groups

On any group $\mathbb{T}$ one can define generalizations of the Minkowski operations [12,14]. We denote elements of $\mathbb{T}$ by g, h, k, etc., and subsets of $\mathbb{T}$ by capitals G, H, K. The product of two group elements g and h is written gh, the inverse of g is denoted by g^{-1} and e is the unit element of $\mathbb{T}$. For $g \in \mathbb{T}$, $H \subseteq \mathbb{T}$, let

$$gH := \{gh : h \in H\}, \quad Hg := \{hg : h \in H\},$$

be the left and right products of a group element with a subset of $\mathbb{T}$. For later use we also define the *inverted* set of a subset G by $G^{-1} = \{g^{-1} : g \in G\}$. Note that *inversion* reduces to *reflection* for subsets of the Euclidean translation group (see Section 2.1).

Definition 3.1. A mapping $\psi = \mathscr{P}(\mathbb{T}) \to \mathscr{P}(\mathbb{T})$ is called *left $\mathbb{T}$-invariant* (or *left-invariant*) when, for all $g \in \mathbb{T}$, $\psi(gG) = g\psi(G), \forall G \in \mathscr{P}(\mathbb{T})$. Similarly, a mapping $\psi : \mathscr{P}(\mathbb{T}) \to \mathscr{P}(\mathbb{T})$ is called *right $\mathbb{T}$-invariant* (or *right-invariant*) when, for all $g \in \mathbb{T}$, $\psi(Gg) = (\psi(G))g, \forall G \in \mathscr{P}(\mathbb{T})$.

Recall that by definition a *dilation* (*erosion*) on $\mathscr{P}(\mathbb{T})$ is a mapping commuting with unions (intersections).

Proposition 3.2. *Let H (the structuring element) be a fixed subset of $\mathbb{T}$. Define*

$$\delta_H^\lambda(G) := G \overset{\mathbb{T}}{\oplus} H := \bigcup_{h\in H} Gh = \bigcup_{g\in G} gH,$$

$$\varepsilon_H^\lambda(G) := G \overset{\lambda}{\ominus} H := \bigcap_{h\in H} Gh^{-1} = \bigcup_{g\in G^c} g\hat{H},$$

where $\hat{H}$ is defined by $\hat{H} = (\check{H})^c$. Then the mapping δ_H^λ defines a left $\mathbb{T}$-invariant dilation on the lattice $\mathscr{P}(\mathbb{T})$, with adjoint erosion ε_H^λ. All left $\mathbb{T}$-invariant adjunctions on $\mathscr{P}(\mathbb{T})$ are of this form.

Duality by complementation is expressed by the formula $(G \overset{\mathbb{T}}{\oplus} H)^c = G^c \overset{\lambda}{\ominus} H^{-1}$.

It is easy to show the following equalities, which provide a geometrical interpretation:

$$G \overset{\mathbb{T}}{\oplus} H = \{k \in \mathbb{T} : (k\check{H}) \cap G \neq \emptyset\} = \{k \in \mathbb{T} : (\check{G}k) \cap H \neq \emptyset\},$$

$$G \overset{\lambda}{\ominus} H = \{g \in \mathbb{T} : gH \subseteq G\}.$$

Remark 3.3. Because of the non-commutativity of the set product $G \overset{\mathbb{T}}{\oplus} H$, one may also introduce a *right-invariant* dilation δ_H^ρ and erosion ε_H^ρ by

$$\delta_H^\rho(G) := H \overset{\mathbb{T}}{\oplus} G := \bigcup_{h\in H} hG = \bigcup_{g\in G} Hg,$$

$$\varepsilon_H^\rho(G) := G \overset{\rho}{\ominus} H := \bigcap_{h\in H} h^{-1}G.$$

There is a connection to the theory of residuated lattices and ordered semigroups [25], which is explained in more detail in Ref. [14]. Only left-invariant dilations and erosions will be used in the remainder of this paper.

From the properties of adjunctions (see Section 2.2) we know that we can build openings and closings by forming products of a dilation and an erosion. In particular, the mapping $\alpha_H^\lambda := \delta_H^\lambda \varepsilon_H^\lambda$ is an opening and the mapping $\phi_H^\lambda := \varepsilon_H^\lambda \delta_H^\lambda$ is a closing. Both mappings are left-invariant. As in the Euclidean case, there is a simple geometrical interpretation of these operations:

$$\alpha_H^\lambda(G) := (G \overset{\lambda}{\ominus} H) \overset{\mathbb{T}}{\oplus} H = \bigcup \{gH : g \in \mathbb{T}, gH \subseteq G\},$$

$$\phi_H^\lambda(G) := (G \overset{\mathbb{T}}{\oplus} H) \overset{\lambda}{\ominus} H = \bigcap \{g\hat{H} : g \in \mathbb{T}, g\hat{H} \supseteq G\},$$

In Fig. 2, we give an example of elementary $\mathbb{T}$-operators for the case of the motion group ($\mathbb{T} = \mathsf{M}$).

A special role is played by the dilation $\tilde{\delta}_\Sigma^\lambda$ and erosion $\tilde{\varepsilon}_\Sigma^\lambda$ by the subgroup Σ:

$$\tilde{\delta}_\Sigma^\lambda(G) = G \overset{\mathbb{T}}{\oplus} \Sigma, \quad \tilde{\varepsilon}_\Sigma^\lambda(G) = G \overset{\lambda}{\ominus} \Sigma.$$

The following lemma was proved in Ref [13].

Lemma 3.4. *The adjunction* $(\tilde{\varepsilon}_\Sigma^\lambda, \tilde{\delta}_\Sigma^\lambda)$ *satisfies* (*a*) $\tilde{\varepsilon}_\Sigma^\lambda = \tilde{\varepsilon}_\Sigma^\lambda \tilde{\varepsilon}_\Sigma^\lambda = \tilde{\delta}_\Sigma^\lambda \tilde{\varepsilon}_\Sigma^\lambda$, (*b*) $\tilde{\delta}_\Sigma^\lambda = \tilde{\delta}_\Sigma^\lambda \tilde{\delta}_\Sigma^\lambda = \tilde{\varepsilon}_\Sigma^\lambda \tilde{\delta}_\Sigma^\lambda$.

This lemma says that $\tilde{\varepsilon}_\Sigma^\lambda$ is not only an erosion but also an opening; and $\tilde{\delta}_\Sigma^\lambda$ is not only a dilation but also a closing. The effect of the closing $\tilde{\delta}_\Sigma^\lambda$ on a subset G of Σ is to make G 'Σ-*closed*', i.e. invariant under right multiplication by Σ. For the case of the motion group, where $\Sigma = \mathsf{R}$

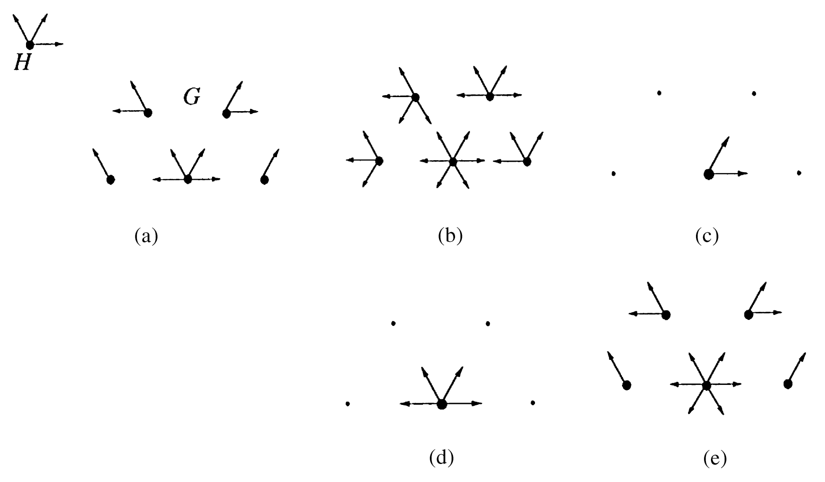


Fig. 2. Morphological operations on the motion group M: (a) set G, structuring element H; (b) dilation of G by H; (c) erosion of G by H; (d) opening of G by H; (e) closing of G by H.

(cf. Example 2.10), any pointer $\tau_x r$ with $r \in \mathsf{R}$, is extended by $\tilde{\delta}_\Sigma^\lambda$ to the set of pointers $\tau_x \Sigma$, see Fig. 3. Similarly, the opening $\tilde{\varepsilon}_\Sigma^\lambda$ extracts all the cosets (i.e., subsets of the form $\tau_x \Sigma$) from a subset G of $\mathbb{T}$.

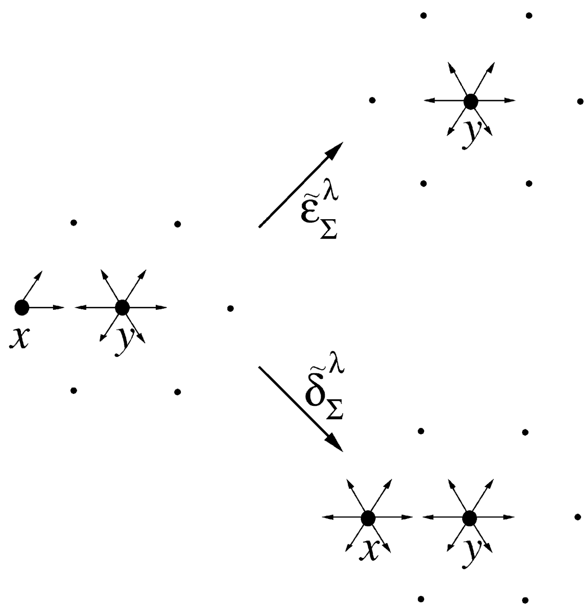


Fig. 3. Action of the erosion and dilation by Σ on a subset of M.

3.2. Boolean lattices with a transitive group action

This subsection summarizes the results obtained in Refs. [13,14] for the Boolean lattice $\mathscr{P}(E)$, with $\mathbb{T}$ acting transitively on E, and presents an application to invariant feature extraction.

3.2.1. Lift and projection operators

Definition 3.5. Let the 'origin' ω be an arbitrary point of E. The *lift* $\vartheta : \mathscr{P}(E) \to \mathscr{P}(\mathbb{T})$ and *projection* π: $\mathscr{P}(\mathbb{T}) \to \mathscr{P}(E)$ are defined by

$$\vartheta(X) = \{g \in \mathbb{T} : g\omega \in X\}, \quad X \in \mathscr{P}(E), \tag{9}$$

$$\pi(G) = \{g\omega : g \in G\}, \quad G \in \mathscr{P}(\mathbb{T}). \tag{10}$$

The mapping ϑ associates to each subset X all group elements which map the origin ω to an element of X. The mapping π associates to each subset G of $\mathbb{T}$ the collection of all points $g\omega$ where g ranges over G. In the graphical representation, π maps G to the set of base-points of the pointers in G (Fig. 4(a)). Conversely, ϑ maps a subset X of E to the set of pointers in $\mathbb{T}$ which have their base-points in X (Fig. 4(b)).

Definition 3.6. Let π be the projection (10) and $\tilde{\varepsilon}_\Sigma^\lambda$ the erosion $\tilde{\varepsilon}_\Sigma^\lambda(G) = G \overset{\lambda}{\ominus} \Sigma$. Then $\pi_\Sigma : \mathscr{P}(\mathbb{T}) \to \mathscr{P}(E)$ is the *modified projection* defined by $\pi_\Sigma = \pi \tilde{\varepsilon}_\Sigma^\lambda$.

The projection π_Σ first extracts the cosets $\tau_x \Sigma$ and then carries out the projection π (Fig. 4(c)).

The operators ϑ, π and π_Σ have several useful properties [14]. The most important ones are given in the next proposition (cf. Fig. 5). Recall that a mapping $\psi : \mathscr{P}(E) \to \mathscr{P}(E)$ is called $\mathbb{T}$-*invariant* or a $\mathbb{T}$-*mapping* if $\psi(gX) = g\psi(X)$ for all $X \in \mathscr{P}(E)$, $g \in \mathbb{T}$.

Proposition 3.7.

(a) π, ϑ, π_Σ *are increasing and* $\mathbb{T}$-*invariant*;
(b) ϑ *and* π *commute with unions*, ϑ *and* π_Σ *commute with intersections*;
(c) $\pi\vartheta = \mathsf{id}_{\mathscr{P}(E)}$; $\pi_\Sigma \vartheta = \mathsf{id}_{\mathscr{P}(E)}$;
(d) $X \subseteq Y \Leftrightarrow \vartheta(X) \subseteq \vartheta(Y)$;
(e) (ϑ, π) *forms an adjunction between* $\mathscr{P}(E)$ *and* $\mathscr{P}(\mathbb{T})$;
(f) (π_Σ, ϑ) *forms an adjunction between* $\mathscr{P}(\mathbb{T})$ *and* $\mathscr{P}(E)$.

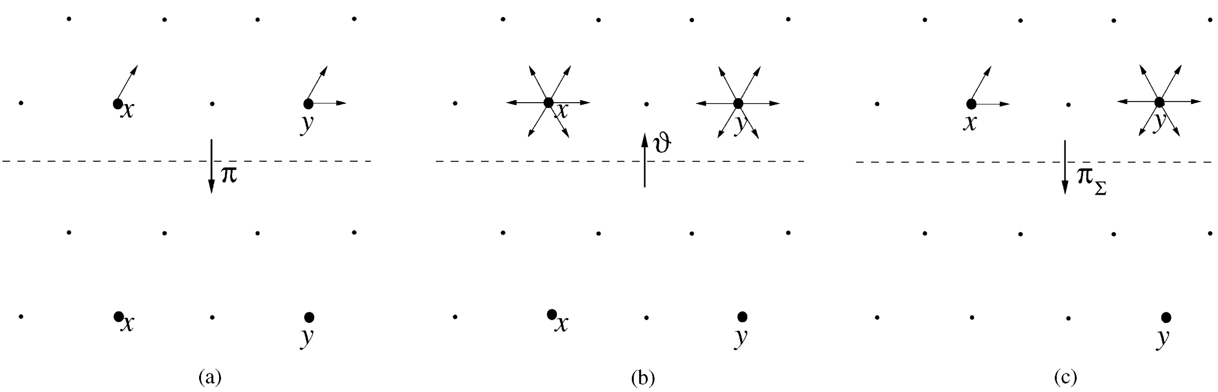


Fig. 4. (a) Action of π on a subset of M. (b) Action of ϑ on a subset of E. (c) action of π_Σ on a subset of M.

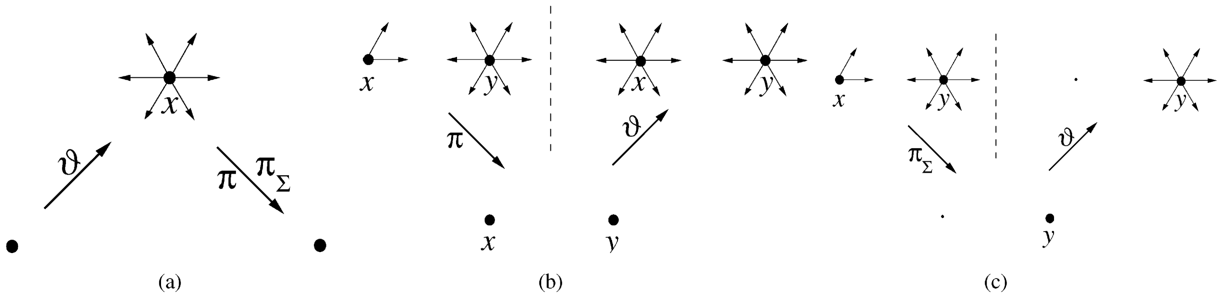


Fig. 5. Illustration of properties of π, ϑ, π_Σ: (a) $\pi\vartheta = \pi_\Sigma\varepsilon = \mathrm{id}_{\mathscr{P}(E)}$. (b) $\vartheta\pi = \tilde{\delta}^{\lambda}_{\Sigma}$. (c) $\vartheta\pi_\Sigma = \tilde{\varepsilon}^{\lambda}_{\Sigma}$.

3.2.2. Construction of $\mathbb{T}$-invariant operators

$\mathbb{T}$-invariant operators can be constructed as follows [13,14]. Given a mapping ψ on $\mathscr{P}(E)$ we 'lift' it to a mapping $\tilde{\psi}$ on $\mathscr{P}(\mathbb{T})$. Then we apply the results of Section 3.1 on $\mathscr{P}(\mathbb{T})$ and finally 'project' the results back to $\mathscr{P}(E)$, see Fig. 6 (left diagram).

Remark 3.8. A first idea to generalize the Minkowski operations is to take a subset G of the group $\mathbb{T}$ (the 'structuring element') and let it act on a subset X of E by defining $GX := \bigcup_{g\in G} gX$. This was applied, for example, in Ref. [26] for the case of the affine group. However, this mapping is in general not $\mathbb{T}$-invariant. For, let $g_0 \in \mathbb{T}$ be arbitrary. Then $G(g_0X) = \bigcup_{g\in G} gg_0X$. If we could interchange g_0 with g, the result would be $\bigcup_{g\in G} g_0gX = g_0GX$, implying group invariance. But this interchange is not allowed if $\mathbb{T}$ is a non-commutative group such as the affine group.

Definition 3.9 Let $\mathbb{T}$ be a group acting on E, with Σ the stabilizer of the origin ω in E. A subset X of E is called Σ-*invariant* if $X = \bar{X}$, where $\bar{X} := \Sigma X = \bigcup_{s\in\Sigma} sX$ is the Σ-*invariant extension* of X.

Proposition 3.10 (Representation of dilations and erosions). *The pair* (ε, δ) *is a* $\mathbb{T}$-*adjunction on* $\mathscr{P}(E)$ *if and only if, for some* $Y \in \mathscr{P}(E)$, $(\varepsilon, \delta) = (\varepsilon^{\mathbb{T}}_Y, \delta^{\mathbb{T}}_Y)$, *where*

$$\delta^{\mathbb{T}}_Y(X) := \pi[\vartheta(X) \overset{\mathbb{T}}{\oplus} \vartheta(Y)] = \bigcup_{g\in\vartheta(X)} gY,$$

$$\varepsilon^{\mathbb{T}}_Y(X) := \pi_\Sigma[\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)] = \bigcap_{g\in\vartheta(X^c)} g\hat{Y}^*,$$

with $\hat{Y}^* = (\pi(\check{\vartheta}(Y)))^c$. *In particular*, $(e^{\mathbb{T}}_Y, \delta^{\mathbb{T}}_Y)$ *is invariant under the substitution* $Y \to \bar{Y}$.

This proposition says that any $\mathbb{T}$-dilation on $\mathscr{P}(E)$ can be reduced to a dilation $\delta^{\mathbb{T}}_Y$ involving a Σ-*invariant* structuring element Y; a similar statement holds for $\mathbb{T}$-erosions. A graphical illustration for the motion group is given in Fig. 7, where the underlined point in the structuring element denotes the origin.

Next we consider openings and closings.

Definition 3.11. The *structural* $\mathbb{T}$-opening $\alpha^{\mathbb{T}}_Y(X)$ and $\mathbb{T}$-closing $\phi^{\mathbb{T}}_Y(X)$ with structuring element $Y \subseteq E$ are defined by

$$\alpha^{\mathbb{T}}_Y(X) = \bigcup\{gY : g \in \mathbb{T}, gY \subseteq X\}, \tag{11}$$

$$\phi^{\mathbb{T}}_Y(X) = \bigcap\{gY : g \in \mathbb{T}, gY \supseteq X\}. \tag{12}$$

In words, $\alpha^{\mathbb{T}}_Y(X)$ is the union of all translates gY which are included in X.

An important consequence of the above proposition is that the adjunctional opening $\delta^{\mathbb{T}}_Y\varepsilon^{\mathbb{T}}_Y$ and closing $\varepsilon^{\mathbb{T}}_Y\delta^{\mathbb{T}}_Y$ are invariant under the substitution $Y \to \bar{Y}$ as well.

Example 3.12. Let X be a union of line segments of varying sizes in the plane and Y a line segment of size L with center at the origin. Let the acting group $\mathbb{T}$ equal the translation-rotation group M. Then $\alpha^{\mathsf{M}}_Y(X)$ consists of the union of all segments in X of size L or larger, but $\delta^{M}_Y\varepsilon^{M}_Y(X) = \alpha^{M}_{\bar{Y}}(X) = \emptyset$, since $\bar{Y} = RY$ is a disc of radius $L/2$, which does not fit anywhere in X, cf. Fig. 8.

So in general we cannot build the opening $\alpha^{\mathbb{T}}_Y$ from a $\mathbb{T}$-erosion $\varepsilon^{\mathbb{T}}_Y$ on $\mathscr{P}(E)$ followed by a $\mathbb{T}$-dilation $\delta^{\mathbb{T}}_Y$ on

$$\begin{array}{ccc} \mathcal{P}(\mathbb{T}) & \xrightarrow{\tilde{\psi}} & \mathcal{P}(\mathbb{T}) \\ \vartheta \uparrow & & \downarrow \pi_\Sigma \mid \pi \\ \mathcal{P}(E) & \xrightarrow{\psi} & \mathcal{P}(E) \end{array} \qquad \begin{array}{ccc} \mathcal{P}(\mathbb{T}) & \xrightarrow{\Psi} & \mathcal{P}(\mathbb{T}) \\ \vartheta \uparrow & & \downarrow \pi \\ \mathcal{P}(\ell) & \xrightarrow{\tilde{\psi}} & \mathcal{P}(\ell) \\ \ell \uparrow & & \downarrow \bigvee \\ \mathcal{L} & \xrightarrow{\psi} & \mathcal{L} \end{array}$$

Fig. 6. Left: relations between mappings on $\mathscr{P}(E)$ and $\mathscr{P}(\mathbb{T})$. Right: relations between mappings on $\mathscr{L}$, $\mathscr{P}(\ell)$ and $\mathscr{P}(\mathbb{T})$.

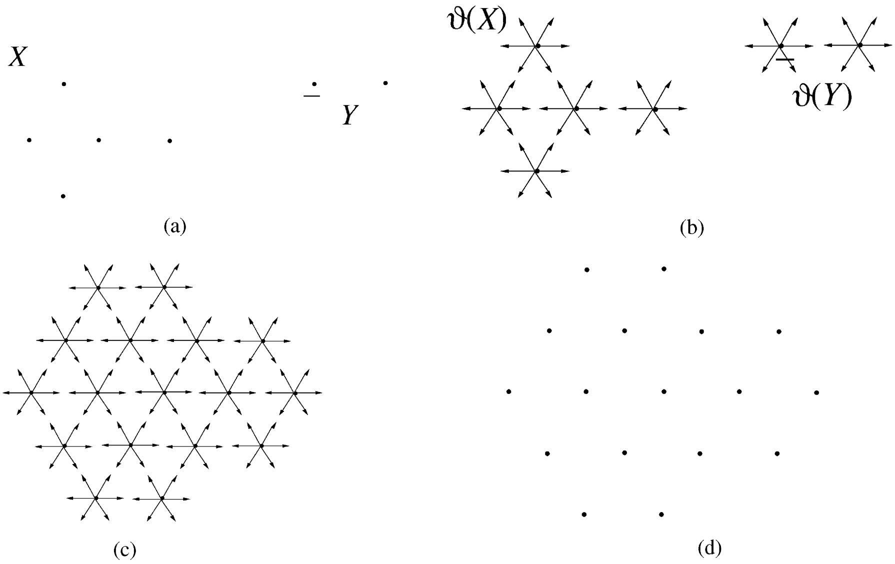


Fig. 7. Construction of an M-invariant dilation. (a) set X, structuring element Y; (b) sets $\vartheta(X)$ and $\vartheta(Y)$ of pointers; (c) set product $\vartheta(X)\overset{\mathsf{M}}{\oplus}\vartheta(Y)$; (d) corresponding set $\pi[\vartheta(X)\overset{\mathsf{M}}{\oplus}\vartheta(Y)]$ of base points.

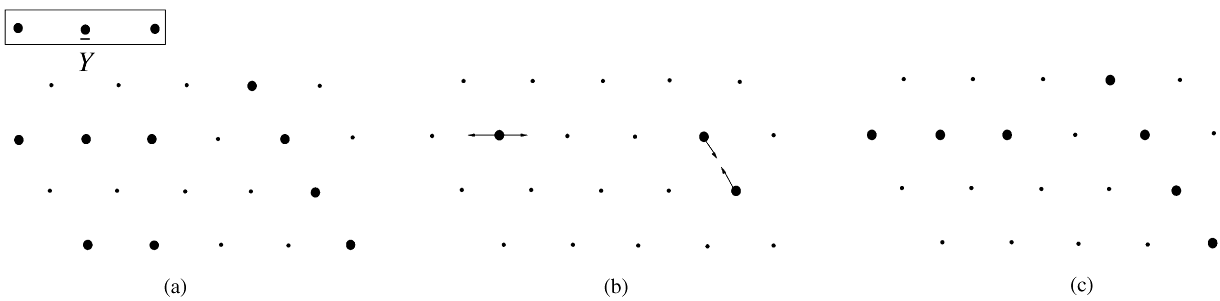


Fig. 8. (a) X: a subset of the hexagonal grid consisting of 'line segments'; set within the rectangle: structuring element Y. (b) Erosion $e_Y^{\uparrow}(X)$ by Y. (c) Dilation $\delta_Y^{\downarrow}$ applied to the result in (b). The opening $\delta_Y^{\mathsf{M}}\varepsilon_Y^{\mathsf{M}}(X) = \alpha_Y^{\mathsf{M}}(X)$ is empty.

$\mathscr{P}(E)$, in contrast to the classical case of the translation group ($\mathbb{T} = \mathsf{T}$), cf. Section 2.1. However, if erosions and dilations between the *distinct* lattices $\mathscr{P}(E)$ and $\mathscr{P}(\mathbb{T})$ are allowed, openings and closings can be decomposed into products of erosion and dilation (this is in agreement with a general result in Ref. [25, Theorem 2.7], see also Ref. [7, Section 6.3]).

Proposition 3.13 (Decomposition of structural $\mathbb{T}$-openings). *The structural $\mathbb{T}$-opening defined by Eq. (11) is the projection of the opening $\tilde{\alpha}_{\vartheta(Y)} = \tilde{\delta}_{\vartheta(Y)}\tilde{\varepsilon}_{\vartheta(Y)}$, with $(\tilde{\varepsilon}_{\vartheta(Y)}, \tilde{\delta}_{\vartheta(Y)})$ the left-invariant adjunction on $\mathscr{P}(\mathbb{T})$ with structuring element $\vartheta(Y)$, i.e.*

$$\alpha_Y^{\mathbb{T}}(X) = (\pi\tilde{\delta}_{\vartheta(Y)}\,\tilde{\varepsilon}_{\vartheta(Y)}\,\vartheta)(X) = \pi((\vartheta(X)\overset{\lambda}{\ominus}\vartheta(Y))\overset{\mathbb{T}}{\oplus}\vartheta(Y)).$$

So, $\alpha_Y^{\mathbb{T}}$ is the product of a $\mathbb{T}$-erosion $\varepsilon_Y^{\uparrow} : \mathscr{P}(E) \to \mathscr{P}(\mathbb{T})$ followed by a $\mathbb{T}$-dilation $\delta_Y^{\downarrow} : \mathscr{P}(\mathbb{T}) \to \mathscr{P}(E)$, where $(\varepsilon_Y^{\uparrow}, \delta_Y^{\downarrow}) := (\tilde{\varepsilon}_{\vartheta(Y)}\vartheta, \pi\tilde{\delta}_{\vartheta(Y)})$ is a $\mathbb{T}$-adjunction between $\mathscr{P}(E)$ and $\mathscr{P}(\mathbb{T})$.

A similar representation holds for structural $\mathbb{T}$-closings [14]. By a general result from Ref. [11], every $\mathbb{T}$-opening on $\mathscr{P}(E)$ is a union of structural $\mathbb{T}$-openings $\alpha_Y^{\mathbb{T}}$, where Y ranges over a subset $\mathscr{Y} \subseteq \mathscr{P}(E)$. Combining this with Proposition 3.13 we therefore can decompose any $\mathbb{T}$-opening into $\mathbb{T}$-openings of the form $\pi\,\tilde{\delta}_{\vartheta(Y)}\tilde{\varepsilon}_{\vartheta(Y)}\vartheta$.

3.2.3. Example: A motion-invariant median filter

Consider the Boolean lattice $\mathscr{L} = \mathscr{P}(\mathbb{Z}^2)$. Let Y be a structuring element containing an odd number of N_Y points. A point x of a subset X is retained by the median filter if the intersection of X and the translated set $\tau_x Y$ contains at least $(N_Y + 1)/2$ points; otherwise the point x disappears. Define a rotation-invariant median filter by allowing rotations of Y around x to get an intersection containing the required number of points. That is, the intersection of X and the set $\tau_x r_\phi Y$ should contains at least $(N_Y + 1)/2$ points for some angle ϕ. This generalized median filter will give the same result as the original median filter if Y is rotation-invariant. Therefore, we give in Fig. 9, an example ($N_Y = 3$) with a structuring element which is not rotation-invariant, and compare the result of the rotation-invariant median filter with that of the classical median filter. As is well known, one often can replace kernels with an infinite number of elements by a finite set of so-called *basis elements* [27]. As an illustration we give in Fig. 10 a decomposition of the M-invariant median filter into a set of nine T-erosions (the structuring element of each erosion is indicated). Notice that even this set of nine erosions is redundant.

3.2.4. Example: *invariant feature extraction*

In computer vision one requires invariance under various groups, such as the Euclidean motion group, the similarity group, the affine group or the projective group [18]. When the group is enlarged, one gradually recovers the various geometric shapes present in the image. The following example is taken from Ref. [28]. Consider Fig. 11, showing a figure containing a number of quadrangles. As the image transformation we take the opening $\alpha_Y^{\mathbb{T}}$, where the structuring element Y is a square (without interior). This extracts from the input image all structures which are 'similar' to the square, where 'similar' means: obtainable from the square by a certain group operation. When $\mathbb{T} = \mathsf{T}$ (translation group), the opening extracts all translates of the square, see Fig. 11(b). When $\mathbb{T}$ is the motion group, the opening extracts all translated and rotated versions of the square, see Fig. 11(c). When $\mathbb{T}$ is the similarity group, also scaled copies of the square are extracted, see Fig. 11(d). When $\mathbb{T}$ is the affine group, the opening extracts all parallelograms from the image, see Fig. 11(e). When $\mathbb{T}$ is the projective group, the opening extracts all quadrangles from the image (i.e., the original image), see Fig. 11 (f).

So morphological operations for feature extraction can be adapted to the type of geometric invariance which is deemed to be appropriate for the application under consideration.

4. Group morphology for non-Boolean lattices

Now we will extend the results of the previous section to non-Boolean lattices. It turns out that in general only part of the results carry over to the non-Boolean case. If the group $\mathbb{T}$ equals the motion group M, or when the lattice has both a sup-generating family ℓ and inf-generating family ℓ', additional characterizations, e.g. of adjunctions, are obtainable, see Section 4.5. As a special case we consider M-operators on the lattice of grey value functions (Section 4.6).

4.1. *Simple transitivity on a sup-generating family*

We start by recalling some results obtained by Heijmans and Ronse [10,11], see also Ref. [7].

Let $\mathscr{L}$ be a complete lattice with an *abelian* automorphism group T and a sup-generating subset ℓ (cf. Section 2.2.3) such that:

(i) ℓ is T-invariant, i.e., for every $\tau \in \mathsf{T}$ and $x \in \ell$, $\tau x \in \ell$;
(ii) T is transitive on ℓ: for every $x, y \in \ell$ there exists $\tau \in T$ such that $\tau x = y$ (since T is abelian this τ is unique).

Given a fixed element ω of ℓ, τ_x is the unique element of T which maps ω to x. This enables to define a binary addition $+$ on ℓ by $x + y = \tau_x \tau_y \omega$, with $-y = \tau_y^{-1}\omega$. Now define binary operations $\oplus$ and $\ominus$ on $\mathscr{L}$ by

$$X \oplus Y = \bigvee_{y \in \ell(Y)} \tau_y X = \bigvee \{x + y : x \in \ell(X), y \in \ell(Y)\}, \quad (13)$$

$$X \ominus Y = \bigwedge_{y \in \ell(Y)} \tau_y^{-1} X = \bigvee \{z \in \ell : \tau_Z Y \leqslant X\}. \quad (14)$$

Proposition 4.1. *For any $Y \in \mathscr{L}$, the pair $(\varepsilon_Y^{\mathsf{T}}, \delta_Y^{\mathsf{T}})$ with $\delta_Y^{\mathsf{T}}(X) = X \oplus Y$, $\varepsilon_Y^{\mathsf{T}}(X) = X \ominus Y$, is a T-adjunction. Every T-adjunction has this form.*

4.2. *Transitivity on a sup-generating family*

To extend the results of Section 4.1 to non-Boolean lattices with a non-abelian automorphism group, we relax the requirement made in Section 4.1 that $\mathbb{T}$ is abelian.

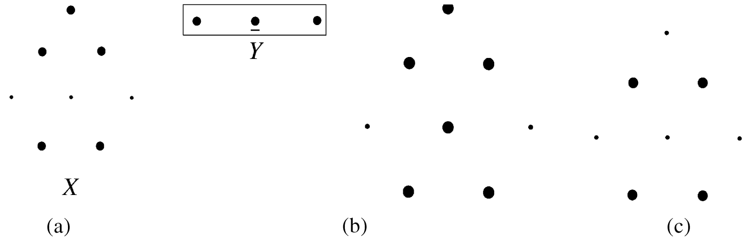


Fig. 9. Median filtering: (a) set X, structuring element Y; (b) result of M-invariant filter; (c) result of T-invariant filter.

Basic Assumption. Let $\mathscr{L}$ be a complete lattice with an automorphism group $\mathbb{T}$ and a sup-generating subset ℓ such that:

(i) ℓ is $\mathbb{T}$-invariant, i.e., for every $\tau \in \mathbb{T}$ and $x \in \ell$, $\tau x \in \ell$;

(ii) $\mathbb{T}$ is transitive on ℓ: for every $x, y \in \ell$ there exists at least one $\tau \in \mathbb{T}$ such that $\tau x = y$.

Various operators can be constructed using an extension of the 'lifting' procedure described in Section 3. This is

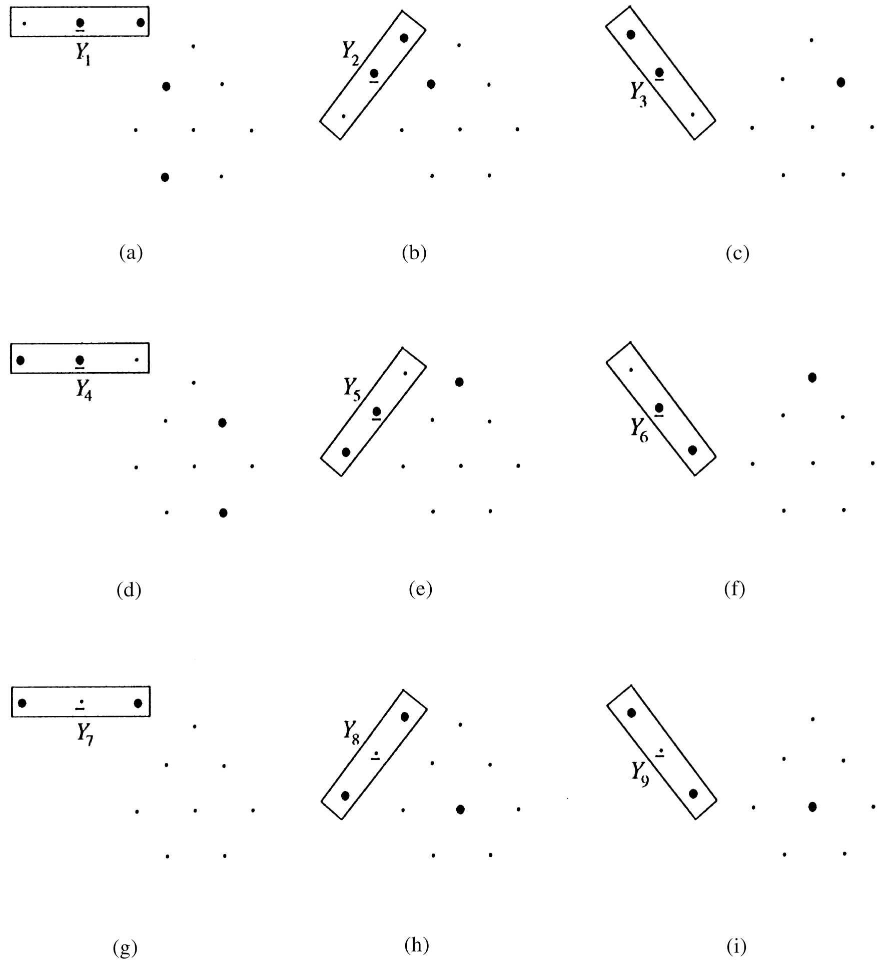


Fig. 10. Decomposition of the M-invariant median filter of Fig. 9 into a set of nine M-erosions. The structuring element of each erosion is indicated within a rectangular box.

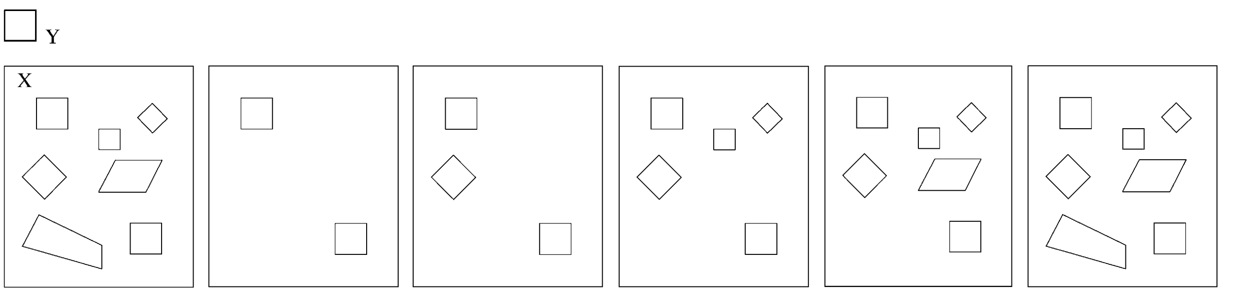


Fig. 11. Opening of the quadrangle image X shown in (a) by a square structuring element Y, using as acting group: (b) translation group; (c) motion group; (d) similarity group; (e) affine group; (f) projective group.

based upon the observation that the pair $(\ell, \vee)$ forms an adjunction between $\mathscr{L}$ and $\mathscr{P}(\ell)$, with $\vee\ell = \mathrm{id}_{\mathscr{L}}$, just as the pair (ϑ, π) forms an adjunction between $\mathscr{P}(\ell)$ and $\mathscr{P}(\mathbb{T})$, with $\pi\vartheta = \mathrm{id}_{\mathscr{P}(\ell)}$.

Given a mapping ψ on $\mathscr{L}$ we lift it to a mapping Ψ on $\mathscr{P}(\mathbb{T})$ as follows. First we go from $\mathscr{L}$ to $\mathscr{P}(\ell)$ by using the operator ℓ. Then we move from $\mathscr{P}(\ell)$ to $\mathscr{P}(\mathbb{T})$ by applying the operator ϑ. Then we apply the results of Section 3.1 on $\mathscr{P}(\mathbb{T})$ and finally project the results back, first to $\mathscr{P}(\ell)$ by using π, then to $\mathscr{L}$ by applying the $\vee$-operator. The procedure is illustrated in Fig. 6 (right diagram). Below we illustrate this approach by developing representations for openings and general increasing $\mathbb{T}$-operators.

For an operator $\psi : \mathscr{L} \to \mathscr{L}$ we define corresponding operators $\tilde{\psi}$ on $\mathscr{P}(\ell)$ and Ψ on $\mathscr{P}(\mathbb{T})$ by

$$\tilde{\psi} = \ell\psi\bigvee, \quad \Psi = \vartheta\tilde{\psi}\pi = \vartheta\ell\psi\bigvee\pi. \tag{15}$$

Using Proposition 3.7(c) and Eq. (4), ψ and $\tilde{\psi}$ can be recovered by

$$\tilde{\psi} = \pi\Psi\vartheta, \quad \psi = \bigvee\tilde{\psi}\ell = \bigvee\pi\Psi\vartheta\ell. \tag{16}$$

The next lemmas give us the necessary tools to derive properties of certain mappings on $\mathscr{L}$ from those on $\mathscr{P}(\mathbb{T})$. These lemmas are generalizations of results for Boolean lattices [13,14]. In the latter case, also results for adjunctions and closings hold, which in general are no longer valid in the non-Boolean case (cf. Remark 4.4).

Lemma 4.2. *Let ψ be an operator on $\mathscr{L}$, and let Ψ be given by Eq. (15). Then:*

(a) *If ψ is an increasing $\mathbb{T}$-mapping, then Ψ is an increasing $\mathbb{T}$-mapping.*
(b) *If ψ is a closing, then Ψ is a closing.*

Proof. (a) Obvious, since $\vartheta, \ell, \vee, \pi$ are all increasing $\mathbb{T}$-operators.

(b) From (a), Ψ is increasing, since ψ, being a closing, is increasing. Also, $\psi \geqslant \mathrm{id}_{\mathscr{L}}$, so $\Psi \geqslant \vartheta\ell \vee\pi \geqslant \vartheta\pi \geqslant \mathrm{id}_{\mathscr{P}(\mathbb{T})}$, because both $\ell\vee$ and $\vartheta\pi$ are closings, hence extensive. Finally, $\Psi^2 = \vartheta\ell\psi\vee\pi\vartheta\ell\psi\vee\pi = \vartheta\ell\psi\psi\vee\pi = \vartheta\ell\psi\vee\pi = \Psi$, where we used that $\pi\vartheta = \mathrm{id}_{\mathscr{P}(\ell)}, \vee\ell = \mathrm{id}_{\mathscr{L}}$, and $\psi^2 = \psi$. So Ψ is increasing, extensive and idempotent, hence a closing. □

Conversely, with an operator Ψ on $\mathscr{P}(\mathbb{T})$ one can associate an operator ψ on $\mathscr{L}$ by

$$\psi = \bigvee\pi\Psi\vartheta\ell. \tag{17}$$

Notice that now Ψ cannot be recovered from ψ. However, we have:

Lemma 4.3. *Let Ψ be an operator on $\mathscr{P}(\mathbb{T})$, and let ψ be given by Eq. (17).*

(a) *If Ψ is an increasing $\mathbb{T}$-mapping, then ψ is an increasing $\mathbb{T}$-mapping.*
(b) *If Ψ is an opening, then ψ is an opening.*

Proof. (a) Obvious, since $\vartheta, \ell, \vee, \pi$ are all increasing $\mathbb{T}$-operators.

(b) From (a), ψ is increasing, since Ψ, being an opening, is increasing. Also, $\Psi \leqslant \mathrm{id}_{\mathscr{L}}$, so $\psi \leqslant \vee\pi\vartheta\ell = \mathrm{id}_{\mathscr{L}}$, since $\pi\vartheta = \mathrm{id}_{\mathscr{P}(\ell)}$ and $\vee\ell = \mathrm{id}_{\mathscr{L}}$, hence ψ is anti-extensive. This also implies that $\psi^2 \leqslant \psi$. On the other hand, using that both $\ell\vee$ and $\vartheta\pi$ are closings, hence extensive, and the fact that $\Psi^2 = \Psi$, we find $\psi^2 = \vee\pi\Psi\vartheta\ell\vee\pi\Psi\vartheta\ell \geqslant \vee\pi\Psi\Psi\vartheta\ell = \vee\pi\Psi\vartheta\ell = \psi$. So we found that $\psi^2 \leqslant \psi$ and $\psi^2 \geqslant \psi$, hence $\psi^2 = \psi$, and we proved the idempotence of ψ. □

Remark 4.4. Note that ϑ is not only an erosion, but also a dilation from $\mathscr{P}(\ell)$ to $\mathscr{P}(\mathbb{T})$ (cf. Section 3.2.1). However, ℓ is not a dilation from $\mathscr{L}$ to $\mathscr{P}(\ell)$. This obstructs the construction of dilations on $\mathscr{L}$ using the lifting technique. For the special case that $\mathbb{T}$ is the Euclidean motion group or the affine group, we do in fact obtain a complete characterization of dilations using the results of Heijmans and Ronse [7,10], see Section 4.5. Another case occurs when $\mathscr{L}$ has both a sup-generating family ℓ and an inf-generating family ℓ' on which $\mathbb{T}$ acts transitively. Then $(\wedge, \ell')$ is an adjunction between $\mathscr{P}(\ell')$ and $\mathscr{L}$, and any dilation δ on $\mathscr{L}$ has the form $\delta(X) = \vee\pi(\vartheta(\ell'(X)) \overset{\mathbb{T}}{\oplus} G)$, with adjoint erosion $\varepsilon(X) = \wedge\pi_\Sigma(\vartheta(\ell(X)) \overset{\lambda}{\ominus} G)$, for some $G \in \mathscr{P}(\mathbb{T})$; cf. Fig. 12. An example is given by the lattice of grey-scale functions (see Section 4.6 below), where grey-level inversion transforms the sup-generating family into an inf-generating family [10].

4.3. Representation of structural $\mathbb{T}$-openings

Definition 4.5. The structural $\mathbb{T}$-opening $\alpha_Y^{\mathbb{T}}$ on $\mathscr{L}$ by $Y \in \mathscr{L}$ is defined by

$$\alpha_Y^{\mathbb{T}}(X) = \bigvee\{gY : g \in \mathbb{T}, gY \leqslant X\}. \tag{18}$$

$$\begin{tikzcd}
\mathcal{P}(\mathbb{T}) \arrow[r, "\tilde{\delta}"] & \mathcal{P}(\mathbb{T}) \arrow[d, "\pi"] \\
\mathcal{P}(\ell') \arrow[u, "\vartheta"] \arrow[r] & \mathcal{P}(\ell) \arrow[d, "\bigvee"] \\
\mathcal{L} \arrow[u, "\ell'"] \arrow[r, "\delta"] & \mathcal{L}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\mathbb{T}) \arrow[r, "\tilde{\varepsilon}"] & \mathcal{P}(\mathbb{T}) \arrow[d, "\pi_\Sigma"] \\
\mathcal{P}(\ell) \arrow[u, "\vartheta"] \arrow[r] & \mathcal{P}(\ell) \arrow[d, "\bigwedge"] \\
\mathcal{L} \arrow[u, "\ell"] \arrow[r, "\varepsilon"] & \mathcal{L}
\end{tikzcd}$$

Fig. 12. Construction of a $\mathbb{T}$-dilation δ (left), and a $\mathbb{T}$-erosion ε (right), on a lattice $\mathscr{L}$ with sup-generating family ℓ and inf-generating family ℓ'.

Proposition 4.6 (Decomposition of structural $\mathbb{T}$-openings). *The structural $\mathbb{T}$-opening $\alpha_Y^{\mathbb{T}}$ defined by Eq.* (18) *is the product of a $\mathbb{T}$-erosion $\varepsilon_Y^{\uparrow} : \mathscr{L} \to \mathscr{P}(\mathbb{T})$ followed by its adjoint $\mathbb{T}$-dilation $\delta_Y^{\downarrow} : \mathscr{P}(\mathbb{T}) \to \mathscr{L}$,* i.e., $\alpha_Y^{\mathbb{T}}(X) = \delta_Y^{\downarrow}\varepsilon_Y^{\uparrow}(X)$, *where*

$$\varepsilon_Y^{\uparrow}(X) = \vartheta(\ell(X)) \overset{\lambda}{\ominus} \vartheta(\ell(Y)), \quad X \in \mathscr{L},$$

$$\delta_Y^{\downarrow}(G) = \bigvee \pi[G \overset{\mathbb{T}}{\oplus} \vartheta(\ell(Y))], \quad G \in \mathscr{P}(\mathbb{T}).$$

Proof. By explicit computation, we find

$$\begin{aligned}
\alpha_Y^{\mathbb{T}}(X) &= \bigvee \{gY : g \in \mathbb{T}, gY \leqslant X\} \\
&= \bigvee \bigcup \{g\ell(Y) : g \in \mathbb{T}, g\ell(Y) \subseteq \ell(X)\} \\
&= \bigvee \bigcup \pi[\{g\vartheta(\ell(Y)) : g \in \mathbb{T}, g\vartheta(\ell(Y)) \subseteq \vartheta(\ell(X))\}] \\
&= \bigvee \pi[\bigcup \{g\vartheta(\ell(Y)) : g \in \mathbb{T}, g\vartheta(\ell(Y)) \subseteq \vartheta(\ell(X))\}] \\
&= \bigvee \pi[(\vartheta(\ell(X)) \overset{\lambda}{\ominus} \vartheta(\ell(Y))) \overset{\mathbb{T}}{\oplus} \vartheta(\ell(Y))] \\
&= \delta_Y^{\downarrow}(\vartheta(\ell(X)) \overset{\lambda}{\ominus} \vartheta(\ell(Y))) = \delta_Y^{\downarrow}(\varepsilon_Y^{\uparrow}(X)),
\end{aligned}$$

where we used the properties of sup-generating families (see Section 2.2.3). □

Again we note that the opening $\alpha_Y^{\mathbb{T}}$ is not an adjunctional opening on $\mathscr{L}$ in the sense of Section 2.2.2. To decompose $\alpha_Y^{\mathbb{T}}$ as a product of an erosion and its adjoint dilation, distinct lattices $\mathscr{L}$ and $\mathscr{P}(\mathbb{T})$ are required.

Finally, to obtain decompositions of structural $\mathbb{T}$-closings one needs a dual Basic Assumption requiring the existence of an inf-generating subset, see Ref. [7, Remark 5.11].

4.4. Representation of increasing $\mathbb{T}$-operators

The lifting approach enables us to obtain a generalization of a theorem by Matheron [1] giving a characterization of $\mathbb{T}$-invariant increasing mappings on $\mathscr{L}$.

Definition 4.7. The *kernel* $\ker(\psi)$ of a mapping $\psi : \mathscr{L} \to \mathscr{L}$ is defined by

$$\ker(\psi) = \{A \in \mathscr{L} : \omega \leqslant \psi(A)\}.$$

Here ω is the origin of the sup-generating family ℓ of $\mathscr{L}$.

Theorem 4.8. *Let $\mathscr{L}$ be a complete lattice with automorphism group $\mathbb{T}$ satisfying the Basic Assumption. Then any increasing $\mathbb{T}$-mapping $\psi : \mathscr{L} \to \mathscr{L}$ has the decomposition*

$$\psi(X) = \bigvee \bigcup_{Y \in \ker(\psi)} \pi[\vartheta(\ell(X)) \overset{\lambda}{\ominus} \vartheta(\ell(Y))]. \tag{19}$$

Proof. The mapping $\tilde{\psi}$ defined by $\tilde{\psi}(G) = \ell(\psi(\vee G)), G \in \mathscr{P}(\ell)$, is an increasing $\mathbb{T}$-operator on $\mathscr{P}(\ell)$. In Ref. [13] we proved that any increasing $\mathbb{T}$-mapping on a Boolean lattice $\mathscr{P}(\ell)$ is a union of projected erosions, i.e., mappings which are projections of erosions on $\mathscr{P}(\mathbb{T})$:

$$\tilde{\psi}(G) = \bigcup_{H \in \ker(\tilde{\psi})} \pi[\vartheta(G) \overset{\lambda}{\ominus} \vartheta(H)],$$

where $\ker(\tilde{\psi}) = \{G \in \mathscr{P}(\ell) : \omega \in \tilde{\psi}(G)\}$ is the kernel of $\tilde{\psi}$. Therefore, Eq. (16) yields

$$\psi(X) = \bigvee \tilde{\psi}(\ell(X)) = \bigvee \bigcup_{H \in \ker(\tilde{\psi})} \pi[\vartheta(\ell(X)) \overset{\lambda}{\ominus} \vartheta(H)].$$

We can relate the kernels of ψ and $\tilde{\psi}$ as follows:

$$\begin{aligned}
\ker(\tilde{\psi}) &= \{G \in \mathscr{P}(\ell) : \omega \in \tilde{\psi}(G)\} \\
&= \{G \in \mathscr{P}(\ell) : \omega \in \ell(\psi(\bigvee G))\} \\
&= \{G \in \mathscr{P}(\ell) : \omega \leqslant \psi(\bigvee G)\} \\
&= \{G \in \mathscr{P}(\ell) : \bigvee G \in \ker(\psi)\}.
\end{aligned}$$

Also, for all $g \in \mathbb{T}$, we have the equivalences

$$\begin{aligned}
g\vartheta(H) \subseteq \vartheta(\ell(X)) &\Leftrightarrow gH \subseteq \ell(X) \Leftrightarrow g\bigvee H \leqslant X \\
&\Leftrightarrow g\ell(\bigvee H) \subseteq \ell(X) \\
&\Leftrightarrow g\vartheta(\ell(\bigvee H)) \subseteq \vartheta(l(X)),
\end{aligned}$$

where we used the properties of ϑ and ℓ summarized in Section 2.2.3 and Section 3.2.1, as well as their $\mathbb{T}$-invariance. This implies that

$$\begin{aligned}
\vartheta(\ell(X)) \overset{\lambda}{\ominus} \vartheta(H) &= \{g \in \mathbb{T} : g\vartheta(H) \subseteq \vartheta(\ell(X))\} \\
&= \{g \in \mathbb{T} : g\vartheta(\ell(\bigvee H)) \subseteq \vartheta(\ell(X))\} \\
&= \vartheta(\ell(X)) \overset{\lambda}{\ominus} \vartheta(\ell(\bigvee H)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi(X) &= \bigvee \bigcup_{Y \in \ker(\psi)} \bigcup_{H : \vee H = Y} \pi[\vartheta(\ell(X)) \overset{\lambda}{\ominus} \vartheta(\ell(\bigvee H))] \\
&= \bigvee \bigcup_{Y \in \ker(\psi)} \pi[\vartheta(\ell(X)) \overset{\lambda}{\ominus} \vartheta(\ell(Y))]
\end{aligned}$$

This completes the proof. □

Note that the mapping $\varepsilon_Y^{\uparrow} : \mathscr{L} \to \mathscr{P}(\mathbb{T})$, with $\varepsilon_Y^{\uparrow}(X) = \vartheta(\ell(X)) \overset{\lambda}{\ominus} \vartheta(\ell(Y)) = \{g \in \mathbb{T} : gY \leqslant X\}$ is an erosion between the lattices $\mathscr{L}$ and $\mathscr{P}(\mathbb{T})$. Again, we remark that to obtain representations of an increasing $\mathbb{T}$-operator as an infimum of projected $\mathbb{T}$-dilations one needs a dual Basic Assumption.

By considering special cases, we recover some of the well-known representations.

1. $\mathbb{T}$ *Abelian*. Using the properties of the operators ϑ and ℓ one finds.

$$\begin{aligned}
&\vartheta(\ell(X)) \overset{\lambda}{\ominus} \vartheta(\ell(Y)) \\
&\quad = \bigcap_{h\in\vartheta(\ell(Y))} \vartheta(\ell(X))h^{-1} = \bigcap_{h\in\vartheta(\ell(Y))} h^{-1}\vartheta(\ell(X)) \\
&\quad = \bigcap_{h\in\vartheta(\ell(Y))} \vartheta(\ell(h^{-1}X)) = \vartheta\Bigg(\ell\Bigg(\bigwedge_{h\in\vartheta(\ell(Y))} h^{-1}X\Bigg)\Bigg) \\
&\quad = \vartheta\Bigg(\ell\Bigg(\bigwedge_{y\in\ell(Y)} \tau_y^{-1}X\Bigg)\Bigg)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi(X) &= \bigvee_{Y\in\ker(\psi)} \pi\Bigg[\vartheta\Bigg(\ell\Bigg(\bigwedge_{y\in\ell(Y)} \tau_y^{-1}X\Bigg)\Bigg)\Bigg] \\
&= \bigvee_{Y\in\ker(\psi)} \ell\Bigg(\bigwedge_{y\in\ell(Y)} \tau_y^{-1}X\Bigg) = \bigvee_{Y\in\ker(\psi)} \Bigg(\bigwedge_{y\in\ell(Y)} \tau_y^{-1}X\Bigg) \\
&= \bigvee_{Y\in\ker(\psi)} X\ominus Y, \qquad (20)
\end{aligned}$$

where $X\ominus Y$ is defined by Eq. (14). This is precisely the representation for increasing $\mathbb{T}$-operators with $\mathbb{T}$ abelian, as derived in Ref. [10, Theorem 3.11], see also Ref. [7, Theorem 5.22].

2. $\mathscr{L}$ *Boolean*. If $\mathscr{L} = \mathscr{P}(E)$ for some set E, then ℓ becomes the identity operator, and $\vee$ becomes union, so

$$\psi(X) = \bigcup_{Y\in\ker(\psi)} \pi[\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)],$$

which is the representation by projected erosions as derived in Ref. [13]. If $\mathbb{T}$ equals the translation group T this representation reduces to that of Matheron [1]. Application of this decomposition to the Boolean dual (3) leads to a representation as intersection of projected dilations.

4.5. M-*invariant operators*

When $\mathbb{T}$ is the motion group M, many formulas simplify considerably, and also some additional characterizations, e.g. for adjunctions, are obtained. Essentially, the same technique applies when M is replaced by other groups which have the translation group T as a transitive subgroup, such as the similarity group or the affine group.

From the results of Section 4.1 we know that a mapping δ is a T-dilation on $\mathscr{L}$ if and only if δ has the form

$$\delta(X) = \delta_Y^{\mathsf{T}}(X) = X\oplus Y = \bigvee_{x\in\ell(X)} \tau_x Y = \bigvee_{y\in\ell(Y)} \tau_y X, \qquad (21)$$

where the structuring element is given by $Y = \delta(\omega)$, with ω the origin of the sup-generating family ℓ. Since every M-dilation is a T-dilation, also all M-dilations have the form (21). But δ has to be R-*invariant* as well, therefore $Y = \delta(\omega) = \delta(r\omega) = r\delta(\omega) = rY, \forall r\in\mathsf{R}$, i.e., Y has to be R-invariant. Conversely, we may ask whether every mapping of the form (21) with R-invariant structuring element Y is an M-dilation. Well, Eq. (21) is a T-dilation, so it remains to prove that δ is R-invariant. For any $r\in\mathsf{R}$,

$$\delta(rX) = \bigvee_{x\in\ell(rX)} \tau_x Y = \bigvee_{x\in r\ell(X)} \tau_x Y = \bigvee_{x'\in\ell(X)} \tau_{rx'} Y.$$

Now M is the semi-direct product of T and R, so from Eq. (8) and the R-invariance of Y, we find

$$\begin{aligned}
\delta(rX) &= \bigvee_{x'\in\ell(X)} r\tau_{x'}r^{-1}Y = \bigvee_{x'\in\ell(X)} r\tau_{x'}Y \\
&= r\bigvee_{x'\in\ell(X)} \tau_{x'}Y = r\delta(X).
\end{aligned}$$

Since adjoints of dilations are unique, we know immediately that the mapping ε given by $\varepsilon_Y^{\mathsf{T}}(X) = X\ominus Y = \bigwedge_{y\in\ell(Y)} \tau_y^{-1}X$ is the M-erosion adjoint to δ. Summarizing:

Proposition 4.9. *For any $Y\in\mathscr{L}$, with Y R-invariant, the pair $(\varepsilon_Y^{\mathsf{M}}, \delta_Y^{\mathsf{M}})$ with*

$$\delta_Y^{\mathsf{M}}(X) = \bigvee_{y\in\ell(Y)} \tau_y X \quad \text{and} \quad \varepsilon_Y^{\mathsf{M}}(X) = \bigwedge_{y\in\ell(Y)} \tau_y^{-1}X,$$

is an M-adjunction. Every M-adjunction has this form.

In the case of the structural M-opening α_Y^{M} by the structuring element Y, we find

$$\begin{aligned}
\alpha_Y^{\mathsf{M}}(X) &:= \bigvee\{gY : g\in\mathsf{M}, gY \leqslant X\} \\
&= \bigvee\{\tau rY : r\in\mathsf{R}, \tau\in\mathsf{T}, \tau rY \leqslant X\} \\
&= \bigvee_{r\in R} \alpha_{rY}^{\mathsf{T}}(X), \qquad (22)
\end{aligned}$$

where $\alpha_{rY}^{\mathsf{T}}(X) := \vee\{\tau rY : \tau\in\mathsf{T}, \tau rY \leqslant X\}$ is the structural T-opening by rY. For the closing one finds similarly,

$$\phi_Y^{\mathsf{M}}(X) = \bigwedge\{gY : g\in\mathsf{M}, gY \geqslant X\} = \bigwedge_{r\in R} \phi_{rY}^{\mathsf{T}}(X),$$

where $\phi_{rY}^{\mathsf{T}}(X) := \wedge\{\tau rY : \tau\in\mathsf{T}, \tau rY \geqslant X\}$ is the structural T-closing by rY.

Remark 4.10. It was proved in Ref. [11] that α_{rY}^{T} is an *adjunctional* T-opening: $\alpha_{rY}^{\mathsf{T}} = \delta_{rY}^{\mathsf{T}}\varepsilon_{rY}^{\mathsf{T}}$, but that, in general, ϕ_{rY}^{T} is not an adjunctional T-closing (cf. Ref. [7]).

Finally, we take a look at the representation of Theorem 4.8 for increasing $\mathbb{T}$-mappings. Since every M-mapping is a T-mapping, Eq. (19) should reduce to the representation (20). For the projected erosions occurring in Eq. (19) we find

$$\begin{aligned}\pi[\vartheta(\ell(X))\overset{\lambda}{\ominus}\vartheta(\ell(Y))] &= \pi\{g\in M : g\vartheta(\ell(Y))\subseteq\vartheta(\ell(X))\}\\ &= \pi\{g\in M : g\ell(Y)\subseteq\ell(X)\}\\ &= \{x\in\ell : \tau_x\ell(rY)\subseteq\ell(X)\\ &\quad \text{for some } r\in R\}\\ &= \bigcup_{r\in R}\ell(X)\overset{\mathsf{T}}{\ominus}(\ell(rY)),\end{aligned}$$

where

$$\ell(X)\overset{\mathsf{T}}{\ominus}(\ell(rY)) = \bigcap_{y\in\ell(rY)}\tau_y^{-1}\ell(X) = \ell\left(\bigwedge_{y\in\ell(rY)}\tau_y^{-1}X\right) \quad (23)$$

denotes the T-dilation of the set $\ell(X)$ by the structuring element $\ell(rY)$. Therefore any increasing M-mapping $\psi : \mathscr{L}\to\mathscr{L}$ has the representation

$$\psi(X) = \bigvee\bigcup_{Y\in\ker(\psi)}\bigcup_{r\in R}\ell(X)\overset{\mathsf{T}}{\ominus}(\ell(rY)). \quad (24)$$

It is easy to show that for any M-mapping ψ the following equivalence holds for all $r\in R : Y\in\ker(\psi)\Leftrightarrow rY\in\ker(\psi)$. This means that the union over all $r\in R$ in Eq. (24) actually can be omitted. So, using Eq. (23) and (7), one obtains

$$\psi(X) = \bigvee\bigcup_{Y\in\ker(\psi)}\ell\left(\bigwedge_{y\in\ell(Y)}\tau_y^{-1}X\right) = \bigvee_{Y\in\ker(\psi)}X\ominus Y,$$

and we recover Eq. (20).

4.6. Group-invariant grey-scale operators

The general approach above can be directly[2] applied to the treatment of $\mathbb{T}$-invariant operators on the lattice $\mathscr{L}$ of grey scale functions. Our approach closely follows that of Ronse and Heijmans [7,10,11].

Let $\mathscr{L}$ denote the complete lattice $\mathrm{Fun}(E,\mathscr{T})$ of grey scale functions with domain E, whose range is a complete lattice $\mathscr{T}$ of grey values. Here E may be $\mathbb{R}^n$ or $\mathbb{Z}^n$, and $\mathscr{T}$ may be $\bar{\mathbb{R}} = \mathbb{R}\cup\{+\infty,-\infty\}$, $\bar{\mathbb{Z}} = \mathbb{Z}\cup\{+\infty,-\infty\}$, or also a finite set of grey values [7, Chapter 11]. In the following we restrict ourselves to the case $n = 2$.

The supremum and infimum of a family $(F_j)_{j\in J}$ of grey-scale functions is given by

$$\left(\bigvee_{j\in J}F_j\right)(x) = \sup_{j\in J}F_j(x), \quad \left(\bigwedge_{j\in J}F_j\right)(x) = \inf_{j\in J}F_j(x), \quad x\in E.$$

The sup-generating family ℓ is now given by the impulse functions $f_{x,t}$, $x\in E$, $t\in\mathscr{T}$ defined by

$$f_{x,t}(y) = \begin{cases} t, & y = x,\\ -\infty, & y\neq x.\end{cases} \quad (25)$$

As indicated in Remark 4.4, one can give complete characterizations of $\mathbb{T}$-invariant grey-scale operators due to the existence of grey-level inversion. We give two examples.

4.6.1. Motion-invariant grey-scale operators

This is the case where $\mathbb{T}$ is the motion group M. Define an automorphism $\gamma_{h,\phi,v}$ on $\mathscr{L}$ by

$$(\gamma_{h,\phi,v}(F))(x) = F(r_\phi^{-1}x - h) + v, \quad F\in\mathscr{L},$$

i.e., $\gamma_{h,\phi,v}$ carries out a motion – consisting of a rotation r_ϕ followed by a translation τ_h – of the graph of F in the plane, and translates it over a distance v along the grey value axis. The group

$$\mathsf{M} := \{\gamma_{h,\phi,v} : h\in E,\ \phi\in[0,2\pi),\ v\in\mathscr{T}\},$$

is an automorphism group of $\mathscr{L}$ acting transitively on ℓ. M is the semi-direct product of the abelian groups T and R where

$$\mathsf{T} = \{\tau_{h,v} : h\in E,\ v\in\mathscr{T}\}, \quad \mathsf{R} = \{r_\phi : \phi\in[0,2\pi)\},$$

with $\tau_{h,v} = \gamma_{h,0,v}$ and $r_\phi = \gamma_{0,\phi,0}$ (note that r_ϕ denotes both an operator on points and on functions). In particular, $r_\phi\tau_{h,v}r_\phi^{-1} = \tau_{r_\phi h,v}$. Note that the group T of translations is transitive on ℓ. So from the results of Section 4.5 we may conclude immediately that all M-dilations have the form $\delta^{\mathsf{T}}(F) = F\oplus G$, for some R-invariant structuring function $G\in\mathscr{L}$, where

$$\begin{aligned}(F\oplus G)(x) &= \bigvee_{(h,v)\in\ell(G)}(\tau_{h,v}F)(x) = \bigvee_{(h,v)\in\ell(G)}F(x-h)+v\\ &= \bigvee_{h\in E}F(x-h)+G(h)\end{aligned}$$

and the R-invariance of the structuring function G is expressed by $rG = G$ for all $r\in R$, i.e.

$$G(r_\phi^{-1}x) = G(x) \quad \forall\phi\in[0,2\pi).$$

The adjoint erosion has the form $\varepsilon^{\mathsf{T}}(F) = F\ominus G$ where

$$(F\ominus G)(x) = \bigwedge_{h\in E}F(x+h) - G(h).$$

Finally, the decomposition (22) of structural M-openings now reads $\alpha_G^{\mathsf{M}}(F) = \vee_{r\in\mathsf{R}}\alpha_{rG}^{\mathsf{T}}(F)$, where $\alpha_{rG}^{\mathsf{T}}(F) := (F\ominus rG)\oplus rG$, with

$$((F\ominus G)\oplus G)(x) = \bigvee_h\bigwedge_{h'}F(x-h+h') - G(h') + G(h)$$

is the structural T-opening with structuring function G.

Decompositions of structural M-closings are possible by the existence of grey-scale inversion, which transforms

[2] An alternative is the umbra approach, which has to be handled with care [7,8].

the sup-generating family (25) into an inf-generating family, cf. Remark 4.4.

Remark 4.11. The chosen group M leads to additive structuring functions. Other choices are possible, leading to multiplicative structuring functions. See Ref. [7,10] for more details.

4.6.2. Grey-scale operators on the sphere

As a second example we consider grey-scale operators on the sphere, invariant under the group SO(3) of rotations in 3-space, cf. Example 2.9. The construction of morphological operators for this case leads to formulas which are completely analogous to the ones for M-invariant operators just considered. So we confine ourselves to illustrating this case by a practical example and making some remarks on the implementation of the spherical operators.

First we recall some facts for the case of binary image operators on the sphere, which was considered in Ref. [23]. We assume that pictures of the sphere are produced by orthographic projection on a plane, which corresponds closely to what happens if pictures of the earth or a planet are taken from a large distance. Only one hemisphere will be visible, so we take a disc on which to map a hemisphere. Let $D := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$ be a disc of radius 1 in the plane. The upper hemisphere is the set $S^2_+ := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geqslant 0\}$. Orthographic projection from the upper hemisphere to the disc D is the map $p_\perp : S^2_+ \to D$ given by $p_\perp(x, y, \sqrt{1-x^2-y^2}) = (x,y)$, with inverse $p_\perp^{-1}(x,y) = (x, y, \sqrt{1-x^2-y^2})$.

Under orthographic projection, the rotations on the sphere induce transformations on the disc D. Consider a disc on the sphere centered at the pole, such that its projection is a disc C of radius $\delta < 1$ with center at the origin of D, cf. Fig. 13. If the disc on the sphere has moved to a location such that the projection of its center is at $(x,y) \in D$, then the image $C_{x,y}$ of the rotated disc consists of those points $(u,v) \in D$ which satisfy the equation

$$1 - xu - yv - \sqrt{(1-x^2-y^2)(1-u^2-v^2)} \leqslant 1 - \sqrt{1-\delta^2}. \tag{26}$$

The boundary of the region $C_{x,y}$ is in general an ellipse, see Fig. 13. The ellipses have their minor axes oriented in the radial direction. Note that (x,y) is not the center of the ellipse $C_{x,y}$: if (x,y) has radial distance r to the origin, then $C_{x,y}$ has its center at radial distance $r\sqrt{1-\delta^2}$. Very close to the boundary of D, $C_{x,y}$ is no longer an ellipse, but a region enclosed between part of an ellipse and the boundary of the disc D, corresponding to the situation that the rotating disc on the sphere moves from one hemisphere to the other.

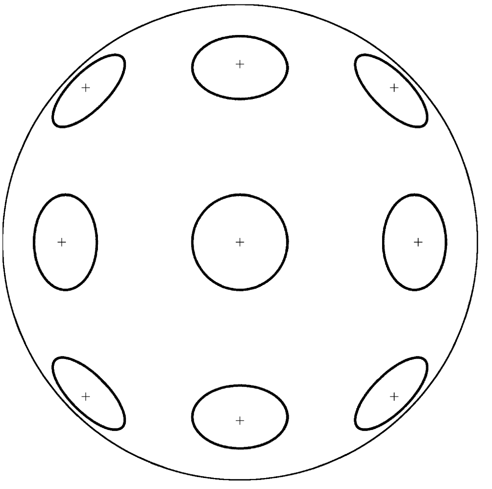
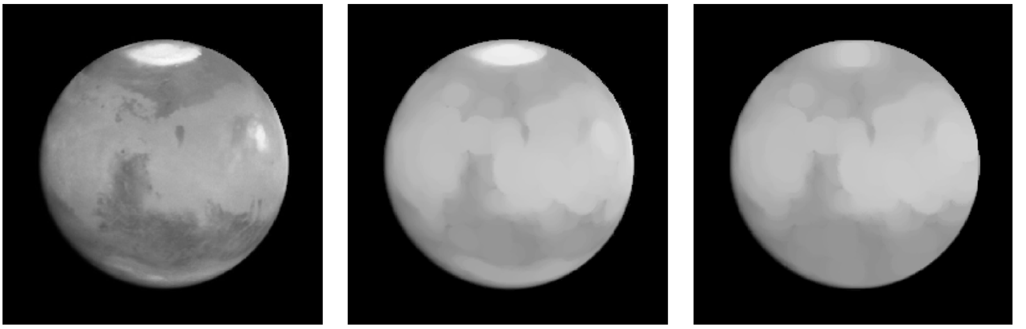


Fig. 13. Disc C centered at the origin of D, and its 'translates' $C_{x,y}$ corresponding to rotated discs on the sphere under orthographic projection. A '+' sign indicates the projection of the center of a rotated disc.

Now we can construct spherical grey-scale operators by a structuring function G with support inside the disc C of radius δ. For simplicity we take a rotationally symmetric structuring function, more in particular a *flat* structuring function with constant value zero. This is implemented in the digital case as follows. The disc D is covered by a square grid of pixels, and for each pixel (x,y) in D, the disc C at the origin is transformed to position (x,y) according to Eq. (26). Then the value of the flat grey-scale dilation or erosion at pixel (x,y) is obtained by computing the maximum (resp. the minimum) of the image values at all pixels inside the region $C_{x,y}$ around (x,y). Products of such an erosion and dilation result in a spherical grey-scale opening or closing.

As an example, we show in Fig. 14(a) a picture of the planet Mars, taken by the Hubble Space Telescope on February 25, 1995 (Source: NASA/National Space Science Data Center; credit: Ph. James (University of Toledo), S. Lee (University of Colorado), NASA). Fig. 14(b) shows its opening by the flat structuring function G defined above, where we have chosen $\delta = 0.1$, i.e. the radius of C equals 10% of the radius of the planet. For comparison, the Euclidean opening with the disc C (for the same value of δ) is shown as well, see Fig. 14(c). Notice the different behavior near the boundary of the planet, in particular with respect to the polar cap: in the Euclidean case, the translates $C_{x,y}$ remain discs of radius δ at all points (x,y). This illustrates that the spherical transformations are better adapted to the geometry than the Euclidean translations.



(a) (b) (c)

Fig. 14. (a) Picture of the planet Mars (for details, see text), (b): spherical grey-scale opening, (c) Euclidean grey-scale opening.

5. Discussion

We have presented a mathematical framework for constructing morphological operators on complete lattices which are invariant under some group $\mathbb{T}$. Starting from the classical operators, like dilation, erosion, opening and closing, which are invariant under the abelian translation group T, a two-stage process was described for constructing $\mathbb{T}$-invariant morphological operators on Boolean lattices with a non-commutative group of automorphisms. First $\mathbb{T}$-invariant morphological operators were defined on the space $\mathscr{P}(\mathbb{T})$ of subsets of $\mathbb{T}$ by generalizing the Minkowski operations to non-commutative groups. Next morphological operators were constructed on the actual object space of interest $\mathscr{P}(E)$ by (i) mapping the subsets of E to subsets of $\mathbb{T}$, (ii) using the results for the lattice $\mathscr{P}(\mathbb{T})$, and (iii) projecting back to the original space $\mathscr{P}(E)$.

Subsequently, we considered non-Boolean lattices with a non-commutative group $\mathbb{T}$ of automorphisms. Following Heijmans and Ronse [10,11] the basic assumption was made that the lattice has a sup-generating family on which $\mathbb{T}$ acts transitively. Differences with the case of Boolean lattices were pointed out. Special attention was given to the case where $\mathbb{T}$ equals the Euclidean motion group M generated by translations and rotations. As another application of special interest we considered $\mathbb{T}$-invariant morphological operators for grey-scale functions.

Examples covered by the general framework are:

- *Polar morphology* [5,10], with applications to models of the visual cortex [29,30].
- *Constrained perspective morphology* [31], where one requires invariance of image operations under object translation parallel to the image plane used for perspective projection.
- *Spherical morphology* [23], which has connections to integral geometry and geometric probability [32,33], see also Section 4.6.2.
- *Translation-rotation morphology* [24], which has applications to robot path planning [34], see also Ref. [35]. Another application is the tailor problem, which concerns the fitting of sets without overlap within a larger set [36], with applications to making cutting plans for clothing manufacture. For similar applications of the classical Minkowski operations to spatial planning and other problems, see Ghosh [37].
- *Projective morphology* [28], which is appropriate for invariant pattern recognition under perspective projection. Invariance may be restricted to subgroups of the projective group, such as the motion group, the similarity group, or the affine group. Other applications concern affine signal models or the inverse problem in fractal modeling [26].
- *Differential morphology* [38]. Shape description of patterns on arbitrary (smooth) surfaces based on concepts of differential geometry may be used to obtain morphological operators which leave the geometry of the surface invariant.

6. Summary

In its original form, mathematical morphology is a theory of binary image transformations which are invariant under the group of Euclidean translations. This paper surveys and extends constructions of morphological operators which are invariant under a more general group $\mathbb{T}$, such as the motion group, the affine group, or the projective group. The motivation for this approach derives from computer vision, where an important question is how to take the projective geometry of the imaging process into account. This is of importance in *invariant pattern recognition*, where the goal is to recognize patterns irrespective of their orientation or location. In *image understanding* one wants to derive information about three-dimensional (3D) scenes from projections on a planar (2D) image screen. In this case it is natural to require invariance of image operations under the 3D camera rotations. So one may require invariance under increasingly larger groups, such as the Euclidean motion group, the similarity group, the affine group or the projective group, which are all non-commutative groups.

We will follow a two-step approach: first we construct morphological operators on the space $\mathscr{P}(\mathbb{T})$ of subsets of the group $\mathbb{T}$ itself; next we use these results to construct morphological operators on the original object space, i.e. the Boolean algebra $\mathscr{P}(E^n)$ in the case of binary images, or the lattice $\mathrm{Fun}(E^n, \mathscr{T})$ in the case of grey value functions $F: E^n \to \mathscr{T}$, where E equals $\mathscr{R}$ or $\mathbb{Z}$, and $\mathscr{T}$ is the grey value set. $\mathbb{T}$-invariant dilations, erosions, openings and closings are defined and several representation theorems are presented. Graphical illustrations are given for the case of the Euclidean motion group generated by translations and rotations. Examples and applications are discussed.

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