REGULAR IMPLEMENTABILITY AND ITS APPLICATION TO STABILIZATION OF SYSTEM BEHAVIORS

M.N. Belur, H.L. Trentelman and J.C. Willems

Abstract

In this paper we study control by interconnection of linear differential systems. We give necessary and sufficient conditions for regular implementability of a given linear differential system. We formulate the problems of stabilization and pole placement as problems of finding suitable, regularly implementable sub-behaviors of the manifest plant behavior. The problem formulations and their resolutions are completely representation free, and specified in terms of the system dynamics only. Control is viewed as regular interconnection. A controller is a system that constrains the plant behavior through a distinguished set of variables, namely, the control variables. The issue of implementation of a controller in the feedback configuration and its relation to regularity of interconnection is addressed. Freedom of disturbances in a plant and regular interconnection with a controller also turn out to be inter-related.

Keywords: Behaviors, regular implementability, stabilization, pole placement, interconnection, controller implementation.

1 Introduction and notation

In this paper we discuss the issue of stabilization of linear dynamical systems. The problem is studied in the behavioral context and control is viewed as interconnection. This view of treating control problems has been used before in, for example, [15], [7], [8] and [2], in an $H_\infty$ control context in [4], [5], [1], [9], [10], [11], [12] and [13], for adaptive control in [8], and for distributed systems in [6]. In contrast to [15] where the problems of stabilization and pole placement were considered for the case that all system variables are available for interconnection (the so-called full information case), we work in the generality that we are allowed to use only some of the system variables for the purpose of interconnection. These variables are called the control variables. Restricting oneself to using only the control variables for interconnection introduces the issue of implementability into the control problem, see [12] and [8]. In the context of stabilization, an important role is played by the notion of regular implementability. We establish necessary and sufficient conditions for a given behavior to be regularly implementable (section 2). This result is then applied to solve the problems of stabilization and pole placement by interconnection (section 3). The general problem formulation reduces to some important special cases. Section 4 contains the case of filtering. Implementation of a controller in a feedback configuration plays a very prominent role in control theory. This issue is addressed in section 5. Finally, in section 6 we give a motivation for the fact that in our problem formulations we restrict ourselves to regular interconnections.

We first discuss some of the notation to be used in this paper, and review some basic facts from the behavioral approach. We use the standard notation $\mathbb{R}^n$ for the $n$-dimensional real Euclidean space. Often, the notation $\mathbb{R}^n$ is used if $w$ denotes a typical element of that vector space, or a typical function taking its value in that vector space. The ring of (one-variable) polynomials with real coefficients in the indeterminate $\xi$ is denoted by $\mathbb{R}[\xi]$. $\mathbb{R}^{n_1 \times n_2}[\xi]$ denotes the set of matrices with $n_1$ rows and $n_2$ columns in which each entry is an element of $\mathbb{R}[\xi]$. We use the notation $\mathbb{R}^{n_1 \times n_2}$ when the number of rows is unspecified.

In this paper, we deal with linear time-invariant differential systems, in short, linear differential systems. A linear differential system is defined as a dynamical system whose behavior $\mathcal{B}$ is equal to the set of solutions of a set of higher order, linear, constant coefficient differential equations. More precisely, there exists a polynomial matrix $R \in \mathbb{R}^{n \times \pi}[\xi]$ such that

$$\mathcal{B} = \{ w \in L_1^{loc}(\mathbb{R}, \mathbb{R}^\pi) \mid R(\frac{d}{dt})w = 0 \}.$$ 

Here, $L_1^{loc}(\mathbb{R}, \mathbb{R}^\pi)$ denotes the space of locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^\pi$, and $R(\frac{d}{dt})w = 0$ is understood to hold in the distributional sense. The set of linear differential systems with manifest variable $w$ taking its value in $\mathbb{R}^\pi$ is denoted by $\mathcal{B}^\pi$.

We make a clear distinction between the behavior as defined as the space of all solutions of a set of (differential) equations, and the set of equations itself. A set of equations in terms of which the behavior is defined, is called a representation of the behavior. Let $R \in \mathbb{R}^{n \times \pi}[\xi]$ be a polynomial matrix. If a behavior $\mathcal{B}$ is represented by $R(\frac{d}{dt})w = 0$ then we call this a kernel representation of $\mathcal{B}$. Further, a kernel representation is
A behavior whose input cardinality is equal to 0 is called autonomous. An autonomous behavior \( \mathcal{B} \) is said to be stable, if for all \( w \in \mathcal{B} \) we have \( w(t) \to 0 \) as \( t \to \infty \). In the context of stability, we often need to describe regions of the complex plane \( \mathbb{C} \). We denote the closed right-half of the complex plane by \( \mathbb{C}^+ \) and the open left-half complex plane by \( \mathbb{C}^- \). A polynomial matrix \( R \in \mathbb{R}^{n \times n}[\xi] \) is called Hurwitz if \( \text{rank}(R(\lambda)) = n \) for all \( \lambda \in \mathbb{C}^+ \). If \( \mathcal{B} \in \mathcal{L}^0 \) is represented by \( R(\frac{d}{dt})w = 0 \) then \( \mathcal{B} \) is stable if and only if \( R \) is Hurwitz.

For autonomous behaviors, we also speak about poles of the behavior. Let \( \mathcal{B} \in \mathcal{L}^0 \) be autonomous. Then there exists an \( R \in \mathbb{R}^{n \times n}[\xi] \) such that \( \mathcal{B} \) is represented minimally by \( R(\frac{d}{dt})w = 0 \). We can choose \( R \) such that \( \det(R) \) is a monic polynomial. This monic polynomial is denoted by \( \chi_\mathcal{B} \) and is called the characteristic polynomial of \( \mathcal{B} \). It can be shown that \( \chi_\mathcal{B} \) depends only on \( \mathcal{B} \), and not on the polynomial matrix \( R \) we used to define \( \mathcal{B} \). The poles of \( \mathcal{B} \) are defined as the roots of \( \chi_\mathcal{B} \). Note that \( \chi_{\mathcal{B}} = 1 \) if and only if \( \mathcal{B} = 0 \). A behavior is stable if and only if all its poles are in \( \mathbb{C}^- \).

Finally, we review the concept of controllability in the context of the behavioral approach. A behavior \( \mathcal{B} \in \mathcal{L}^0 \) is controllable if for all \( w_1, w_2 \in \mathcal{B} \), there exists a \( T \geq 0 \) and a \( w \in \mathcal{B} \) such that \( w(t) = w_1(t) \) for \( t < 0 \) and \( w(t + T) = w_2(t) \) for \( t \geq 0 \). A weaker notion is stabilizability, which is defined as follows. A behavior \( \mathcal{B} \) is stabilizable if for all \( w_1 \in \mathcal{B} \), there exists a \( w \in \mathcal{B} \) such that \( w(t) = w_1(t) \) for \( t < 0 \), and \( w(t) \to 0 \) as \( t \to \infty \). Thus every trajectory in a stabilizable behavior \( \mathcal{B} \), can be steered to zero asymptotically.

Often, we encounter behaviors \( \mathcal{B} \in \mathcal{L}^0 \) that are neither autonomous nor controllable. The controllable part of a behavior \( \mathcal{B} \) is defined as the largest controllable sub-behavior of \( \mathcal{B} \). This is denoted by \( \mathcal{B}_{\text{cont}} \). A given \( \mathcal{B} \in \mathcal{L}^0 \) can always be decomposed as \( \mathcal{B} = \mathcal{B}_{\text{cont}} \oplus \mathcal{B}_{\text{out}} \), where \( \mathcal{B}_{\text{cont}} \) is the (unique) controllable part of \( \mathcal{B} \), and \( \mathcal{B}_{\text{out}} \) is a (non-unique) autonomous sub-behavior of \( \mathcal{B} \). For details we refer to [7].
2 Regular implementability

Suppose we have a plant to be controlled, with two types of variables. In the given plant, the variables whose trajectories we intend to shape (called the to-be-controlled variables), are denoted by \( w \). These to-be-controlled variables can be controlled through a set of control variables \( c \), over which we can "attach" a controller. These are the variables, that can be measured and/or actuated upon. Often we have some common components in \( w \) and \( c \). We formulate the problem, however, for the general case, in which we have access to just the control variables \( c \).

Before the controller acts, there are two behaviors of the plant that are relevant: \( P_{\text{full}} \) (called the full plant behavior) that formalizes the dynamics of the variables \( w \) and \( c \), and the behavior \( P \) (called the manifest plant behavior) that formalizes the dynamics of the to-be-controlled variables \( w \) only. Thus

\[
P_{\text{full}} = \{ (w, c) \in \mathcal{L}_4^{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \mid (w, c) \text{ satisfies the plant equations}\},
\]

\[
P = \{ w \in \mathcal{L}_4^{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \mid \exists c \text{ such that } (w, c) \in P_{\text{full}} \}^\text{closure}.
\]

In this paper, we assume that the plant is a linear differential system, i.e., \( P_{\text{full}} \in \mathcal{L}^{w+c} \). The particular representation by which it is given, is immaterial to us. The manifest plant behavior \( P \) is obtained by eliminating \( c \) from \( P_{\text{full}} \), so, by the elimination theorem, \( P \in \mathcal{L}^w \).

A controller restricts the trajectories that \( c \) can assume and is described by a controller behavior \( \mathcal{C} \in \mathcal{L}^c \):

\[
\mathcal{C} = \{ c \in \mathcal{L}_4^{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \mid c \text{ satisfies the controller equations}\}.
\]

The full controlled behavior \( \mathcal{X}_{\text{full}} \) is obtained by taking the interconnection of \( P_{\text{full}} \) and \( \mathcal{C} \) through the variable \( c \) and is defined as:

\[
\mathcal{X}_{\text{full}} = \{ (w, c) \mid (w, c) \in P_{\text{full}} \text{ and } c \in \mathcal{C} \}.
\]

![Figure 1: The plant and controller after interconnection](image)

The manifest controlled behavior \( \mathcal{X} \) is obtained from \( \mathcal{X}_{\text{full}} \) by eliminating \( c \) and is defined as:

\[
\mathcal{X} = \{ w \mid \exists c \in \mathcal{C} \text{ such that } (w, c) \in P_{\text{full}} \}^\text{closure}.
\]

In that case we say that \( \mathcal{X} \) is implemented by \( \mathcal{C} \), or \( \mathcal{C} \) implements \( \mathcal{X} \) through \( c \). A given \( \mathcal{X} \in \mathcal{L}^w \) is called implementable with respect to \( P_{\text{full}} \) by interconnection through \( c \), if there exists a controller \( \mathcal{C} \in \mathcal{L}^c \), such that \( \mathcal{X} \) is implemented by \( \mathcal{C} \). If it is clear from the context, we often suppress the specifications 'w.r.t. \( P_{\text{full}} \) and 'through \( c \).

An important issue is the question which \( \mathcal{X} \in \mathcal{L}^w \) are implementable, i.e. for which \( \mathcal{X} \in \mathcal{L}^w \) there exists a controller \( \mathcal{C} \in \mathcal{L}^c \) such that (1) holds. A crucial concept to answer this question is the notion of hidden behavior: the hidden behavior \( \mathcal{N} \) is the behavior consisting of the plant trajectories that occur when the control variables are zero:

\[
\mathcal{N} = \{ w \in \mathcal{L}_4^{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \mid (w, 0) \in P_{\text{full}} \}.
\]

We have access to only the control variables \( c \) - hence the notion of \( \mathcal{N} \) being hidden from the control variables.

The following proposition from [12] settles the question of implementability for a given \( \mathcal{X} \in \mathcal{L}^w \). We refer to this proposition as the controller implementability theorem.

**Proposition 1:** Let \( P_{\text{full}} \in \mathcal{L}^{w+c} \) be a given full plant behavior, and let \( \mathcal{P}, \mathcal{N} \in \mathcal{L}^w \) be the manifest plant behavior and hidden behavior, respectively. Then \( \mathcal{X} \in \mathcal{L}^w \) is implementable w.r.t. \( P_{\text{full}} \) by interconnection through \( c \) if and only if

\[
\mathcal{N} \subseteq \mathcal{X} \subseteq \mathcal{P}.
\]

In addition to implementability issues, the hidden behavior \( \mathcal{N} \) plays a role in observability and detectability of \( P_{\text{full}} \). It can be easily seen that, in \( P_{\text{full}} \), \( w \) is observable from \( c \) if and only if \( \mathcal{N} = 0 \), and \( w \) is detectable from \( c \) if and only if \( \mathcal{N} \) is stable.

Roughly speaking, for a given \( P_{\text{full}} \) we want to find a controller \( \mathcal{C} \) such that the manifest controlled behavior \( \mathcal{X} \) has desired properties. However, we shall restrict ourselves to \( \mathcal{C} \)'s such that the interconnection of \( P_{\text{full}} \) and \( \mathcal{C} \) is regular. A motivation for this is provided in section 6. The interconnection of \( P_{\text{full}} \) and \( \mathcal{C} \) through \( c \) is called regular if

\[
p(\mathcal{X}_{\text{full}}) = p(P_{\text{full}}) + p(\mathcal{C}),
\]

i.e., if the output cardinalities of \( P_{\text{full}} \) and \( \mathcal{C} \) add up to that of \( \mathcal{X}_{\text{full}} \).

A given \( \mathcal{X} \in \mathcal{L}^w \) is called regularly implementable if there exists a \( \mathcal{C} \in \mathcal{L}^c \) such that \( \mathcal{X} \) is implemented by \( \mathcal{C} \), and if the interconnection of \( P_{\text{full}} \) and \( \mathcal{C} \) is regular. Similar to plain implementability, an important question is under what conditions a given sub-behavior \( \mathcal{X} \) of \( \mathcal{P} \) is regularly implementable. The following theorem is the main result of this section, and provides necessary and sufficient conditions for this:

**Theorem 2:** Let \( P_{\text{full}} \in \mathcal{L}^{w+c} \). Let \( \mathcal{P}, \mathcal{N} \in \mathcal{L}^w \) be the corresponding manifest plant behavior and hidden behavior respectively. Let \( P_{\text{cont}} \) be the controllable part of \( \mathcal{P} \). Let \( \mathcal{X} \in \mathcal{L}^w \). Then, \( \mathcal{X} \) is implementable w.r.t. \( P_{\text{full}} \)
The manifest controlled behavior $X$ is stable if and only if $N$ is stable (equivalently: in $\mathcal{P}_{\text{full}}$, $w$ is detectable from $c$).

The main results of this section are the following theorems, which establish necessary and sufficient conditions for pole placement and stabilization.

**Theorem 3:** Let $\mathcal{P}_{\text{full}} \in \mathbb{L}^{*+c}$. For every monic $r \in \mathbb{R}[z]$, there exists a regularly implementable $\mathcal{K} \in \mathbb{L}^s$ such that $X_{\mathcal{K}} = r$ if and only if $N = 0$ and $\mathcal{P}$ is controllable, equivalently, if and only if:

- in $\mathcal{P}_{\text{full}}$, $w$ is observable from $c$,
- $\mathcal{P}$ is controllable.

**Theorem 4:** Let $\mathcal{P}_{\text{full}} \in \mathbb{L}^{*+c}$. There exists a regularly implementable stable $\mathcal{K} \in \mathbb{L}^s$ if and only if $N$ is stable and $\mathcal{P}$ is stabilizable, equivalently, if and only if:

- in $\mathcal{P}_{\text{full}}$, $w$ is detectable from $c$,
- $\mathcal{P}$ is stabilizable.

Note that, neither in the problem formulations nor in the conditions appearing in theorems 3 and 4, do representations of the given plant appear. Indeed, our problem formulations and their resolutions are completely representation free, and are formulated purely in terms of properties of the behavior $\mathcal{P}_{\text{full}}$. Thus, our treatment of the pole placement and stabilization problems is genuinely behavioral. Of course, theorems 3 and 4 are applicable to any particular representation of $\mathcal{P}_{\text{full}}$ as well.

In both the stabilization problem and the pole placement problem, we have restricted ourselves to regular interconnections. We give an explanation for this in section 6. At this point we note that if in the above problem formulations we replace “regularly implementable” by merely “implementable”, then in the stabilization problem a necessary and sufficient condition for the existence of $\mathcal{K}$ is that $N$ is stable (equivalently: in $\mathcal{P}_{\text{full}}$, $w$ is detectable from $c$). In the pole placement problem, necessary and sufficient conditions are that $N = 0$ (i.e., in $\mathcal{P}_{\text{full}}$, $w$ is observable from $c$) and that $\mathcal{P}$ is not autonomous.

4 The filtering problem

Our general problem formulation of finding a regularly implementable, stable $\mathcal{K} \in \mathbb{L}^s$, for a given $\mathcal{P}_{\text{full}} \in \mathbb{L}^{*+c}$, includes also a problem that is, strictly speaking, not a control problem, but rather a filtering problem.

Consider the set-up of figure 2. The observed plant $\mathcal{P}_{\text{obs}} \in \mathbb{L}^{*+y}$ has two types of variables, $w$ and $y$. $w$ is a variable that we want to estimate and $y$ is a variable that we measure.

A filter is a system $\mathcal{F} \in \mathbb{L}^{*+y}$, with variables $(y, \hat{w})$. The idea is to find a filter $\mathcal{F}$ such that in the interconnection of $\mathcal{P}_{\text{obs}}$ and $\mathcal{F}$ through $y$ (the measured variable), $\hat{w}$
becomes an estimate of \( w \). In order to formalize this, for a given filter \( \mathcal{F} \) we define the associated estimation error behavior \( \mathcal{E} \) by

\[
\mathcal{E} = \{ e \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^2) \mid \exists w, \tilde{w}, y \text{ such that:} \\
(w, y) \in \mathcal{P}_{\text{obs}}, (y, \tilde{w}) \in \mathcal{F} \text{ and } e = w - \tilde{w}\} \text{ closure}. \tag{2}
\]

Here, as before, the closure is taken with respect to the topology of \( L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^2) \). If \( \mathcal{E} \), \( \mathcal{P}_{\text{obs}} \) and \( \mathcal{F} \) are related via equation (2), we say that \( \mathcal{E} \) is implemented by the filter \( \mathcal{F} \). Given \( \mathcal{P}_{\text{obs}} \in \mathbb{L}^{2+T} \), a given behavior \( \mathcal{E} \in \mathbb{L}^{2} \) is called implementable (with respect to \( \mathcal{P}_{\text{obs}} \)) if there exists a filter \( \mathcal{F} \in \mathbb{L}^{2+T} \) such that \( \mathcal{E} \) is implemented by \( \mathcal{F} \). The question what \( \mathcal{E} \)'s are implementable is answered in the following lemma. In the following, let \( N \) be the hidden behavior associated with \( \mathcal{P}_{\text{obs}} \), i.e.,

\[ N = \{ w \mid (w, 0) \in \mathcal{P}_{\text{obs}} \}. \]

Lemma 5: Let \( \mathcal{P}_{\text{obs}} \in \mathbb{L}^{2+T} \). Then we have:

1. The behavior \( \mathcal{E} \in \mathbb{L}^{2} \) is implementable if and only if \( N \subseteq \mathcal{E} \).
2. If \( \mathcal{E} \) is autonomous and implementable, it can be implemented by a filter \( \mathcal{F} \in \mathbb{L}^{2+T} \) such that, in \( \mathcal{F}, \)
   \( y \) is input and \( \tilde{w} \) output.

The problem we want to consider in this section is to find a filter that makes the estimation error behavior stable. The following theorem states when such a filter exists.

Theorem 6: Let \( \mathcal{P}_{\text{obs}} \in \mathbb{L}^{2+T} \). There exists a filter \( \mathcal{F} \in \mathbb{L}^{2+T} \) such that the estimation error \( \mathcal{E} \) is stable if and only if, in \( \mathcal{P}_{\text{obs}}, w \) is detectable from \( y \). In that case, there exists a filter such that the measured variable \( y \) is input and the estimate \( \tilde{w} \) is output.

5 Input/output partition

In the classical view of control, a controller is, in general, considered to be a feedback processor that generates control inputs for the plant on the basis of measured outputs of the plant. In our set-up, controller behaviors are obtained directly from the full plant. It is important to know a priori when such controlled behavior is implementable by a feedback processor. Results on this have been obtained in [15], [11] and [12]. We extend these results here for the problems considered in this paper.

Our first result states that if \( \mathcal{X} \in \mathbb{L}^{2+} \) is regularly implementable and autonomous (so, in particular, if it is stable or has prescribed characteristic polynomial), then for any controller \( \mathcal{C} \in \mathbb{L}^{2} \) that implements \( \mathcal{X} \) there exists a partition of the control variable \( e \) such that the interconnection of \( \mathcal{P}_{\text{full}} \) and \( \mathcal{C} \) is, in fact, a feedback interconnection:

\[ \text{Figure 3: Feedback interconnection of } \mathcal{F} \text{ and } \mathcal{C} \]

Theorem 7: Let \( \mathcal{P}_{\text{full}} \in \mathbb{L}^{2+T} \). Let \( \mathcal{X} \in \mathbb{L}^{2} \) be autonomous and regularly implementable through \( e \), and let \( \mathcal{C} \in \mathbb{L}^{2} \) be a controller that regularly implements \( \mathcal{X} \). Then, possibly after permuting its components, there exists a partition of \( e \) into \( e = (y, u_1, u_2) \) such that:

- for \( (w, y, u_1, u_2) \in \mathcal{P}_{\text{full}}, (u_1, u_2) \) is input and \( (w, y) \) is output,
- for \( (y, u_1, u_2) \in \mathcal{C}, (y, u_2) \) is input and \( u_1 \) is output,
- for \( (w, y, u_1, u_2) \in \mathcal{X}_{\text{full}}, u_2 \) is input and \( (w, y, u_1) \) is output.

As a special case, when \( \mathcal{X}_{\text{full}} \) is autonomous, we interpret \( u_2 \) as having zero components. Figure 3 depicts how the control variables are partitioned into inputs and outputs in order to implement the controller behavior in a feedback configuration.

The above theorem assigns an input/output partition without modifying the controller itself. Often, we are not allowed to choose an input/output partition, because we are given \( \text{a priori} \) that some variables are sensors, while others are actuators. Hence, necessarily, the sensors are plant outputs and should, correspondingly, be controller inputs. The actuators, then, are inputs to the plant. In the following theorem we show that if our plant \( \mathcal{P}_{\text{full}} \) has an \( \text{a priori} \) given input/output structure with respect to sensors and actuators, and if \( \mathcal{X} \in \mathbb{L}^{2} \) is regularly implementable and autonomous, then \( \mathcal{X} \) can be regularly implemented by a controller \( \mathcal{C} \in \mathbb{L}^{2} \) that takes the sensors as input, and actuates part of the plant actuators. Since \( \mathcal{X}_{\text{full}} \) is again not necessarily autonomous, some control variables remain free. These can be interpreted as plant actuators which are not being used for the control of the to-be-controlled variables.
Theorem 8: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\omega+r+2}$ with to-be-controlled variable $w$ and control variable $c = (\eta, u)$. Assume, in $\mathcal{P}_{\text{full}}$, $u$ is input and $(w, \eta)$ is output. Then, for every regularly implementable, autonomous $K \in \mathcal{L}^{\omega}$, there exist a controller $\mathcal{C} \in \mathcal{L}^{\omega}$ that implements $K$ through $c$, and a partition $u = (u_1, u_2)$ such that

- in $\mathcal{C}$, $(y, u_2)$ is input and $u_1$ is output,
- in $\mathcal{K}_{\text{full}}$, $u_2$ is input and $(w, y, u_1)$ is output.

In general, the feedback transfer functions obtained in the above two theorems, are singular. In [15] it has been argued that many applications of control do not require the properness condition of the feedback transfer function and that the properness condition is, nevertheless, a very important special case.

6 Disturbances and regular interconnection

In section 3 we have formulated the problems of stabilization and pole placement for a given plant $\mathcal{P}_{\text{full}}$ with to-be-controlled variable $w$ and control variable $c$. In most system models, an unknown external disturbance variable, $d$, also occurs. The stabilization problem is then to find a controller acting on $c$ such that whenever $d(t) = 0 (t \geq 0)$, we have $w(t) \to 0 (t \to \infty)$. Typically, the disturbance $d$ is assumed to be free, in the sense that every $\mathcal{L}^{\omega}$ function $d$ is compatible with the equations of the model. As an example, think of a model of a car suspension system given by $R_1 \frac{\partial w}{\partial t} + R_2 \frac{\partial c}{\partial t} + R_3 \frac{\partial d}{\partial t} = 0$, where $d$ is the road profile as a function of time. In the stabilization problem, one puts $d = 0$ and solves the stabilization problem for the full plant $\mathcal{P}_{\text{full}}$ represented by $R_1 \frac{\partial w}{\partial t} + R_2 \frac{\partial c}{\partial t} = 0$. In doing this, one should make sure that the stabilizing controller $\mathcal{C}$: $C\left(\frac{\partial}{\partial t}\right)c = 0$, when connected to the actual model, does not put restrictions on $d$. The notion of regular interconnection captures this, as explained below:

Consider the full plant behavior $\mathcal{P}_{\text{full}} \in \mathcal{L}^{\omega+r+2}$. An extension of $\mathcal{P}_{\text{full}}$ is a behavior $\mathcal{P}^{\text{ext}}_{\text{full}} \in \mathcal{L}^{\omega+r+2}$ (with $d$ an arbitrary positive integer), with variables $(w, c, d)$, such that

- $d$ is free in $\mathcal{P}^{\text{ext}}_{\text{full}}$,
- $\mathcal{P}_{\text{full}} = \{(w, c, 0) \mid (w, c, 0) \in \mathcal{P}^{\text{ext}}_{\text{full}}\}$.

Thus, $\mathcal{P}^{\text{ext}}_{\text{full}}$ being an extension of $\mathcal{P}_{\text{full}}$ formalizes that $\mathcal{P}_{\text{full}}$ has exactly those signals $(w, c)$ that are compatible with the disturbance $d = 0$ in $\mathcal{P}_{\text{ext}}$. Of course, a given full behavior $\mathcal{P}_{\text{full}}$ has many extensions.

For a given extension $\mathcal{P}^{\text{ext}}_{\text{full}}$ and a given controller $\mathcal{C} \in \mathcal{L}^{\omega}$, we define the extended controlled behavior by

$\mathcal{K}^{\text{ext}}_{\text{full}} = \{(w, c, d) \mid (w, c, d) \in \mathcal{P}^{\text{ext}}_{\text{full}} \text{ and } c \in \mathcal{C}\}$.

A controller $\mathcal{C}$ shall be acceptable only if the disturbance $d$ remains free in $\mathcal{K}^{\text{ext}}_{\text{full}}$ for any possible extension $\mathcal{P}^{\text{ext}}_{\text{full}}$. It turns out that this is guaranteed exactly, by the regularity of the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$.

Theorem 9: The following statements are equivalent.

- The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular,
- for any extension $\mathcal{P}^{\text{ext}}_{\text{full}}$ of $\mathcal{P}_{\text{full}}$, $d$ is free in $\mathcal{K}^{\text{ext}}_{\text{full}}$.

References