Chapter 3

The parametrically forced pendulum: a case study in $1\frac{1}{2}$ degree of freedom

This chapter is concerned with the global coherent (i.e., non-chaotic) dynamics of the parametrically forced pendulum. The system is studied in a $1\frac{1}{2}$ degree of freedom Hamiltonian setting with two parameters, where a spatio-temporal symmetry is taken into account. Our explorations are restricted to sufficiently large regions of coherent dynamics in phase space and parameter plane. At any given parameter point we restrict to a bounded subset of phase space, using KAM theory to exclude an infinitely large region with trivial dynamics.

In the absence of forcing the system is integrable. Analytical and numerical methods are used to study the dynamics in a parameter region away from integrability, where the results of a perturbation analysis of the nearly integrable case are used as a starting point. We organize the dynamics by dividing the parameter plane in fundamental domains, guided by the linearized system at the upper and lower equilibria.

Away from integrability some features of the nearly integrable coherent dynamics persist, while new bifurcations arise. On the other hand, the chaotic region increases.

3.1 Introduction

We consider a parametrically forced pendulum in a Hamiltonian $1\frac{1}{2}$ degree of freedom setting, given by the equation of motion

$$\ddot{x} = (\alpha + \beta \cos t) \sin x.$$  \hspace{1cm} (3.1)

Here $x \in \mathbb{S}^1$ is the deviation from the upper equilibrium and ranges over the whole circle. The equation is given in ‘inverted pendulum format’, meaning that $x = 0$ corresponds to the upper equilibrium, i.e., where the pendulum stands up, while $x = \pi$ denotes the lower equilibrium, i.e., where it hangs down. The independent variable is $t \in \mathbb{S}^1$. The parameters $\alpha, \beta \in \mathbb{R}$ correspond to the square of the eigenfrequency of the (free) pendulum at
the lower equilibrium, and the amplitude of the forcing, respectively. Indeed, \( \alpha = \sqrt{g/\ell} \), where \( g \) denotes the gravitational acceleration and \( \ell \) the length of the pendulum. Without loss of generality we restrict to \( \alpha, \beta \geq 0 \).

Figure 3.1: Numerically computed stability diagram of the parametrically forced pendulum (3.1) on a large scale, revealing a checkerboard structure. On the curves going from the left towards the top side the stability of the upper equilibrium changes, on the curves coming from the \( \alpha \) axis the lower equilibrium changes stability. The stable regions above the line \( \alpha = \beta \) are extremely small, and not visible at this scale. Below this line the upper equilibrium is unstable and the lower is mostly stable.

This system is widely studied within the context of classical perturbation theory, in several regions in phase and parameter space. The stability of the upper and lower equilibria of the pendulum is determined by the Mathieu equation, that is, the linearized equation at these points. See figure 3.1 for a stability diagram, also compare van der Pol and Strutt [PS28], Stoker [Sto50], Meixner and Schäffke [MS54], Hale [Hal63, Hal69], Weinstein and Keller [WK85, WK87], Levi et al. [Lev88, LW95], and Broer et al. [BL95, BS98, BS00]. Except near the diagonal \( \alpha = \beta \), every line in figure 3.1 consists, in fact, of two stability curves, that are extremely close together. Hence the stable regions of the upper equilibrium are extremely narrow, as are those of the lower equilibrium above the diagonal, while below this diagonal the unstable regions of the lower equilibrium are very small. At the diagonal, consider up to some \( \alpha_{\max} \) the measure of the intervals where the lower equilibrium is stable. It tends to \( \alpha_{\max}/2 \) as \( \alpha_{\max} \to +\infty \), cf. [BLS].

For small \( \beta/\alpha \) the system is nearly integrable. This permits a local nonlinear bifurcation analysis at the two equilibria, near the resonance points of the lower equilibrium on
the α axis, see Broer and Vegter [BV92], and near the degenerate point \((\alpha, \beta) = (0, 0)\), see chapter 2 and Broer et al. [BHN98, BHN99].

For sufficiently large velocity \(y = \dot{x}\), depending on the parameters, the existence of invariant circles can be shown using KAM theory, e.g., see Moser [Mos89a, Mos89b] and You [You90], also compare Levi [Lev90] for a discussion of a similar system. These circles form a Cantor foliation that tends to full measure as \(y \to +\infty\).

### 3.1.1 Setting of the problem

The goal of this chapter is to explore the coherent (i.e., non-chaotic) dynamics of the forced pendulum, in dependence of the parameters \((\alpha, \beta)\), as a case study in 1 \(\frac{3}{2}\) degree of freedom. By coherent dynamics we mean all non-chaotic phenomena, that is, all periodic and quasi-periodic dynamics, and their bifurcations. Emphasis lies on orbits of low period, since these usually generate the largest stability islands, cf. [LL92].

In this study we combine perturbation theory and numerical tools, adopting the programme of Broer et al. [BST98]. Indeed, analytical results obtained in certain parameter regions serve as a framework and a starting point for numerical continuation to a larger part of parameter space. Furthermore, at representative parameter points phase portraits are computed. Such an interaction between analytical and numerical methods seems a fruitful approach in the study of complicated systems.

\[
\begin{align*}
\text{PF: pitchfork bifurcation} & \quad \Diamond \quad \text{unstable period 2 points} \\
\text{PD: period doubling bifurcation} & \quad \Box \quad \text{stable period 2 points} \\
\text{SC: saddle-center bifurcation} & \quad \Downarrow \quad \text{unstable fixed points} \\
\text{TC: transcritical bifurcation} & \quad \Delta \quad \text{stable fixed points} \\
\text{HC: heteroclinic bifurcation} & \\
\hline
\text{--- stable manifold upper equilibrium} & \\
\text{---- unstable manifold upper equilibrium} & \\
\text{----- stable manifold lower equilibrium} & \\
\text{------ unstable manifold lower equilibrium} & \\
\end{align*}
\]

Figure 3.2: Coding of periodic points, invariant manifolds of equilibria, and bifurcations of the Poincaré map, used in phase portraits, bifurcation and stability diagrams.
Our explorations mostly deal with the Poincaré map $P$ of the forced pendulum, defined on the cylindrical section $t = 0 \mod 2\pi \mathbb{Z}$. This map has a spatial and a temporal symmetry. For any parameter point $(\alpha, \beta)$, the coherent dynamics of $P$ at sufficiently large $|y|$ consists of invariant circles with strings of islands in between, as illustrated in figure 3.3, also see figure 3.2 for the coding of periodic points and bifurcations that will be used throughout this chapter. By KAM theory we exclude this region, thereby restricting to a 'region of interest' in phase space. (For the free pendulum this coincides with the region between the separatrices.)

In the 'region of interest', normal form theory yields an integrable approximation to $P$, valid for small $\beta/\alpha$. This is continued numerically to larger parameter values. As the parameters increase, new bifurcations arise, that are studied as well. Furthermore, the 'region of interest' fills with chaos, and stable and unstable manifolds of saddle points are computed to visualize its structure.

3.1.2 Method and sketch of the results

Let us describe the results of this chapter and the methods by which they are obtained. First the bound on the 'region of interest' is discussed. Then we introduce the concept of fundamental domains in the parameter plane. Finally the dynamics is studied in some of these domains.

Region of interest

As mentioned before, for any parameter point we restrict to a bounded 'region of interest' in phase space. Let us present some quantitative results on the size of this region. It seems to be bounded by $y \approx \pm 2\sqrt{\alpha + \beta}$ for all $(\alpha, \beta) \in \mathbb{R}_{\geq 0}$. This estimate follows from a numerical computation on a large part of the parameter plane. The same estimate, valid on parts of the parameter plane, is obtained by adiabatic theory. The estimate is very accurate for large $\alpha$ and $\beta$, but not so good near the origin of the parameter plane. We observe that for some $(\alpha, \beta)$ the 'region of interest' also includes orbits that in the integrable case $\beta = 0$ lie outside this region. For example, the large stable domains above and below the lower equilibrium in figure 3.3 lie inside the 'region of interest', but correspond to a resonant invariant circle with rotation number one in the integrable system.

A rigorous bound on the 'region of interest' can be obtained by quantitative KAM theory. We pass to action-angle coordinates to get a well-defined perturbation problem, where the fast pendulum, with unbounded $y$, $\alpha$ and $\beta$, is the unperturbed case. We sketch how such a bound can be computed, for more concrete results referring to future research, cf. [BNS].

Fundamental domains

We organize our explorations by dividing the parameter plane into so-called fundamental domains, based on the stability types of the upper and lower equilibria. Figure 3.4 shows the corresponding stability diagrams in the $(\alpha, \beta)$ plane, where shading indicates
Figure 3.3: Phase portraits of the Poincaré map at $\alpha = 0.079$ and various $\beta$, as indicated. Outside a ‘region of interest’, the dynamics consists mainly of invariant circles winding around the cylinder. Inside, chaos increases with $\beta$, but coherent dynamics remains present. Both equilibria undergo period doubling and pitchfork bifurcations. Parts of invariant manifolds of (unstable) equilibria are also plotted. Periodic points and invariant manifolds are marked according to the coding in figure 3.2. There are (unmarked) fixed points at $x = \pi$, corresponding to a resonant orbit of the vector field $X$ with frequency 1 in the $x$-direction.
stability. The stability boundaries are curves of pitchfork (PF) or period doubling (PD) bifurcations, as indicated. The pitchfork bifurcation has codimension one due to the spatial symmetry of the system, see below.

Combining the two stability diagrams we see that the parameter plane consists of regions of four different types, depending on the stability types of the two equilibria. These regions are collected mostly in groups of four, one of each type, forming what will be called fundamental domains, again see figure 3.4, also compare figure 3.5 for a sketch of one domain. Most fundamental domains belong to four types, depending on the bifurcation type of their boundaries. Indeed, these domains have four sides, where opposite sides correspond to different bifurcation types, either pitchfork or period doubling, leading to a total of four possibilities.

The fundamental domains mostly appear in blocks of $2 \times 2$ domains, one of each type, as illustrated in figure 3.6. Moving across such a block, both `horizontally' and `vertically', one intersects two stability boundaries of PF type (where the trace of the linearized Poincaré map at the relevant equilibrium equals 2), two of PD type (where the trace is $-2$), and finally another one of PF type, in that order.

The stability types and bifurcations of the upper and lower equilibria are the same in fundamental domains of the same type. Since the stability islands around centers of lower period are usually the largest, the upper and lower equilibria are expected to influence the dynamics mostly. Therefore we restrict our study to a few fundamental domains near the origin of the parameter plane and farther away, and conjecture that the dynamics in other domains is similar. This conjecture is supported by the fact that the dynamics in the domains away from the origin is mainly chaotic (in the `region of interest') and does not vary much. For the domains near the origin this is not true. Let us discuss both kinds of domains separately.

**Remark 20**: Below the diagonal $\alpha = \beta$ there is a single row of fundamental domains. As noted before, the lower equilibrium is mostly stable here, and hence this region contains the largest part of the coherent, near integrable dynamics of the full problem.
Figure 3.5: Sketch of one fundamental domain, with an indication of the stability types and bifurcations of the two equilibria. The bifurcations on the stability boundaries depend on the type of fundamental domain.

Coherent dynamics near the origin of the parameter plane

The nearly integrable system for small $\beta$ serves as a starting point for our study. This can be treated as a perturbation problem, in a neighborhood of any given point $(\alpha, \beta) = (\alpha_0, 0)$ on the $\alpha$-axis, with $\beta$ and a ‘detuning’ parameter $\delta = \alpha - \alpha_0$ serving as perturbation parameters. It yields an integrable approximation to the Poincaré map, see [BV92, BHN98, BHNV99] and chapter 2. In the present chapter the results of the perturbation analysis are extended to the non-integrable case by numerical continuation, where the symmetries are taken into account. We also simply compute phase portraits of $P$ at representative parameter points.

Let us first briefly discuss the nearly integrable dynamics. In [BV92] normal forms for the integrable approximating map are given at the resonance points of the lower equilibrium $(\alpha, \beta) = (\frac{1}{4}k^2, 0)$, $k = 1, 2, \ldots$. Their local phase portraits at the lower equilibrium are reproduced in figure 3.7, compare [BV92, figure 8]. Up to a conjugacy the map $P$ is an exponentially small perturbation of the integrable approximation as the parameters go to the resonance point on the $\alpha$ axis and $(x, y)$ tend to the lower equilibrium. Similar phase portraits for the degenerate point $(\alpha, \beta) = (0, 0)$ of the upper equilibrium are given in figure 2.4 in chapter 2, [BHN98, figures 2, 6] and [BHNV99, figures 2, 4], and reproduced in figure 3.8. In this case the integrable approximation is exponentially good as $(y, \alpha, \beta) \to (0, 0, 0)$.

Some features of the nearly integrable dynamics, like periodic points and their bifur-
cations, persist as the perturbation increases. Because of the symmetries, the periodic points of $P$ bifurcating off at upper and lower equilibria have to move either in the $x$ or in the $y$ direction. Numerical continuation shows that as $\beta$ increases, they either go towards the upper equilibrium, or escape from the equilibria in the $y$ direction. As an example, figure 3.9 displays bifurcation diagrams of both equilibria for $\alpha = 0.079$, with $\beta$ as bifurcation parameter.

We also find secondary bifurcation points on the branches bifurcating from the upper and lower equilibria, that are not found in the perturbation analysis of chapter 2 and [BHN98, BHNV99, BV92], since this analysis only concerns a small neighborhood in phase-parameter space of a resonant or degenerate equilibrium. These bifurcations are also pitchfork or period doubling points, again see figure 3.9, and because of the spatio-temporal symmetry the crossing branches have to be tangent to the $x$- or $y$-axis at the bifurcation point. There is numerical evidence that they are the start of period doubling cascades, see [KH98, McL81].

More bifurcations can be found. For example, figure 3.3 shows large stable islands above and below the lower equilibrium, with the central fixed point undergoing 1:3 and 1:2 resonance bifurcations. In section 3.4.4.3 below we find that it also has a period doubling bifurcation. Apparently the eigenvalues of the linearized Poincaré map at the fixed point move along the complex unit circle from the saddle-center position (where both eigenvalues are 1) at $\beta = 0$ to the period doubling position (where the eigenvalues are -1) at some nonzero $\beta$, passing through all resonances $e^{\pm 2\pi i p/9}$ in between.
Figure 3.7: Stability diagram and phase portraits of an integrable approximation to $P$ at a resonance point $(\alpha, \beta) = (\frac{1}{4}k^2, 0)$, $k \in \mathbb{Z}_{>0}$, of the lower equilibrium. The stability boundaries have $k$-th order of contact at the resonance point (the case $k = 1$ is displayed). The bifurcations on the boundaries are of pitchfork type if $k$ is even, and of period doubling type otherwise, and thus generate fixed and period two points, respectively. Apart from this, in our case of spatio-temporal symmetry, the phase portraits of the integrable approximation are qualitatively the same for all $k$.

Figure 3.8: Stability diagram and phase portraits of an integrable approximation to $P$ at the degenerate point $(\alpha, \beta) = (0, 0)$ of the upper equilibrium. The stability boundary consists of pitchfork bifurcations.
Figure 3.9: Bifurcation diagrams on the parameter line $\alpha = 0.079$ in $(\beta, x, y)$-space, and projected onto the $(\beta, x)$- and $(\beta, y)$-planes. They display period one and two points of $P$ bifurcating from the upper and lower equilibria. The lines parallel to the $\beta$-axis correspond to the positions of the equilibria. For simplicity only the branches in one quadrant of $(\beta, x, y)$-space are displayed, the others are their symmetric counterpart under reflection in the $(\beta, x)$- or $(\beta, y)$-plane.
Coherent dynamics away from the origin of the parameter plane

As the perturbation increases, chaos seems to fill the ‘region of interest’ in phase space, compare figure 3.3. However, coherent dynamics does remain with small measure for all $\alpha, \beta$. Moreover, the total relative measure of the set of islands is of order one as $(\alpha, \beta) \to \infty$ with $\alpha > \beta$, as shown by Neishtadt et al. [NST97], compare also [BLS] for the case $\alpha < \beta$. In a qualitative sense the dynamics does not seem to vary much with the parameters.

3.1.3 Overview

Let us give a short overview of the remainder of this chapter. In section 3.2 the system is introduced and some properties are discussed. We also present a codification of the fundamental domains. Section 3.3 applies KAM theory to show persistence of Diophantine invariant circles of $P$, and gives estimates on the measure of these circles and on the ‘region of interest’. Appendix 3.A discusses a numerical method for obtaining such an estimate. In section 3.4 the Poincaré map $P$ is studied in several fundamental domains. The integrator used in the numerical explorations is based on Taylor series expansions, and for completeness is discussed in appendix 3.B, see also [BS98].

3.2 Preliminaries

In this section some properties of the system, like symmetries, are discussed. Moreover, a codification of the fundamental domains is given.

Properties of the system

The system (3.1) of the parametrically forced pendulum can be written as a vector field

$$X(x, y, t; \alpha, \beta) = \frac{\partial}{\partial t} + y \frac{\partial}{\partial x} - (\alpha + \beta \rho(t)) V'(x) \frac{\partial}{\partial y},$$  

(3.2)

where $y = \dot{x}$ denotes velocity as before, $(x, y, t) \in S^1 \times \mathbb{R} \times S^1$, $\rho(t) = \cos t$, $V(x) = \cos x - 1$, and $(\alpha, \beta) \in \mathbb{R}^2$. This vector field is Hamiltonian, with time-dependent Hamilton function

$$H(x, y, t; \alpha, \beta) = \frac{1}{2} y^2 + (\alpha + \beta \rho(t)) V(x),$$  

(3.3)

implying that it is divergence free.

The vector field $X$ has several symmetries. Since $V$ is even, it has a spatial symmetry given by $S : (x, y, t) \mapsto (-x, -y, t)$. This means that $S_* X = X$, and $X$ is called $S$-equivariant. Furthermore, $X$ is $R$-reversible, meaning that $R_* X = -X$, where the temporal symmetry $R : (x, y, t) \mapsto (x, -y, -t)$ is due to the evenness of $\rho$. There are other symmetries, involving both phase and parameter space. Indeed, since $V(x) = -V(x + \pi)$, $X$ is $T$-equivariant, where $T : (x, y, t; \alpha, \beta) \mapsto (x + \pi, y, t; -\alpha, -\beta)$. Finally, $X$ is $U$-equivariant, with $U : (x, y, t; \alpha, \beta) \mapsto (x, y, t + \pi; \alpha, -\beta)$, because $\rho(t) = -\rho(t + \pi)$. By
these two symmetries without loss of generality we can restrict to the first quadrant \( \alpha, \beta \geq 0 \) of the parameter plane.

**Remark 21**: In chapter 2 and in [BHN98, BHNV99, BV92] more general cases of \( V \) and \( \rho \) are considered, where the symmetries \( S \) and \( R \) are optional, while \( T \) and \( U \) play no role. In the present chapter we restrict to the simplest case with maximal symmetry, and take the potential function \( V(x) = \cos x - 1 \) of the pendulum and the forcing function \( \rho(t) = \cos t \) of the classical Mathieu case.

Since the system is \( 2\pi \)-periodic in time, it it natural to consider its Poincaré or stroboscopic map \( P \) on the section \( t = 0 \), corresponding to the flow over time \( 2\pi \) of \( X \). The map \( P \) is defined implicitly by

\[
(P(x, y), 2\pi) = X^{2\pi} (x, y, 0),
\]

where \( X^t \) is the flow over time \( t \) of \( X \). Since \( X \) is divergence free, \( P \) is area and orientation preserving. Moreover it inherits the symmetries of \( X \). This means that \( P \) is \( S \)-equivariant and \( R \)-reversible, that is, \( SPS = P \) and \( RPR = P^{-1} \), where \( S : (x, y) \mapsto (-x, -y) \) and \( R : (x, y) \mapsto (x, -y) \). The symmetries \( T \) and \( U \) also carry over to symmetries of \( P \) in a trivial way.

**Codification of the fundamental domains**

![Figure 3.10: The stability diagram of the parametrically forced pendulum is a checkerboard of fundamental domains, that mostly consist of four stability regions. The domains are bounded by thickened (black) curves and codified as indicated. In the shaded regions the lower or upper equilibrium is stable. (In many domains these regions are too narrow to be seen.)](image-url)
As said before, the parameter plane is divided into fundamental domains, based on the stability types of the upper and lower equilibria. These are determined by the linearized system, given by the so-called Mathieu equation
\[ \dot{x} = (\alpha + \beta \rho(t))x. \]

The Poincaré map of the Mathieu equation equals the linearization of the map \( P \) at the upper equilibrium. Thus the Mathieu equation at the parameter point \((\alpha, \beta)\) determines the stability of the upper equilibrium \((x, y, t) = (0, 0, t)\) at \((\alpha, \beta)\), and, by the symmetry \(T\) of the nonlinear system, the stability of the lower equilibrium \((x, y, t) = (\pi, 0, t)\) at \((-\alpha, -\beta)\). Hence, combining the stability diagram of the Mathieu equation with a copy rotated around the origin over angle \(\pi\) results in a diagram where each parameter point shows the stability type of both equilibria. (Because of the symmetry \(U\) the copy can also be reflected in the \(\beta\) axis instead of rotated). This turns the parameter plane into a checkerboard, symmetric with respect to the \(\alpha\) and \(\beta\) axes, see figures 3.1 and 3.4. The ‘squares’ of the checkerboard will be called ‘fundamental domains’, and mostly consist of four stability regions, as explained in section 3.1.2 and shown in figure 3.4(c).

We introduce a codification of the checkerboard. Each fundamental domain can be identified by a pair (column number, row number), starting with \((1, 1)\), as illustrated in figure 3.10. The stability regions in a fundamental domain can be identified by a third label equal to one of the strings ‘UU’, ‘US’, ‘SU’, ‘SS’, where the former (latter) letter determines the stability of the lower (upper) equilibrium; ‘U’ means unstable and ‘S’ stable.

**Remark 22**: Some of the domains, namely those along the \(\beta\) axis, do not have four stability regions. We can add these fundamental domains with their symmetric counterparts (with respect to the \(\beta\) axis) to get squared domains. A disadvantage is that the codification of the four stability regions in such a square is different from that in squares lying entirely in the right half plane. Indeed, in the top left there is an ‘SU’ region (where the system has a stable lower and unstable upper equilibrium), while the top left regions of the ordinary squares are of type ‘US’ (i.e, with reversed stability types). Therefore we shall not use this.

### 3.3 Invariant circles and the ‘region of interest’

We restrict our study of the coherent dynamics of the map \( P \) to a ‘region of interest’ in phase space, by excluding an infinitely large region of quasi-periodic dynamics interlaced with thin strings of islands. At any given parameter point, the ‘region of interest’ is bounded.

It is well known from KAM theory that for \( y \) sufficiently large and satisfying a Diophantine condition, the dynamics of \( P \) is almost completely quasi-periodic, compare Moser [Mos89a, Mos89b], Chierchia and Zhelnder [CZ89], and You [You90], also see Levi [Lev90, Lev91, LL91] for a similar system. In between these invariant circles of quasi-periodic motion one generically expects only strings of islands, also called Poincaré-Birkhoff chains. Indeed, the Poincaré-Birkhoff theorem [Bir13, Bir25] implies that in
between any two invariant circles there exist periodic points of all intermediate rational rotation numbers. Moreover, the invariant circles form a Whitney smooth Cantor foliation of phase space that tends to full measure as $y \to +\infty$, see e.g., Broer et al. [BST98] and Pöschel [Pö82]. Thus the dynamics for $y$ sufficiently large is well known, and we can restrict to a compact ‘region of interest’ in phase space by excluding the region of invariant circles and strings of islands, as announced in the introduction.

**Remark 23**: Another well known perturbative setting is where the integrable case $\beta = 0$ serves as unperturbed system, and persistence of Diophantine invariant circles can be shown for $\beta$ sufficiently small, by a direct application of KAM theory, see Kolmogorov [Kol54], Arno’ld [Arn63], and Moser [Mos62], also compare Huang [Hua98].

We obtain the following results. At a given parameter point $(\alpha, \beta)$, invariant circles of $P$ with Diophantine rotation number exist for $y > C(\alpha, \beta)$, for some increasing function $C > 0$. Moreover, the relative measure of invariant circles is $1 - O(1/y)$ as $y \to +\infty$, i.e., for all parameter values the invariant circle of $P$ at $y = +\infty$ is a Lebesgue density point of the set of invariant circles.

**Remark 24**: In some cases a better, exponential estimate can be obtained by averaging to a higher order in $\varepsilon$, see e.g. Jorba and Villanueva [JV97] for a discussion of real-analytic Hamiltonian systems with a quasi-periodic reducible invariant torus. We expect that a similar approach will be successful in our case, and will report on this in [BHN⁺].

Using a quantitative KAM theorem the location of the ‘lowest’ invariant circle of $P$ can be estimated, thus providing a bound on the ‘region of interest’ for sufficiently large $y$ at each $\alpha$ and $\beta$. Here ‘lowest’ invariant circle means an invariant circle winding around the cylindrical phase space, and sufficiently close to the circle $y = 0$. However, a quantitative KAM theory is rather complicated, involving normalization of a vector field given by elliptic integrals. Therefore we only indicate how such a bound can be obtained, and refer for sharp results to future research [BNS].

The location of the ‘lowest’ invariant circle of $P$ also can be estimated by numerical means. We demonstrate that, in the region $[0,1000] \times [0,1000]$ in the parameter plane, the ‘lowest’ invariant circle passes approximately through the point $(x, y) = (\pi, 2\sqrt{\alpha} + \beta)$ in phase space. Thus the ‘region of interest’ is approximately bounded by $y = 2\sqrt{\alpha} + \beta$. This result is supported by estimates $y = 2\sqrt{\alpha}$, to first approximation as $\beta \to 0$, and $y = 2\sqrt{\alpha} + \beta$, to first approximation as $\alpha > \beta \to \infty$. These estimates are based on adiabatic theory.

In the next subsection the persistence of invariant circles is shown qualitatively, and we obtain the asymptotics of the relative measure. Subsections 3.3.2 and 3.3.3 are concerned with the adiabatic and numerical estimates, respectively.

### 3.3.1 Persistence and measure of invariant circles

To obtain a persistence result for large $y$ we first need a well defined perturbation problem. The vector field of the forced pendulum is rescaled to a slowly varying system, and for this
system action-angle coordinates are constructed. After some averaging transformations, the vector field at large \( y \) can be seen as a small perturbation of an integrable vector field, corresponding to the system at \( y = +\infty \) in the original coordinates, that has a family of invariant tori corresponding to level sets of the action. In the perturbed setting many of these still give rise to quasi-periodic dynamics, and their asymptotical relative measure can be estimated for \( y \to +\infty \). Furthermore, in this perturbative setting a rigorous theoretical bound on the location of the 'lowest' invariant circle may be obtained, valid for large \( y \), and we sketch how such a bound can be found. For simplicity we work with the vector field of the forced pendulum, but similar results hold for the corresponding Poincaré map.

In the following lemma the vector field is rescaled to a system depending on a slowly varying parameter \( t \). This can be seen as a perturbation of the frozen system, where this parameter is kept fixed. The frozen system is (in the original variables) located at \( y = +\infty \).

**Lemma 25 (rescaling)** Let \( X = y \frac{\partial}{\partial x} + (\alpha + \beta \cos t) \sin x \frac{\partial}{\partial y} + \frac{\partial}{\partial t} \) be the vector field of the forced pendulum. Let the rescaling

\[
(x, y, t; \alpha, \beta) \mapsto (\tilde{x}, \tilde{y}, \tilde{t}; \tilde{\alpha}, \tilde{\beta}, \varepsilon), \quad X \mapsto \tilde{X}
\]

be given by

\[
\tilde{x} = x, \quad \tilde{y} = \varepsilon y, \quad \tilde{t} = t, \quad \tilde{\alpha} = \varepsilon^2 \alpha, \quad \tilde{\beta} = \varepsilon^2 \beta, \quad \varepsilon = (\alpha + \beta)^{-1/2}, \quad \tilde{X} = \varepsilon X.
\]

Then the rescaled vector field is given by

\[
X(x, y, t; \alpha, \beta, \varepsilon) = y \frac{\partial}{\partial x} + (\alpha + \beta \cos t) \sin x \frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial t},
\]

where the tildes are dropped for simplicity. It is Hamiltonian, with time-dependent Hamilton function

\[
H(x, y, t; \alpha, \beta, \varepsilon) = \frac{1}{2} y^2 + (\alpha + \beta \cos t)(\cos x - 1).
\]

The proof is a straightforward computation, and is therefore omitted. The new parameters \( \alpha, \beta \) satisfy \( \alpha + \beta = 1 \). The rescaled vector field \( \tilde{X} \) can be seen as a perturbation of the integrable, frozen vector field

\[
X(x, y, t; \alpha, \beta, 0) = y \frac{\partial}{\partial x} + (\alpha + \beta \cos t) \sin x \frac{\partial}{\partial y}.
\]

The dynamics of the frozen system in the \((x, y)\)-cylinder depends on the value of the 'parameter' \( t \), see figure 3.11, while its integral curves coincide with the level sets of the Hamiltonian. The region above the frozen separatrix is filled with invariant circles of the frozen system, and we want to show that some of these circles persist under the perturbation.
Figure 3.11: Sketch of the phase portrait of the unperturbed, frozen system $X_t = 0$. If \( \alpha + \beta \cos t > 0 \), then the upper equilibrium is unstable and the lower is stable. If \( \alpha + \beta \cos t < 0 \) (can only occur if \( \beta > \alpha \)), the stability types are reversed. If \( \alpha + \beta \cos t = 0 \) (only possible if \( \beta \geq \alpha \)), then the system is highly degenerate: all points on the $x$ axis are in equilibrium.

Since the frequency in $t$ is zero, in the current coordinates the unperturbed system is still highly degenerate. We therefore introduce action-angle coordinates $(\varphi, I)$, depending\(^1\) on $(x, y, t)$, in the region above the frozen separatrix, by a well-known procedure, cf. e.g., Arnol’d [Arn89, §52E] or Neishtadt et al. [NST97]. For given $(x, y, t)$ in this region, let $h = H(x, y, t)$ be the (constant) value of the Hamiltonian on the frozen integral curve through the point $(x, y, t)$. The action $I$ is defined to be the area in $(x, y)$-space enclosed by the frozen integral curve $H = h$ and the $x$-axis, divided by $2\pi$. The angle $\varphi$ is the time it takes to travel (in the frozen system) along the curve from $x = 0$ to the given point $(x, y)$, divided by the period $T$ of the curve, and multiplied by $2\pi$. It is convenient to consider $\varphi$ and $I$ as functions of $(x, h, t)$ rather than $(x, y, t)$. Moreover, $I$ does not depend on $x$. Thus $\varphi$ and $I$ are defined to be

\[
\varphi = \varphi(x, h, t) = \frac{2\pi}{T} \int_0^x \frac{1}{y} \, dx, \quad \text{where} \quad T = T(h, t) = \int_0^{2\pi} \frac{1}{y} \, dx
\]

\[
I = I(h, t) = \frac{1}{2\pi} \int_0^{2\pi} y \, dx.
\]

The factor $2\pi$ ensures that $\varphi$ is $2\pi$-periodic. Since $h = h(I, t)$, the period $T$ can be seen as a function of $I$ and $t$: $T = T(I, t)$. In all the integrals, $x \mapsto (x, y)$, with $y = y(x, h, t)$, is a parameterization of the frozen integral curve $H = h$, i.e.,

\[
y(x, h, t) = \sqrt{2h - 2(\alpha + \beta \cos t) \cos x}.
\]

Thus the integrals are elliptic, and by the substitution $u = \cos x$ they can be written in the standard form, that is, as an integral of a rational function in $u$ and the square root of a polynomial expression in $u$.

\(^1\)The action-angle coordinates also depend on the parameters $\alpha$ and $\beta$. For simplicity we suppress this dependence.
Lemma 26 Let \((\varphi, I)\) be the action-angle coordinates introduced above. Then the coordinate change \((x, y, t) \mapsto (\varphi, I, t)\) is symplectic, meaning that \(dx \wedge dy = d\varphi \wedge dI\). In the new coordinates the rescaled vector field of lemma 25 is given by

\[
X(\varphi, I, t) = X_0(I, t) + X_1(\varphi, I, t), \quad \text{where}
\]

\[
X_0(I, t) = \omega(I, t) \frac{\partial}{\partial \varphi} + \varepsilon \frac{\partial}{\partial t},
\]

\[
X_1(\varphi, I, t) = O \left( \frac{\varepsilon}{\omega(I, t)^2} \right) \frac{\partial}{\partial \varphi} + O \left( \frac{\varepsilon}{\omega(I, t)} \right) \frac{\partial}{\partial t} \quad \text{as} \; \varepsilon \to 0, \; \omega(I, t) \to +\infty.
\]

Here \(\omega(I, t) = \frac{2\pi}{T(I, t)}\). The Hamiltonian \(H\) of \(X\) is defined by

\[
H(\varphi, I, t) = H_0(I, t) + H_1(\varphi, I, t), \quad \text{where}
\]

\[
H_0(I, t) = \int^t \omega(I, t) dt, \quad H_1 = O \left( \frac{\varepsilon}{\omega(I, t)} \right).
\]

The proof is straightforward and hence omitted. We observe that \(\omega \to +\infty\) as \(I \to +\infty\); in fact, \(\omega \approx I\) near infinity.

The unperturbed system \(X_0\) has invariant tori \(I = \text{constant}\). Its frequency in \(\varphi\)-direction depends on \(I\), obstructing a direct application of KAM theory. Therefore we average \(X\), first with respect to \(\varphi\) to make the perturbation in \(I\)-direction smaller (i.e., of higher order in \(\varepsilon\) and \(\omega^{-1}\)), and then with respect to \(t\). Indeed, since the \(\frac{\partial}{\partial \varphi}\)-component of \(X\) equals \(-\frac{\partial}{\partial \varphi} H_1\), its average with respect to \(\varphi\) is zero, and hence there exists a symplectic near-identity analytic averaging transformation of the phase variables \((\varphi, I, t)\) that maps \(X\) to a Hamiltonian vector field \(\tilde{X}\) of the form

\[
\tilde{X}(\varphi, I, t) = \tilde{X}_0(I, t) + \tilde{X}_1(\varphi, I, t),
\]

with \(\tilde{X}_0(I, t)\) the same as above, and \(\tilde{X}_1 = O(\varepsilon \omega^{-2}) \frac{\partial}{\partial \varphi} + O(\varepsilon^2 \omega^{-2}) \frac{\partial}{\partial t}\).

Now let \(\tilde{\omega}\) be the average of \(\omega\) with respect to \(t\):

\[
\tilde{\omega}(I) = \frac{1}{2\pi} \int^2 \omega(I, t) dt.
\]

Then for \(\varepsilon \neq 0\) there exists a further smooth symplectic averaging transformation, mapping \(\tilde{X}\) to

\[
\tilde{X}(\varphi, I, t) = \tilde{X}_0(I) + \tilde{X}_1(\varphi, I, t),
\]

where

\[
\tilde{X}_0(I) = \tilde{\omega}(I) \frac{\partial}{\partial \varphi} + \varepsilon \frac{\partial}{\partial t}, \quad \text{and} \quad \tilde{X}_1 = O(\varepsilon \omega^{-2}) \frac{\partial}{\partial \varphi} + O(\varepsilon^2 \omega^{-2}) \frac{\partial}{\partial t}.
\]

Indeed, take the transformation \((\varphi, I, t) \mapsto (\tilde{\varphi}, I, t)\), where \(\tilde{\varphi} = \varphi + f(I, t)\), and \(f\) is defined by

\[
\varepsilon \frac{\partial}{\partial t} f = \tilde{\omega} - \omega.
\]
Then the diffeomorphism is well-defined for all $\varepsilon \neq 0$, and the $\frac{\partial}{\partial z}$-component of the transformed vector field is given by the equation

$$
\dot{\varphi} = \dot{\varphi} + \frac{\partial f}{\partial I} \dot{I} + \frac{\partial f}{\partial t} \dot{t},
$$

$$
= \omega(I, t) + \varepsilon \frac{\partial f}{\partial t} + O \left( \frac{\varepsilon^2}{\omega^2} \right),
$$

$$
= \hat{\omega}(I) + O \left( \frac{\varepsilon^2}{\omega^2} \right).
$$

In the second line we use that $\dot{I} = O(\varepsilon^2 \omega^{-2})$. For simplicity we drop the tildes from now on.

The unperturbed vector field $X_0$ has invariant tori $I = \text{constant}$, with frequency vector $(\dot{\varphi}, \dot{t}) = (\hat{\omega}(I), \varepsilon)$. We prefer to have for any parameter value just a single torus to check the persistence of. Therefore an additional ‘localization parameter’ $\omega > 0$ is introduced, such that for any $(\omega, \varepsilon)$ only one torus of $X_0$ is considered, namely the one with frequency vector $(\omega, \varepsilon)$. This construction is possible since $\frac{\partial f}{\partial t} \hat{\omega}(I) \neq 0$ for all $I$, and hence $\hat{\omega}$ is invertible as a function of $I$. We refer to [BHT90, Hui88, §5a] for more details on the construction.

A simple translation puts all the tori of the unperturbed system at $I = 0$. Define $I_\omega$ to be the action corresponding to the frequency $\hat{\omega} = \omega$, then the desired translation is

$$
(\varphi, I, t) \mapsto (\varphi, I - I_\omega, t).
$$

The vector field now takes the form

$$
X(\varphi, I, t; \omega) = X_0(\omega) + X_1(\varphi, I, t; \omega),
$$

with

$$
X_0(\omega) = \omega \frac{\partial}{\partial \varphi} + \varepsilon \frac{\partial}{\partial t}, \quad \text{and} \quad X_1 = O \left( \frac{\varepsilon^2}{\omega^2} \right) \frac{\partial}{\partial \varphi} + O \left( \frac{\varepsilon^4}{\omega^4} \right) \frac{\partial}{\partial I}.
$$

The unperturbed system $X_0$ has invariant tori $I = 0$ with frequency vector $(\omega, \varepsilon)$. We now prove that the tori with Diophantine frequency vector persist under the perturbation $X_1$ if it is sufficiently small. The parameter $\varepsilon$ is first considered to be fixed, while $\omega$ ranges over an interval $\Omega = [\omega_m, \omega_M]$, with $\omega_M > \omega_m > 0$. Let

$$
D = \{(\varphi, I, t; \omega)\} = S^1 \times \{0\} \times S^1 \times \Omega
$$

be a region in phase-parameter space, that contains the invariant tori $I = 0$ of the unperturbed system. In this region the frequency ratio $\dot{\varphi}/\dot{t}$ varies with $\omega$, and in this sense is not degenerate.

For $\sigma, \eta > 0$, let $D_{\sigma, \eta}$ be a complex neighborhood of $D$, given by

$$
D_{\sigma, \eta} = (S^1 + \eta) \times B(0, \sigma) \times (S^1 + \eta) \times (\Omega + \sigma),
$$

where $B(0, \sigma)$ is the closed ball in $\mathbb{C}$ with center $0$ and radius $\sigma$, and, for any set $S \subset \mathbb{C}^n$ and any $\eta > 0$,

$$
S + \eta = \bigcup_{x \in S} \{z \in \mathbb{C}^n : |z - x| < \eta\}.
$$
Furthermore, define the set $\Omega^\gamma$ of Diophantine frequencies in $\Omega$ by

$$\Omega^\gamma = \{ \omega \in \Omega : |p\omega + q\varepsilon| \geq \gamma |(p,q)|^{-\tau} \text{ for all } (p,q) \in \mathbb{Z}^2 \setminus \{0\} \}. \quad (3.5)$$

Here $\gamma > 0$ is a parameter and $\tau > 2$ is a constant. The dependence of $\Omega^\gamma$ on $\tau$ and $\varepsilon$ is suppressed in the notation.

A KAM theorem in the context of divergence free vector fields with codimension 1 invariant tori, cf. [Mos67, BHT90], shows the following: for $\varepsilon \neq 0$ sufficiently small, all invariant 2-tori of $X_0$ in $D$ with frequency $\omega$ in $\Omega^\gamma$ and not too near the boundary of $\Omega$, correspond to invariant 2-tori of the perturbed system $X$ with the same frequency ratio $\omega/\varepsilon$. The perturbed invariant tori lie in a neighborhood of $D$ in $\{(\varphi, I, t; \omega)\} = S^1 \times \mathbb{R} \times S^1 \times \Omega$. The tori of $X$ and $X_0$ are equivalent by a Whitney smooth time-preserving diffeomorphism $\Psi_{\alpha,\beta,\varepsilon}$ that approximately maps $D$ into $D$. The diffeomorphism $\Psi_{\alpha,\beta,\varepsilon}$ is the identity in the $t$ component, see [BHT90, Hui88, §7f].

The smallness condition on $\varepsilon$ amounts to the following. By KAM theory there exists a $\delta > 0$, depending only on $\eta$ (so in particular independent of $\gamma$, $\tau$, $\varepsilon$, $\sigma$, and $\Omega$), such that all tori $I = 0$ of the unperturbed system $X_0$ with frequency $\omega$ in $\Omega^\gamma$ persist under the perturbation $X_1$ if

$$|X_1|_{\sigma, \eta} := \sup_{D_{\sigma, \eta}} |X_1| < \gamma \delta, \text{ and } 0 < \gamma \leq \sigma < 1, \quad (3.6)$$

see, e.g., [BHS96, §5.2]. Since $|X_1|_{\sigma, \eta} = O(\varepsilon \omega_m^2)$, this condition is obviously satisfied for constant $\gamma$ and $\sigma$, and $\varepsilon \neq 0$ sufficiently small, depending on $\omega_m$.

**Remark 27:** In the usual KAM setting, persistence of a torus also requires the frequency vector to be non-degenerate, compare, e.g., [Kol54, Kol57, Arn88, Arn89]. One then finds a *conjugacy* between the tori of the perturbed and unperturbed systems. In our case, the frequency vector of $X_0$ has a constant component and is therefore degenerate in this strict sense. However, this non-degeneracy condition can be relaxed to the requirement that the frequency *ratio* is non-degenerate, i.e., varies with $\omega$, see, e.g., [Mos67, BHT90]. In this setting the perturbed and unperturbed tori are just equivalent in general.

**Remark 28:** An alternative way to show persistence of invariant tori is a direct construction by Poincaré-Lindstedt perturbation series (instead of constructing a conjugacy or equivalence), cf., e.g., Celletti and Chierchia [CC88a, CC88b], Greene and Percival [GP81], and Jorba et al. [JdILZ99, JM99].

**Remark 29:** The persistence result can also be obtained for the return map $P$ of $X$ on the section $t = 0$. In action-angle coordinates, it is a perturbation of the return map $P_0$ of $X_0$, given by

$$P_0 : (\varphi, I) \mapsto (\varphi + \frac{2\pi \omega}{\varepsilon}, I),$$

where $\omega$ depends on $I$. For all $\varepsilon \neq 0$ this is a pure twist map, having invariant circles $I = 0$ with rotation number $\omega/\varepsilon$, corresponding to the invariant 2-tori $I = 0$ of $X_0$, and we have to check persistence of these circles.
One now considers a region \( \tilde{D} = \{ (\varphi, I; \omega) \} = S^1 \times \{0\} \times \Omega \) in the phase space of \( P \), that is completely foliated by invariant circles of \( P_0 \). Since \( P \) is area preserving, the twist theorem, cf. [Mos62], is applicable, and shows that for \( \varepsilon \) sufficiently small, the invariant circles of \( P_0 \) in \( \tilde{D} \) with Diophantine rotation number (except those with \( \omega \) near the boundary of \( \Omega \)) are conjugate to invariant circles of \( P \) with the same rotation number. The perturbed invariant circles lie in a neighborhood of \( \tilde{D} \). The rotation number is exactly equal to the frequency ratio of the vector field, and the Diophantine condition on \( \omega/\varepsilon \) is the same as in the previous case. The conjugacy between the invariant circles of \( P_0 \) and \( P \) is given by the same diffeomorphism \( \Psi_{\alpha, \beta, \varepsilon} \), restricted to the section \( t = 0 \) where \( P \) is defined.

It remains to derive the asymptotics of the relative measure of invariant tori of \( X \) as \( \omega_m \to +\infty \). Under condition 3.6, the relative measure of frequencies in \( \Omega \) corresponding to persistent tori satisfies

\[
\mu_{\text{rel}}(\Omega) := \frac{\mu(\tilde{\Omega}^\mu)}{\mu(\Omega)} = 1 - O \left( \frac{\gamma}{\omega_m - \omega_m} \right) \quad \text{as} \quad \frac{\gamma}{\omega_M - \omega_m} \to 0,
\]

cf. [BHS96, §5.2]. Taking \( \gamma \) and \( \omega_M/\omega_m \) constant, we obtain the following theorem.

**Theorem 30** Let \( X \) be the vector field of the forced pendulum in action-angle coordinates, given by (3.4). Let \( \varepsilon > 0 \) be some constant, and let \( \Omega = [\omega_m, \omega_m] \) be an interval of frequencies. Choose constants \( \tau > 2, \eta > 0 \) and \( \gamma \), \( \sigma \) satisfying \( 0 < \gamma < \sigma < 1 \). Fix \( \varepsilon > 0 \), sufficiently small such that condition (3.6) is satisfied, and let \( \tilde{\Omega}^\mu \) be the \((\varepsilon \text{ dependent})\) set of Diophantine frequencies in \( \Omega \), defined by (3.5). Then then relative measure of persistent tori in \( \Omega \), defined as above, satisfies

\[
\mu_{\text{rel}}(\Omega) = 1 - O \left( \frac{1}{\omega_m} \right) \quad \text{as} \quad \omega_m \to +\infty.
\]

This proves that the torus with infinite frequency \( \omega \) is a Lebesgue density point of the set of persistent tori. For the original system (3.2) this means the following: for large \( \omega \), one has \( \omega \approx \varepsilon y \), where \( y \) is the phase variable in the original setting. Moreover, if \( \varepsilon, \alpha \) and \( \beta \) are constant, then the same holds for the parameters in the original setting. Thus the invariant torus \( y = +\infty \) of the original system is a Lebesgue density point of the set of invariant tori, and the relative measure is \( 1 - O(1/y) \) as \( y \to +\infty \), for constant parameter values.

Furthermore, for fixed \( \varepsilon \) the persistence condition (3.6) is satisfied for sufficiently large \( \omega \), i.e., for sufficiently large \( y \) in the original setting. This shows that at any parameter point invariant circles exist at sufficiently large \( y \).

To obtain a quantitative result, condition (3.6) has to be evaluated explicitly. This involves estimates on the perturbation \( X_1 \) and on \( \delta \). The first requires estimates on elliptic integrals, that can be improved by further averaging transformations. For the latter one can use, for example, the explicit expression obtained by Delshams and Gutiérrez [DG96]. For other quantitative KAM theorems, designed mostly for specific systems, see, e.g., [Ben88, CC91, CF87, CGL00, Loc99]. We plan to report on this in more detail in future research, cf. [BNS].
3.3.2 Adiabatic estimates of the ‘region of interest’

Adiabatic estimates for the size of the ‘region of interest’ are obtained in two regions in the parameter plane: for $\beta$ small compared to $\alpha$, and for large $(\alpha, \beta)$, ‘near infinity’, with $\alpha > \beta$. In these regions the adiabatic estimates are

$$y = 2\sqrt{\alpha},$$

to first approximation as $\beta \to 0$, and

$$y = 2\sqrt{\alpha + \beta},$$

to first approximation as $\alpha > \beta \to +\infty$, respectively. We first discuss the estimate for small $\beta/\alpha$.

The stability diagram of figure 3.1 shows that the upper equilibrium is unstable if $\alpha > \beta \geq 0$, also see [PS28, MS54]. For $\alpha > 0$ and $\beta$ small compared to $\alpha$, the ‘region of interest’ is approximately bounded by the invariant manifolds of the unstable upper equilibrium of $P$. Indeed, for $\beta = 0$ the system is integrable, and the separatrices of the upper equilibrium exactly bound the ‘region of interest’, while outside this region only invariant circles exist. Thus by KAM theory the map $P$ has invariant circles arbitrarily close to the invariant manifolds of the upper equilibrium, if $\beta > 0$ is sufficiently small. Neishtadts theorem [Nei84, Gel97] implies that these manifolds lie in a neighborhood of the separatrices of the integrable system that vanishes as $\beta \to 0$. This proves the following lemma.

**Lemma 31** For $\alpha > 0$ the maximal $y$ coordinate of the invariant manifolds of the unstable upper equilibrium satisfies

$$y(\alpha, \beta) = 2\sqrt{\alpha} + f(\alpha, \beta),$$

where the remainder $f(\alpha, \beta)$ vanishes as $\beta \to 0$. The ‘region of interest’ is bounded by

$$y(\alpha, \beta) = 2\sqrt{\alpha},$$

to first approximation, as $\beta \to 0$.

To obtain an estimate for $\alpha, \beta \to +\infty$, with $\alpha > \beta$, we use the slowly varying system of lemma 25, given by

$$X(x, y, t; \alpha, \beta, \varepsilon) = y \frac{\partial}{\partial x} + (\alpha + \beta \cos t) \sin x \frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial t},$$

where $t$ is a slowly varying parameter. The frozen system $\varepsilon = 0$ is (in the original parameters and phase variables) located at unbounded $(y, \alpha, \beta)$, and the slowly varying vector field $X$ can be seen as a perturbation.

Adiabatic theory can now be applied to get an estimate on the ‘lowest’ invariant circle of the slowly varying system. Since $\alpha > \beta$, the unperturbed system has an unstable upper
and a stable lower equilibrium, compare figure 3.11, and hence its lowest invariant circle is arbitrarily close to the separatrix

\[ y = \sqrt{2(\alpha + \beta \cos t)(1 - \cos x)}. \]

By adiabatic theory, an orbit of the perturbed system remains \(O(\varepsilon)\) close to the frozen orbit with the same initial condition for infinite time, if it does not intersect the frozen separatrix, see, e.g., [Arn62, Arn88, EE91, NST97]. The ‘lowest’ frozen orbit that does not remain above the separatrix lies on the manifold

\[ y = \sqrt{2(\alpha + \beta)(1 - \cos x)}. \]

Thus the maximal \(y\)-coordinate of the invariant manifolds of the upper equilibrium of the perturbed system is given by

\[ y = 2\sqrt{\alpha + \beta} + \varepsilon \tilde{g}(\alpha, \beta, \varepsilon), \]

for some function \(\tilde{g}\). Rescaling back to the original phase variables and parameters, and applying KAM theory, we arrive at the following lemma:

**Lemma 32** In the original coordinates, for \(\alpha > \beta\) the maximal \(y\) coordinate of the invariant manifolds of the unstable upper equilibrium satisfies

\[ y(\alpha, \beta) = 2\sqrt{\alpha + \beta} + g(\alpha, \beta), \]

where \(g(\alpha, \beta)\) is bounded as \(\alpha, \beta \to +\infty\). To first approximation as \(\alpha, \beta \to +\infty\), the ‘region of interest’ is bounded by

\[ y(\alpha, \beta) = 2\sqrt{\alpha + \beta}. \]

### 3.3.3 Numerical estimates of the ‘region of interest’

We estimate numerically the size of the ‘region of interest’, by computing the location of the ‘lowest’ invariant circle of \(P\) on three evenly spaced grids in the parameter plane, namely in the regions \(\{(\alpha, \beta)\} = [0, 10] \times [0, 10]\) with step 0.05 in both variables, \(\{(\alpha, \beta)\} = [0, 100] \times [0, 100]\) with step 0.5, and \(\{(\alpha, \beta)\} = [0, 1000] \times [0, 1000]\) with step 10. By location we mean the \(y\)-coordinate of the invariant circle at \(x = \pi\). For a description of the method of computation we refer to appendix 3.A, also compare [ST].

Figure 3.12 shows the location \(y = y_{\text{num}}\) of the ‘lowest’ invariant circle in the region \([0, 10] \times [0, 10]\) in the parameter plane. The graph of the intersection of the ‘lowest’ invariant circle with \(x = 0\) is displayed in figure 3.13. The latter graph turns out to lie below the plane \(y = 2\) (this is also true for the larger grids), while the adiabatic estimates, see section 3.3.2, suggest that the former is close to \(y = y_{\text{adiab}} = 2\sqrt{\alpha + \beta}\). The corresponding absolute difference \(\varepsilon_{\text{abs}} = |y_{\text{num}} - y_{\text{adiab}}|\) is displayed in figure 3.14.

In the three figures 3.12 − 3.14 one clearly sees jumps in the location of the ‘lowest’ invariant circle. These are due to destruction of the ‘lowest’ invariant circle as \(\beta\) grows,
Figure 3.12: Graph over \((\alpha, \beta) \in [0,10] \times [0,10]\) of the location of the ‘lowest’ invariant circle of \(P\), i.e., its intersection with \(x = \pi\).

Figure 3.13: Graph over \((\alpha, \beta) \in [0,10] \times [0,10]\) of the intersection of the ‘lowest’ invariant circle of \(P\) with \(x = 0\).
Figure 3.14: Absolute difference $\varepsilon_{\text{abs}} = |y_{\text{num}} - y_{\text{adiab}}|$ between the numerical and adiabatic estimates of the location of the ‘lowest’ invariant circle, on the domain $\{(\alpha, \beta)\} = [0, 10] \times [0, 10]$.

Figure 3.15: Graph over $\beta \in [0, 10]$ of the intersection of the ‘lowest’ invariant circle with $x = 0$ (lower graph) and $x = \pi$ (upper graph), for fixed $\alpha = 0$. The graphs clearly show where the ‘lowest’ invariant circle jumps.
resulting in a jump in the estimate to a higher invariant circle. This can be seen very well in figure 3.15, where the intersections of the ‘lowest’ invariant circle with the lines \( x = 0 \) and \( x = \pi \) are displayed as graphs over \( \beta \in [0,10] \), while \( \alpha = 0 \) is fixed. This results in a ‘devil’s staircase’, where the largest discontinuities in the graphs correspond to jumps of the ‘lowest’ invariant circle over an island of low period.

Figures 3.16 and 3.17 show the location of the ‘lowest’ invariant circle and the absolute difference \( \varepsilon_{\text{abs}} \) on the large grid \( \{(\alpha, \beta)\} = [0,1000] \times [0,1000] \). Again we can see the location of the jumps, and it is clear that the relative difference decreases as \( \alpha \) and \( \beta \) increase.

<table>
<thead>
<tr>
<th>grid</th>
<th>( \varepsilon_{\text{abs}} )</th>
<th>( \varepsilon_{\text{rel}} )</th>
<th>( \varepsilon_{\text{rel}}(5) )</th>
<th>( \varepsilon_{\text{rel}}(50) )</th>
<th>( \varepsilon_{\text{rel}}(500) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 \times 10</td>
<td>0.77</td>
<td>0.92</td>
<td>0.11</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>100 \times 100</td>
<td>0.67</td>
<td>0.32</td>
<td>0.10</td>
<td>0.021</td>
<td>-</td>
</tr>
<tr>
<td>1000 \times 1000</td>
<td>0.35</td>
<td>0.049</td>
<td>0.049</td>
<td>0.014</td>
<td>0.0026</td>
</tr>
</tbody>
</table>

Table 3.1: The maximal relative difference \( \varepsilon_{\text{rel}} \) between the numerical and adiabatic estimates of the location of the ‘lowest’ invariant circle, in all three grids. Further, \( \varepsilon_{\text{rel}}(k) \) is the same difference restricted to the region \( \|\{\alpha, \beta\}\| > k \) of the grid.

Table 3.1 presents for the three grids the maximal absolute and relative differences \( \varepsilon_{\text{abs}} \) and \( \varepsilon_{\text{rel}} \) between the numerical and adiabatic estimates. By relative difference we
Figure 3.17: Absolute difference $\varepsilon_{\text{abs}} = |y_{\text{num}} - y_{\text{adiab}}|$ between the numerical and adiabatic estimates of the location of the ‘lowest’ invariant circle, on the domain $\{(\alpha, \beta)\} = [0, 1000] \times [0, 1000]$. 
mean the absolute difference divided by the maximum of $y_{\text{num}}$ and $y_{\text{adiab}}$. In formula:

$$
\varepsilon_{\text{rel}} = \left| \frac{y_{\text{num}} - y_{\text{adiab}}}{\max(\|y_{\text{num}}\|, \|y_{\text{adiab}}\|)} \right|.
$$

The relative difference decreases rapidly when $\alpha$, $\beta$ increase, compare figure 3.14. To illustrate this, the table also states, for several $k$, the maximal relative difference $\varepsilon_{\text{rel}}(k)$ in the part of the grid where $\|(\alpha, \beta)\| > k$.

![Graph](image)

Figure 3.18: Graph over $\lambda \in [0, 200]$ of the maximal and minimal values of the location of the ‘lowest’ invariant circle in grid points on the line $\lambda = \alpha + \beta$.

The adiabatic estimate $y_{\text{adiab}}$ is constant on lines of constant $\alpha + \beta$. For the region $[0, 100] \times [0, 100]$ in the parameter plane, figure 3.18 shows the maximal and minimal values $y_{\max}$ and $y_{\min}$ of the numerical estimate in grid points on lines $\alpha + \beta = \text{constant}$, plotted against $\alpha + \beta$. In formula:

$$
y_{\max}(\lambda) = \max_{\alpha, \beta} (y_{\text{num}}(\alpha, \beta)),
$$

where the maximum is taken over all $(\alpha, \beta)$ in the grid with $\alpha + \beta = \lambda$, and $y_{\min}$ is defined analogously. As $\alpha + \beta$ increases the two graphs seem to converge, which again suggests that for large $\alpha$, $\beta$ the adiabatic estimate $y_{\text{adiab}}$ is very accurate.

### 3.4 Numerical study of the coherent dynamics

The goal of this section is to study the coherent (and to a lesser extent the non-coherent) dynamics of the Poincaré map in several fundamental domains in the parameter plane,
and to determine to what extent this dynamics differs from one domain to another. The concept of fundamental domains was introduced in section 3.2. In some fundamental domains near the origin of the parameter plane and some further away, this dynamics is described using both analytical and numerical means. An integrable approximation to the Poincaré map \( P \), valid for small \( \beta \), is used as a starting point for the investigation of the dynamics. Analytical results regarding periodic orbits, their stability type and bifurcations can be extended to a larger part of the parameter plane by numerical continuation. Numerical methods (like continuation) also serve to describe aspects of the coherent dynamics not covered by the integrable model.

As said before, in section 3.1.2, it is our conjecture that fundamental domains of the same type show similar dynamics. This is true for the stability types and bifurcations of the two equilibria. Near the origin of the parameter plane the dynamics of the nonlinear system varies largely from one domain to the other. On the other hand, in domains farther away the ‘region of interest’ in the parameter plane is filled with chaos, and thus the conjecture seems to hold trivially.

### 3.4.1 Method and sketch of the results

Let us now describe our method of research and its results in more detail. To describe the dynamics of \( P \) in an arbitrary fundamental domain, we start with the integrable approximation, obtained in chapter 2 and by Broer et al. [BHN98, BHNV99, BV92], using perturbation theory. This yields a ‘skeleton’ for the dynamics in any fundamental domain in the parameter plane, that is, it describes the stability types and bifurcations of the upper and lower equilibrium of the forced pendulum. The dynamics of \( P \) is restricted by its symmetries \( S \) and \( R \), see section 3.2.

Using these analytical ingredients, the dynamics of the Poincaré map is investigated numerically. First, to get an overview of the phenomena that can occur, we simply compute phase portraits of the Poincaré map by numerical integration, where the locations of periodic points bifurcated from the equilibria and stable and unstable invariant manifolds of the equilibria (when unstable) are shown. Then follows an analysis of the observed dynamical objects. Here numerical continuation is used to obtain bifurcation diagrams, locate periodic points, and compute curves of codimension one bifurcation points in the parameter plane.

We consider some fundamental domains near the origin of the parameter plane, and some farther away. In the fundamental domains away from the origin of the parameter plane the Poincaré map restricted to the ‘region of interest’ is mainly chaotic, and varies little (in a qualitative sense) with the parameters. The two equilibria are unstable except in very narrow regions in the parameter plane. Thus we conclude that, for fundamental domains far away from \((\alpha, \beta) = (0, 0)\) the Poincaré map has little coherent dynamics and is qualitatively the same in domains of the same type.

In the domains near the origin the coherent dynamics is very rich. To start with, we retrieve the bifurcations of the upper and lower equilibria that were found in the integrable skeleton. Because of the symmetries \( S \) and \( R \) the location of a periodic point born in a bifurcation at one of the equilibria is restricted to the lines \( y = 0, x = 0 \) and
$x = \pi$.

There are several phenomena in these phase portraits that are not found in the skeleton. Numerically computed bifurcation diagrams show that the periodic points bifurcating from the upper and lower equilibrium rapidly undergo further bifurcations if they are stable. For numerical evidence that this is the start of period doubling cascades we refer to Kim and Hu [KH98] and McLaughlin [McL81]. All periodic points seem either to go towards the upper or lower equilibrium or to escape from the equilibria in the $y$ direction as the parameter $\beta$ increases.

Furthermore, there are large stable regions above and below the lower equilibrium, coming from the broken invariant circle winding around phase space with rotation number $1$, i.e., consisting of points where the underlying vector field has frequency $1$ in the $x$-direction. The fixed point in the center of the stable region undergoes period doubling and pitchfork bifurcations. Moreover, we find $1 : 3$ resonance bifurcations here, and expect also $p : q$ resonances of higher order.

There are more stable regions on the $x$-axis, originating from a saddle-center bifurcation. Both the stable and unstable period two points born at this bifurcation undergo pitchfork bifurcations.

At any periodic point subharmonic bifurcations occur when the eigenvalues of the linearized Poincaré map pass through a resonance. We investigate this for the two equilibria, concentrating on the $1 : 3$ resonance. All bifurcations mentioned above take place on curves in the parameter plane, that for large parameter values either converge to the $\alpha$ axis or to one of the stability boundaries of the upper equilibrium.

Finally there is chaos. Invariant circles around stable periodic points and winding around the cylindrical phase space are interlaced with resonant dynamics. Indeed, the Poincaré-Birkhoff theorem [Bir13, Bir25] implies that in between any two invariant circles there exist periodic points of all intermediate rational rotation numbers. Moreover, there is a ‘chaotic sea’ in the ‘region of interest’, that seems to grow as $\beta$ increases. At the same time the stable domains in this region seem to get smaller, although their relative measure remains positive, see Neishtadt et al. [NST97]. The ‘chaotic sea’ can be associated with a transversal hyperbolic periodic point that generates it by homoclinic intersections, as follows. When invariant manifolds of hyperbolic points have transversal intersections, most nearby orbits form clouds. It is generally conjectured that the closure of these orbits is an ergodic set of positive Lebesgue measure, see Arnold and Avez [AA68]. On the other hand, generically the ‘chaotic sea’ is densely filled with islands of elliptic periodic points, see Robinson [Rob70]. So the ‘chaotic sea’ probably is a nowhere dense set of positive area. A similar phenomenon was first observed in the Hénon-Heiles system, cf. Hénon and Heiles [HH64] and, e.g., Moser [Mos68], and can be found in many other examples.

**Remark 33**: In the dissipative case of the Hénon family, Benedicks et al. [BC91, BY93] show that there exists a set of parameter points $(a, b)$ of positive Lebesgue measure for which the map has a strange attractor, which is the closure of the unstable manifold of the saddle point, and is ergodic with respect to an SBR measure. For other parameter values it has a periodic attractor.
Figure 3.19: Sketch of the phase portraits of the integrable skeleton in a fundamental domain. The bifurcations on the stability boundaries may differ depending on the type of domain. In each phase portrait the upper equilibrium is in the center, while the lower corresponds to the two points on the extreme left and right.

Remark 34: For a parameter set of positive Lebesgue measure the Hénon attractor has a positive Lyapunov exponent almost everywhere, cf. [BY93, You98]. It is widely conjectured that the same holds for 'chaotic seas' of other conservative and dissipative systems, in particular the standard map, compare Viana [Via00].

Let us present an overview of the remainder of this section. The integrable skeleton is described in section 3.4.2, the phase portraits of $P$ are computed in section 3.4.3, and section 3.4.4 analyzes their dynamical features. Section 3.4.5 presents an extended stability diagram in the parameter plane, including curves of codimension one bifurcation points.

### 3.4.2 Integrable skeleton

As mentioned before, the integrable approximation of the Poincaré map of the forced pendulum, cf. chapter 2 and Broer et al. [BHN98, BHNV99, BV92], is valid for small $\beta$, and by continuation yields a skeleton for the dynamics on the whole parameter plane, that describes the stability types and bifurcations of the upper and lower equilibrium, as well as the configuration of the periodic points bifurcating from them. In this section the skeleton is described, first for an arbitrary fundamental domain, then in more detail for the fundamental domains that are studied in the following subsections.
We recall that figures 3.7 and 3.8 show stability diagrams and phase portraits of the integrable approximations near the lower and upper equilibrium, respectively. The following lemma shows that all the bifurcations (except subharmonics of period ≥ 3) of the two equilibria are given by these integrable approximations, and hence the stability types and bifurcations of the equilibria (and the configuration of periodic points resulting from the bifurcations) are known at any parameter point.

**Lemma 35** The eigenvalues of the linearized Poincaré map at the upper or lower equilibrium are ±1 on curves in the parameter plane of the form

\[ \mathbb{R} \ni \beta \mapsto (\alpha, \beta) = (f(\beta), \beta) \in \mathbb{R}^2, \]

for some analytic function \( f \). At these curves the stability of the equilibrium changes.

The lemma implies that all stability curves intersect the \( \alpha \)-axis (possibly at negative \( \alpha \)-values), and hence are already present in the integrable approximation. A proof of the lemma is given in appendix 3.C.

The stability boundaries form the boundaries of the fundamental domains, and depending on the bifurcation types at these boundaries, there are four types of fundamental domains (and some special cases near the \( \alpha \) and \( \beta \) axes), see section 3.2. Figure 3.19 shows for one of the types a stability diagram with phase portraits of the skeleton. We note that the stability types of the periodic points bifurcating from the equilibria can change due to further bifurcations. Furthermore, the schemes are not in proportion, as in the actual phase portraits the periodic points are mostly very close to the upper equilibrium, or far away (in \( y \)-direction) from both equilibria.

The diagram only shows the periodic points that bifurcate in this domain. Of course there may be other periodic points present that bifurcated from the equilibria in other domains. For the other types of fundamental domains the picture looks the same, except that the bifurcations and periods of the points bifurcating from the equilibria differ.

In the next subsection phase portraits of \( P \) are computed in the three fundamental domains \((1, 1), (2, 1)\) and \((2, 2)\) near the origin, and the domains \((12, 6)\) and \((21, 9)\) farther away. For an explanation of the codification we refer back to figure 3.10. Figure 3.20 contains the skeletons for the first three domains. The domains are not proportional, in order to show the phase portraits of the skeletons more clearly. The phase portraits also indicate the direction in which the bifurcated periodic points go. Because of the symmetries \( S \) and \( R \) of \( P \) this has to be either the \( x \)- or \( y \)-direction. It is not useful to show skeletons for the last two domains, since they have too many periodic points.

**Remark 36** : The lowest stability boundary of the upper equilibrium is a curve passing through \((\alpha, \beta) = (0, 0)\). The others are obtained by computing a stability boundary of the lower equilibrium and reflecting it in the \( \beta \) axis, as explained in section 3.2, also see figure 3.4. Thus the bifurcations of the upper equilibrium can easily be deduced from those of the lower equilibrium. Because of the symmetry involved here, the local phase portraits at the upper equilibrium have to be rotated over 90 degrees compared to those of the lower equilibrium in figure 3.7, see Broer and Vegter [BV92].
3.4.3 Phase portraits

To get an overview of the dynamics of $P$ we study its phase portraits at representative parameter points in three fundamental domains $(1, 1)$, $(2, 1)$, and $(2, 2)$ near the origin of the parameter plane, and two domains $(12, 6)$ and $(21, 9)$ farther away.

For the first three domains, representative parameter points are selected by computing phase portraits at lattice points in the parameter plane, and sorting out interesting ones. Since the regions of stability in the parameter plane are very small for large $\beta$, each region in the domain has its own lattice, instead of a single lattice for the entire domain or the entire parameter plane. In the other two domains the stable regions are too tiny to detect, and therefore phase portraits are computed at a few randomly chosen parameter points, in the region where both equilibria are unstable. Since there is a big difference between the first three domains and the last two, we treat them separately.
The fundamental domains near the origin of the parameter plane

The phase portraits in the fundamental domains \((1, 1)\), \((2, 1)\) and \((2, 2)\) are displayed in figures 3.21 to 3.23. Each figure shows a stability diagram and some phase portraits. The stability diagram displays the stability curves in the fundamental domain under consideration, drawn in black, while stability curves outside this domain are grayed. As before, regions (in the current fundamental domain) where the upper or lower equilibrium is stable are shaded. The stability diagram further displays, in each stability region, the phase portrait of the integrable skeleton, compare the previous subsection, especially figure 3.20 where the same skeletons are shown.

The phase portraits show a representative collection of orbits of \(P\), obtained simply by iteration using an integrator based on Taylor series expansion, cf. Broer and Simó [BS98] or appendix 3.B for a discussion of this method. The integrator is very efficient and preserves the symmetries of the system, but (formally) not the symplectic structure, see appendix 3.B.

In each phase portrait, the origin corresponds to the upper equilibrium, while the point \((x, y) = (\pm \pi, 0)\) corresponds to the lower equilibrium. Each diagram (except for the blowups) contains about 46000 points; its caption states the parameter point \((\alpha, \beta)\) where the phase portrait is taken, and a diagram number corresponding to that in the stability diagram. In case the upper or lower equilibrium is unstable, (finite parts of) its invariant manifolds, computed by DsTool [BGM\textsuperscript{+}92, KO00], are displayed. The locations and stability types of the upper and lower equilibria and the period one and two points bifurcated from them are marked. These are found using numerical continuation.

We now discuss the fundamental domains one by one.

**Fundamental domain** \((1, 1)\). In this domain there are only two stability regions. A stability diagram and phase portraits are displayed in figure 3.21. In the lower region (‘SU’) the upper equilibrium is unstable, while the lower is stable, surrounded by invariant circles, see diagram 1 in figure 3.21. The chaotic sea can be associated to the unstable upper equilibrium, as explained in section 3.4.

In the upper region (‘SS’) both equilibria are stable, see diagram 2, but with smaller stability domains than in diagram 1. The four satellites around the upper equilibrium have rotation number \(\frac{1}{4}\). They have been observed in another numerical study, see Acheson [Ach95], and are likely to come from a broken invariant circle. We note that both figures show two stable domains above and below the lower equilibrium, both around a stable fixed point. These are due to a broken invariant circle winding around the cylindrical phase space.

**Fundamental domain** \((2, 1)\). Figure 3.22 shows a stability diagram and phase portraits. In the lower left region (‘UU’) the upper and lower equilibria are unstable. As suggested by the location of their invariant manifolds, see diagram 3 in figure 3.22, a large part of the chaotic sea can be associated to these two points. There is a stable period two orbit on the x-axis, see diagram 3 and the blowup in diagram 4. This is the period two point that bifurcated from the lower equilibrium at the curve of period dou-
Figure 3.21: Stability diagram and phase portraits of the Poincaré map $P$ in fundamental domain $(1,1)$. The stability diagram also shows the bifurcation scheme. The stability curves in the fundamental domain under consideration are drawn in black, outside this domain they are grayed. The dashed vertical lines in the stability diagram indicate the parameter lines where bifurcation diagrams are made, see section 3.4.4.1 for details. See section 3.4.3 for more comments, and figure 3.2 for the coding of periodic points and invariant manifolds.
bling bifurcations to the left of region ‘UU’. Its stability domain decreases as \( \beta \) increases. Again there are two stable domains above and below the lower equilibrium.

In the lower right region (‘SU’) the lower equilibrium is stable, while the upper is unstable, causing a chaotic sea. For small \( \beta \) the Poincaré map is as in diagram 5. For larger \( \beta \) the stability domain of the lower equilibrium becomes smaller and smaller. Diagram 6 displays the situation near the upper edge of the region. The stable period two points already present in diagram 3 have destabilized (in a pitchfork bifurcation, compare figures 3.26 and 3.27 below), and move towards the upper equilibrium as the parameters \((\alpha, \beta)\) increase. An unstable period two orbit has bifurcated from the lower equilibrium at the stability boundary between regions ‘UU’ and ‘SU’, and moves away from the equilibrium in y direction. At some parameter points a stable period two orbit, that did not bifurcate from one of the equilibria, is found on the \( x \) axis, like in diagram 6.

In the upper left region (‘US’) the chaotic sea corresponds to the unstable lower equilibrium. Compared with diagram 3 there are two additional unstable fixed points on the \( x \)-axis that bifurcated from the upper equilibrium, that has become stable at this bifurcation. There are 4 satellites around the upper equilibrium, just as in diagram 2 of figure 3.21. The stable fixed points above and below the lower equilibrium of diagrams 1, 2 and 3 have bifurcated, creating a stable period 2 orbit, that moves towards the upper equilibrium as \( \alpha \) and \( \beta \) increase, see diagram 8 (also taken from region ‘US’).

In the upper right region (‘SS’) both stable equilibria have very small stability domains, see diagram 9. This region in the parameter plane is very narrow, and the Poincaré map does not vary much in it. Therefore just one diagram is shown. The saddle fixed points bifurcated from the upper equilibrium at the stability boundary between regions ‘SU’ and ‘SS’ are very close to it and hardly distinguishable. Two unstable period two orbits, bifurcated from the lower equilibrium, can be found on the \( x \)-axis and on the line \( x = \pi \). Like in region ‘SU’ there are two period two centers on the \( x \), in this case surrounded by a period 6 satellite. The stable domains above and below the lower equilibrium are also present.

**Fundamental domain (2, 2).** This domain consists of three regions. The stability diagram and Poincaré maps are displayed in figure 3.23. In the lower left region (‘UU’) the two unstable equilibria cause a large chaotic sea, see diagrams 10 and 11. Near the upper equilibrium there are two saddles on the \( x \)-axis and a period two orbit on the \( y \)-axis, all bifurcated from the upper equilibrium, and a period two orbit on the \( x \)-axis bifurcated from the lower equilibrium. Both period two orbits were stable when they bifurcated from the equilibria, but have destabilized in further bifurcations. At some parameter values we find four period two points on the \( x \) axis, as in diagram 11. If we count the periodic points from left to right, then the first and third form an orbit, and so do the second and fourth.

In the lower right region (‘SU’) the lower equilibrium has stabilized, with a small stability domain. Two unstable period two points have bifurcated off in \( y \) direction. The chaotic sea can now be associated with the unstable upper equilibrium. Indeed, its manifolds seem to fill up a large part of the sea, see diagrams 12 to 14. In diagram 12
Figure 3.22: (a) Stability diagram and phase portraits of $P$ at parameter points in domain $(2, 1)$. Diagram 4 is a blowup of diagram 3, as indicated by the rectangle. Continued on next page.
Figure 3.22: (b) Continued from previous page. Phase portraits of $P$ at parameter points in domain (2,1).
Figure 3.23: (a) Stability diagram and phase portraits of $P$ at parameter points in domain $(2, 2)$. To be continued on the next page.
Figure 3.23: (b) Continued from previous page. Phase portraits of $P$ at parameter points in domain $(2, 2)$. Diagram 15 is a blowup of diagram 14, as indicated. Diagram 16 is taken from region ‘SS’.
we find a stable period two orbit on the $x$ axis, surrounded by two (hardly visible) stable period four orbits. As $\beta$ increases the period 2 points undergo a pitchfork bifurcation, resulting in two additional period 2 centers and saddles, compare diagram 13. As in diagram 11, the first and third center form one orbit, as do the second and fourth. Diagram 14 and its zoom in diagram 15 show that the centers and saddles move away from each other as $\beta$ increases.

Region ‘SS’ is very small and the Poincaré map varies little in it. Both stable equilibria have very small stability domains. At some parameter points a stable period two orbit is found on the $x$-axis, as in diagram 16, where it lies close to the lower equilibrium. Apart from that there seems to be only chaos in the ‘region of interest’.

The fundamental domains away from the origin of the parameter plane

Figures 3.24 and 3.25 show stability diagrams and phase portraits for the fundamental domains $(12, 6)$ and $(21, 9)$. The stability diagrams do not show integrable skeletons, since these contain too many periodic points, but simply display the fundamental domain under consideration, with an indication of the location of the computed phase portraits. The stable regions in the stability diagram are too narrow to be detected, therefore all phase portraits are taken from the region ‘UU’ where both equilibria are unstable.

The phase portraits display a representative collection of orbits of $P$. The periodic points that bifurcated from the upper and lower equilibrium are not shown, because there are so many. The phase portraits show only chaotic dynamics, except in diagram 17 in figure 3.24, where two stable regions surrounding a period two orbit are found on the $x$-axis. We conclude that the dynamics is qualitatively the same in different domains (of the same type) away from the origin of the parameter plane.

3.4.4 Numerical analysis

In this section some of the dynamical features encountered in the phase portraits of the Poincaré map are studied in more detail. The analysis is restricted to the domains near the origin of the parameter plane, since they show the most interesting dynamics. We first give an overview of the phenomena that we want to study, and then discuss them one by one.

The equilibria undergo bifurcations as determined by the skeleton. Section 3.4.4.1 presents numerically computed bifurcation diagrams of the equilibria that give more quantitative information, and show the stability types and bifurcations of the periodic points that bifurcated from the equilibria.

In some phase portraits, e.g. diagrams 6 and 9 in figure 3.22 or diagrams 12 and 16 in figure 3.23, there is a stable period two orbit on the $x$ axis. It originates from a saddle-center bifurcation. This bifurcation, as well as a subsequent pitchfork bifurcation, resulting in a pair of period two orbits that can be observed in diagrams 11, 13, 14 and 15 in figure 3.23, is studied in section 3.4.4.2.

The large stable regions above and below the lower equilibrium, see diagrams 1 and 2 in figure 3.21, diagrams 3 and 9 in figure 3.22 and diagrams 10 and 12 in figure 3.23,
Figure 3.24: Stability diagram and phase portraits of the Poincaré map at parameter points in fundamental domain (12, 6). See section 3.4.3 for more comments on the dynamics.
Figure 3.25: Stability diagram and phase portraits of the Poincaré map at parameter points in fundamental domain (21, 9). See section 3.4.3 for more comments on the dynamics.
are caused by a broken invariant circle. Section 3.4.4.3 presents a bifurcation analysis.

Finally, around stable periodic points satellites are born at resonances. For the equilibria and their satellites this is discussed in more detail in section 3.4.4.4.

### 3.4.4.1 Bifurcation diagrams of the equilibria

![Bifurcation Diagrams](image)

**Figure 3.26:** Schematic view of the bifurcation diagrams on lines $\alpha = \text{constant}$. The straight lines parallel to the $\beta$-axis correspond to the positions of the equilibria. The distances between bifurcation points and solution branches are exaggerated for clarity. The presented diagram of the upper equilibrium is valid on any line $\alpha = \text{constant}$, that of the lower equilibrium only if the constant is in $(0, \frac{1}{4})$, and has to be adapted otherwise. For simplicity only the branches in one quadrant of $(\beta, x, y)$-space are displayed, the others are their $S$- and $R$-symmetric counterparts.

To study the period one and two points of the Poincaré map $P$ bifurcating from the upper and lower equilibria of the pendulum, we compute some bifurcation diagrams at lines $\alpha = \text{constant}$ in the parameter plane, with $\beta$ as bifurcation parameter. We choose the lines $\alpha = 0.15, \alpha = 0.4$ and $\alpha = 0.7$. They intersect the three domains near the origin of the parameter plane considered in subsection 3.4.3, as indicated by vertical lines in the stability diagrams of figures 3.21 – 3.23. Other lines give rise to similar bifurcation diagrams. Each diagram is invariant under reflection in the $(x, \beta)$ and $(y, \beta)$ planes due to the symmetries $S$ and $R$.

The bifurcation diagrams are displayed in figures 3.27 – 3.29 for the respective parameter lines. They show branches of periodic points emanating from bifurcation points of the upper and lower equilibria, called primary branches and bifurcation points. The bifurcation points on the primary branches and the periodic points born there are called secondary. We note that for simplicity only the branches in one quadrant of $(\beta, x, y)$-space are displayed, the others are their symmetric counterpart with respect to the sym-
Figure 3.27: Bifurcation diagrams on the line $\alpha = 0.15$, and their projections onto the $(\beta, x)$- and $(\beta, y)$-planes, displaying primary and secondary branches of period one and two points of $P$. The straight lines parallel to the $\beta$-axis correspond to the positions of the equilibria. For simplicity only the branches in one quadrant of $(\beta, x, y)$-space are displayed, the others are their symmetric counterpart under reflection in the $(\beta, x)$- or $(\beta, y)$-plane. Both diagrams are in agreement with the schemes of figure 3.26.
Figure 3.28: Bifurcation diagrams on the line \( \alpha = 0.4 \), and their projections onto the \((\beta, x)\)- and \((\beta, y)\)-planes, displaying primary and secondary branches of period one and two points of \( P \). As in figure 3.27, the lines parallel to the \( \beta \)-axis correspond to the positions of the equilibria, and only the branches in one quadrant are displayed. The leftmost stability boundary of the lower equilibrium is not intersected, while the next one is intersected twice. Thus the leftmost period doubling point of the lower equilibrium in figure 3.26 is absent, but another period doubling has appeared, with a branch turning back towards \( \beta = 0 \).
Figure 3.29: Bifurcation diagrams on the line $\alpha = 0.7$, and their projections onto the $(\beta, x)$- and $(\beta, y)$-planes, displaying primary and secondary branches of period one and two points of $P$. The lines parallel to the $\beta$-axis correspond to the positions of the equilibria, and only the branches in one quadrant are displayed. Both stability boundaries of the lower equilibrium originating from the first resonance point $(\alpha, \beta) = (\frac{1}{4}, 0)$ are not intersected, so the two period doublings of this equilibrium shown in figure 3.26 are absent here.
metries $S$ and $R$. Indeed, at each primary bifurcation point, and at some secondary bifurcation points, just one of the two newly born periodic points is shown. The diagrams are obtained by numerical continuation of periodic points. For general background and examples on numerical continuation we refer to Simó [Sim89], Doedel et al. [DKK91a, DKK91b], Kuznetsov [Kuz95] and Castellá and Jorba [CJ00].

Since some of the bifurcation points and their branches are very close together, in figure 3.26 we present a sketch of the bifurcation diagrams, where the points and branches are easier to distinguish. The bifurcation diagram of the upper equilibrium is qualitatively the same on all lines $\alpha = \text{constant}$, since on any such line one encounters the same bifurcation sequence as $\beta$ increases from 0. This can be concluded from the stability diagram, see figures 3.1 and 3.4. For the lower equilibrium this is not true. For example, both stability curves originating from the first resonance point $(\alpha, \beta) = (\frac{1}{4}, 0)$ intersect the line $\alpha = 0.15$, none intersect the line $\alpha = 0.7$, and for $\alpha = 0.4$ the right hand curve is intersected twice, the other one not at all. The sketch in figure 3.26 is valid for constant $\alpha \in (0, \frac{1}{4})$, i.e., where any stability curve of the lower equilibrium is intersected exactly once. The examples in figures 3.28 and 3.29 illustrate how this diagram should be modified to match other cases.

The bifurcations from the upper and lower equilibria found in chapter 2 and by Broer et al. [BHN98, BHNV99, BV92], using perturbation theory, agree with the numerical results, compare figures 3.7 and 3.8. Because of the symmetries $S$ and $R$ the location of the primary periodic points is restricted to the lines $y = 0$, $x = 0$ and $x = \pi$. In all diagrams we see similar behavior: for the upper equilibrium, as $\beta$ increases the primary branches first move away from the equilibrium along the $x$ or $y$ axis, but at higher $\beta$ values they turn back. The secondary branches also go towards the upper equilibrium, but their location is not restricted to the axes.

As $\beta$ increases, the primary branches of the lower equilibrium move either towards the upper equilibrium along the $x$-axis, or they escape in $y$ direction along the line $x = \pi$, and do not seem to return. Some of the secondary branches return to the lower equilibrium, while others go to the upper equilibrium or go away in $y$ direction. (Observe that the range of plotted $y$ values is much larger for the lower equilibrium than for the upper equilibrium). Moreover it seems that no periodic point can be stable for a large interval of parameter values. Indeed, any stable periodic point rapidly bifurcates, while unstable points remain in existence over a large parameter range. This is true on primary and secondary branches.

### 3.4.4.2 The saddle-center bifurcation

In some of the phase portraits of section 3.4.3 there are two or four stable period 2 orbits on the $x$ axis that did not bifurcate from one of the equilibria. We analyze these points by computing their bifurcation diagrams and some phase portraits at relevant parameter values.

The bifurcation diagrams are shown in figures 3.30 – 3.32. They have $\beta$ as bifurcation parameter, while $\alpha$ is constant equal to 0.2836, 0.5021 and 0.7738, respectively. (These lines pass through the parameter points where the phase portraits of diagrams 10, 14
Figure 3.30: Bifurcation diagram on the line $\alpha = 0.2836$, displaying a saddle-center bifurcation. On the stable branch a pitchfork bifurcation occurs.

Figure 3.31: Bifurcation diagram of the saddle-center bifurcation, on the line $\alpha = 0.5021$. Like in the previous diagram there is a pitchfork bifurcation point on the stable branch.
Figure 3.32: Bifurcation diagram on the line $\alpha = 0.7738$. Apart from the saddle-center and pitchfork bifurcation points found in the previous two diagrams, we also see another pair of pitchfork points on the stable and unstable branches. The three-dimensional bifurcation diagrams to the right, with different view points, show that the branches coming from these two points do not lie in the plane $y = 0$, unlike all other branches. Stability types and bifurcation points are indicated as in the previous two diagrams.
and 9 in figures 3.22 and 3.23 are taken). In any diagram, except the last one, $y$ equals 0 along all the curves. A three-dimensional bifurcation diagram of the line $\alpha = 0.7738$, including the $y$-coordinate, is displayed in figure 3.32.

In all bifurcation diagrams we analyze the period 2 points to the left of the upper equilibrium. Those to the right are their images under $P$, and their bifurcation diagrams can be obtained by reflection in the $y$-axis. The diagrams show that the periodic points come from a saddle-center bifurcation. The stable orbit born at this bifurcation bifurcates further in a pitchfork bifurcation, creating the pair of stable period 2 orbits seen in for example in the phase portrait in figure 3.23, diagram 14. In the diagram at $\alpha = 0.7738$ more pitchfork bifurcation points on the stable and unstable branch of the saddle-center bifurcation are found. These have branches with nonzero $y$-coordinate. The saddle-center bifurcation has codimension 1 and takes place at a curve in the parameter plane. This curve and the curve of pitchfork bifurcation points are computed in section 3.4.5 below.

Phase portraits of the Poincaré map are shown in figure 3.33, for several parameter points on the line $\alpha = 0.2836$. Their captions state the parameter values, and the diagrams also display primary periodic points and invariant manifolds of (unstable) equilibria. In the first phase portrait, at $(\alpha, \beta) = (0.2836, 1.2134)$ — this is the same as diagram 10 in figure 3.23 — the saddle-center bifurcation has not yet occurred. The bifurcation takes place at $\beta \approx 1.4989$, corresponding to the second diagram. For higher $\beta$ values there is one stable and one unstable period two orbit on the $x$-axis, see the diagram at $\beta = 1.6$. The stable orbit undergoes a pitchfork bifurcation at $\beta \approx 1.9423$, that creates a pair of stable period two orbits, see the last diagram. Here the first and third period two point (counted from the left) form an orbit, and so do the second and fourth.

In the last diagram two very small stable regions can be seen at $(x, y) \approx (\pi, \pm 2.9)$. These are due to another saddle-center bifurcation. Its bifurcation diagram is given in figure 3.34. This bifurcation diagram is very similar to the previous one. In fact, the curve of saddle-center bifurcations mentioned above yields, by the symmetry $T \circ U$, involving reflection in the $\beta$-axis, another curve of saddle-center bifurcations in the parameter plane. It can be shown that the second saddle-center bifurcation lies on the reflected curve, which explains the similarity. See section 3.4.5 below for more details.

### 3.4.4.3 The broken invariant circle

The Poincaré map of the integrable system at $\beta = 0$ has a continuous family of invariant circles winding around the cylindrical phase space, and filling the region above the separatrix. (For simplicity we restrict to circles in the upper half cylinder $y > 0$.) For any $\beta > 0$, generically only circles satisfying a certain Diophantine condition persist, compare section 3.3.1 and references therein. In particular, the resonant circle with rotation number 1 does not exist for any $\beta > 0$, and gives rise to the large stable domains above the lower equilibrium that can be seen in many phase portraits, for example diagrams 1 and 2 in figure 3.21. This phenomenon is studied by bifurcation diagrams at lines of constant $\alpha$ in the parameter plane. Phase portraits are shown at relevant parameter values.

Figures 3.35 - 3.37 display bifurcation diagrams at the lines $\alpha = 0.079$, $\alpha = 0.16$
Figure 3.33: Phase portraits of the Poincaré map $P$ at several parameter points on the line $\alpha = 0.2836$, showing a saddle-center and subsequent pitchfork bifurcation. All centers and saddles involved are of period 2, jumping from the right half plane to the left one, and back. See section 3.4.4.2 for more comments.
Figure 3.34: Bifurcation diagram on the line $\alpha = 0.2836$ of a second saddle-center bifurcation, taking place at a higher $\beta$ value than the previous one, compare figure 3.30. Qualitatively the bifurcation diagram is the same. The branches of the saddle-center point lie in the plane $x = \pi$, but the other branches do not. Therefore three-dimensional bifurcation diagrams from different view points are added for clarity. Stability types and bifurcation points are indicated as in previous diagrams.
Figure 3.35: Bifurcation diagram on the line $\alpha = 0.079$, displaying the center and the saddle coming from the broken invariant circle with rotation number 1. The center undergoes a period doubling bifurcation.

Figure 3.36: Bifurcation diagram of the same center and saddle as in figure 3.35, but now on the line $\alpha = 0.16$. We again find a period doubling bifurcation.
and \( \alpha = 0.28 \), respectively. In all three cases, the invariant circle under consideration is broken for all \( \beta > 0 \), yielding one stable and one unstable fixed point, at \( x = \pi \) and \( x = 0 \), respectively. The saddle point does not seem to bifurcate, while the center undergoes a period doubling bifurcation. At the line \( \alpha = 0.28 \) one more period doubling and a pitchfork bifurcation are found. The first period doubling bifurcation yields a stable period two orbit moving away from the line \( x = \pi \). This can be seen for example in diagrams 7 and 8 of figure 3.22.

More phase portraits, at parameter points on the line \( \alpha = 0.79 \), are displayed in figure 3.38. On this line the period doubling bifurcation takes place at \( \beta \approx 0.43 \). Furthermore, a subharmonic saddle-center bifurcation occurs at \( \beta = 0.2148 \ldots \), creating for higher \( \beta \) values a stable and an unstable period three orbit, near the stable fixed point coming from the broken invariant circle. The stable period three orbit can clearly be seen in the phase portrait at \( \beta = 0.22 \). It moves away from the stable fixed point as \( \beta \) increases. The unstable period three orbit intersects the stable fixed point at \( \beta \approx 0.218 \) and \( (x,y) \approx (\pi,1.14) \). At this point the eigenvalues of the linearized Poincaré map \( DP \) are (non-real) third roots of unity. This is a generic codimension one bifurcation sequence, compare Duistermaat [Dui84]. A three-dimensional bifurcation diagram is presented in figure 3.39.

### 3.4.4.4 Satellites

This section deals with satellites of the equilibria, that is, strings of stable and unstable periodic orbits of period at least three, that encircle one of the equilibria. Orbits of period one and two are not discussed here: they are born at pitchfork and period doubling
Figure 3.38: Phase portraits of the Poincaré map $P$ at several parameter points on the line $\alpha = 0.079$, showing a period doubling bifurcation of the stable fixed points above and below the lower equilibrium. See section 3.4.4.3 for more comments.
Figure 3.39: Bifurcation diagram on the line $\alpha = 0.079$ of a stable and an unstable period three orbit that are created in a saddle-center bifurcation point at $\beta = 0.2148 \ldots$, from two different viewpoints. The unstable periodic points intersect the stable fixed point (from the broken invariant circle) at $\beta \approx 0.218$. The curves of periodic points passing through the saddle-center points lie in three different manifolds. Due to symmetry, one is the plane $x = \pi$.

Figure 3.40: Diagram in the parameter plane showing curves of subharmonic bifurcation points, where period 3 or 6 points are created at the upper or lower equilibrium, as indicated. For clarity the stability boundaries of the equilibria are also indicated.
bifurcations, respectively, and are already dealt with above. For the forced pendulum
satellites of period $q \geq 3$ are created in two different ways, where the symmetries and
non-dissipativity of the system play an important role, compare [BV92].

The first is at a Poincaré-Birkhoff bifurcation, cf. [Bir13, Bir25, BV92], where a re-
onant invariant circle of $P$ breaks. This bifurcation takes place at the $\alpha$-axis, where the
map is integrable and hence has resonant invariant circles. For $\beta \neq 0$ these generically
break into satellites, with transversely intersecting stable and unstable manifolds of
the saddle points, as follows from subharmonic Melnikov theory, see e.g. Guckenheimer
and Holmes [GH83] and Broer and Takens [BT89]. On the $\alpha$ axis the lower equilibrium is
surrounded by a region of invariant circles, bounded by the separatrices. At a point
$\alpha = \alpha_0$ on the $\alpha$-axis the range of rotation numbers of these invariant circles is
$(0, \sqrt{\alpha_0})$. Thus if $\beta$ is increased from this point we only expect $p : q$ resonance bifurcations with
$p/q < \alpha_0^2$.

The second is at a resonance of a fixed or a periodic point. As the eigenvalues of
the linearized Poincaré map at this point pass through a resonance, satellites are born
there, see, e.g., Meyer [Mey70], Takens [Tak74b] and Duistermaat [Dui84], also compare
Takens [Tak74a] and Arno1'd [Arn88, Arn94] for the dissipative case. For the lower equilibrium
the $p : q$ resonance bifurcation takes place at a curve in the parameter plane that
passes through the point $\alpha = (p/q)^2$ on the $\alpha$ axis. The resonance curves of the
upper equilibrium are the images of those of the lower equilibrium under reflection in the
$\beta$ axis (like the stability curves, cf. section 3.2). At a $p : q$ resonance curve the eigenvalues
of the linearized Poincaré map pass through $e^{\pm 2\pi i p/q}$, but stay on the unit circle since
$P$ is area preserving. Thus the fixed point can only be unstable at the resonance.

In figure 3.41 we sketch (for the lower equilibrium) where the two bifurcations take
place in the case of the 1 : 3 resonance, and also the corresponding phase portraits. The
latter only give a qualitative description of the dynamics involved, whereas the real map
$P$ of course has no heteroclinic connections, etc. Moreover, the width of the ‘annulus’
of $p : q$ resonant dynamics is $O(\sqrt{\beta})$ as $\beta \to 0$, hence resonant dynamics are a small
phenomenon, cf. Duistermaat [Dui84], Arno1'd [Arn93] and Wiggins [Wig90]. Finally,
we observe that, due to the spatial and temporal symmetries $S$ and $R$ of $P$, at both
resonance bifurcations two period 3 satellites are created simultaneously, leading to a
total of 6 stable and 6 unstable periodic points.

Thus, if one starts for example at $(\alpha, \beta) = (1/4, 0)$, there is an invariant circle of
rotation number 1/3 around the lower equilibrium. When $\beta$ is increased, say to 1/10,
this circle breaks into a string of islands. If we then go the left in the parameter plane
until we hit the 1 : 3 resonance curve, the radius of the broken invariant circle shrinks to
zero, and at the resonance curve it vanishes at the equilibrium.

Figure 3.40 shows curves in the parameter plane where the linearized map has a
resonance of order 3 or 6, as indicated. Curves passing through the $\alpha$-axis correspond to
resonances of the lower, the others to resonances of the upper equilibrium. The stability
boundaries of the equilibria are displayed for clarity. Obviously the curves of resonance
points have to lie in the stable regions of the parameter plane. They converge to the
stability boundaries as $\beta$ increases. An easy computation reveals that the resonance
Figure 3.41: Sketch of the two types of $1:3$ resonance bifurcations in the parameter plane, with phase portraits. For simplicity only the lower equilibrium and the $1:3$ resonant dynamics are shown, for the regions to the left and right of the $1:3$ resonance curve, and for the $\alpha$-axis to the right of this curve. On the $\alpha$ axis (for $\alpha > 1/9$) a Poincaré-Birkhoff bifurcation takes place, and on the $1:3$ resonance curve passing through $\alpha = 1/9$ the lower equilibrium goes through a $1:3$ resonance bifurcation. Observe that there are two stable and two unstable period 3 orbits, due to the symmetries $S$ and $R$. 
points of order \( k \) on the \( \alpha \)-axis are given by

\[
\alpha = \left( n + \frac{m}{k} \right)^2,
\]

with \( m \in \{1, 2, \ldots, k - 1\} \) relatively prime with \( k \), and \( n = 0, 1, \ldots \). Thus the resonance points of order 3 on the \( \alpha \)-axis are given by

\[
\alpha = \frac{1}{9}, \frac{4}{9}, \frac{16}{9}, \frac{25}{9}, \ldots,
\]

and those of order 6 are given by

\[
\alpha = \frac{1}{36}, \frac{25}{36}, \frac{49}{36}, \frac{121}{36}, \ldots,
\]

as can be seen in the diagram.

Figure 3.42: Bifurcation diagram on the line \( \alpha = 0.0612 \) of the stable and an unstable period three orbits (four in total) that are created in a bifurcation at the lower equilibrium point at \( \beta \approx 0.25 \). All orbits are created at the same bifurcation point due to symmetries. The stable periodic points undergo a period doubling bifurcation.

Figure 3.42 displays a bifurcation diagram on the line \( \alpha = 0.0612 \), showing four period 3 orbits, two stable and two unstable, branching off of the lower equilibrium in a subharmonic bifurcation. The satellites are born in a third order resonance at the lower equilibrium, at \( \beta \approx 0.25 \). Figure 3.43 shows Poincaré maps just before and after the bifurcation. In the preceding sections we encountered satellites of other periods in several phase portraits, e.g., period 4 satellites around the upper equilibrium in diagram 2 in figure 3.21 and diagram 7 in figure 3.22.

Remark 37: The \( p:q \) subharmonic bifurcations of the forced pendulum take place in a Hamiltonian equivariant context with a discrete \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry group, generated by the
Figure 3.43: Phase portraits of the Poincaré map $P$ at two parameter points on the line $\alpha = 0.0612$, before and after the subharmonic bifurcation at the lower equilibrium. The bottom left diagram is a blowup of the one at the top right, that shows the two stable period 3 orbits more clearly. See section 3.4.4.4 for more comments.
symmetries $S$ and $R$. A complete analysis of the $p : q$ resonance in this setting requires an equivariant version of the subharmonic bifurcation theory developed by [Mey70, Tak74b, Dui84]. As far as we know no systematic study of these equivariant resonance bifurcations is available at this moment. However, for partial results in this direction, see [BLV98, BHLV98, Lun99].

3.4.5 The extended stability diagram

![Stability diagram](image)

Figure 3.44: Curves of codimension one bifurcations in the parameter plane. The curves correspond to stability changes of the equilibria, a period doubling bifurcation of a broken invariant circle, bifurcations on the primary branches, and a saddle-center bifurcation, as indicated. See section 3.4.5 for more comments.

In the previous subsection we studied several bifurcations of saddle-center, pitchfork and period doubling type. They are all codimension one bifurcations, and hence take place on curves in the parameter plane. (For the bifurcations of the upper and lower equilibrium these are just the stability boundaries shown, e.g., in figure 3.4.) The curves are computed numerically using a continuation method, summarized in figure 3.44. It contains the following curves of bifurcation points:

- pitchfork and period doubling bifurcations of the equilibria, and bifurcations of the same type on the primary branches of the equilibria. For the lower equilibrium, we consider two secondary bifurcations, namely the pitchfork on the primary branch
coming from the leftmost stability boundary, and the period doubling on the branch born at the third boundary from the left, compare the sketch in figure 3.26. The corresponding bifurcations are displayed for the upper equilibrium, namely the pitchfork and period doubling points on the second and fourth primary branches from below, respectively. See section 3.4.4.1 for details on these bifurcations.

- saddle-center bifurcations corresponding to the large stable regions on the $x$-axis (and smaller ones above and below the lower equilibrium), discussed in section 3.4.4.2. Curves of the pitchfork bifurcations on the stable and unstable branches of the saddle-center are not computed.

- period doubling bifurcations of the stable fixed point coming from the broken invariant circle with rotation number 1, cf. section 3.4.4.3.

Curves of subharmonic bifurcations are shown in figure 3.40, and are omitted in the present diagram for clarity. Like the stability diagram, figure 3.44 is symmetric in both coordinate axes because of the symmetries $T$ and $U$ of $P$, and therefore only the first quadrant $\alpha, \beta \geq 0$ is displayed. As $\alpha$ increases, all curves seem to tend either to a stability boundary of the upper equilibrium, or to the $\alpha$-axis.

**Remark 38**: The symmetries $T$ and $U$ not only involve reflections and rotations in the parameter plane but also a transformation in phase space. Hence a bifurcation curve and its reflection in the $\beta$ axis correspond to bifurcations at different locations in phase space. This is clearly illustrated in section 3.4.4.2, where it is shown that a saddle-center bifurcation on the lower branch of the bifurcation curve takes place at the $x$-axis, while on the upper part it takes place at the line $x = \pi$, see figures 3.30, 3.33, and 3.34.
3.A Numerical computation of the ‘lowest’ invariant circle

This appendix describes the numerical method to compute the location of the ‘lowest’ invariant circle, used in subsection 3.3.3. This location is computed at grid points in the parameter plane. In fact, we compute the intersection of the ‘lowest’ invariant circle with the line $x = 0$, since the islands in between the circles are much larger there than at $x = \pi$. At each parameter point we start with an estimate for $y$ which is below the location of the ‘lowest’ invariant circle, e.g., $y = 0$. We then increase $y$ with a fixed increment until an invariant circle is obtained. Then the step is refined to improve the estimate. We start with a step size between $10^{-4}$ and $10^{-3}$, depending on the parameter point, and refine each time by a factor of 10 until the step size is $10^{-6}$.

At each $y$, we compute 2500 iterates of the corresponding orbit of $P$, and supplement them with their symmetric counterparts (under the reversible symmetry $SR : (x, y) \mapsto (-x, y)$) to get a total of 5000 points. The $x$-coordinates of the corresponding orbit of the lift of $P$ on $\{(x, y)\} = \mathbb{R} \times \mathbb{R}$ are also computed. Further, to estimate the Lyapunov exponent, the orbit of a random vector under the map $DP$ is tracked. Each time that the next iterate is computed, the following tests are performed to check whether the orbit is an invariant circle:

- If the $y$-coordinate is negative, the orbit is definitely not an invariant circle (of rotational type), and hence is discarded.

- If the norm of the transported random vector is larger than some given value (typically of the order $10^6$), the orbit is considered chaotic, so it is discarded.

- If the maximal difference between $x$-coordinates of successive points on the lift is more than $2\pi$ larger than the minimal difference, the orbit can not be an invariant circle, and is discarded.

If the orbit is discarded, we increase $y$ and try the next orbit. Otherwise, at each tenth iterate the orbit points are sorted by their $x$-coordinate and the following tests are done:

- The orbit is discarded if its slope is larger than a given value, depending on the parameters. For this value we take $2 \sqrt{\alpha + \beta}$ (this is twice the maximal slope of the ‘frozen’ separatrix, compare section 3.3.1). The slope is estimated by the difference of neighboring points, and this test is only done for $x$ close to zero (we take $|x| < 0.1$), where the slope of the invariant circles is large.

- If the ordering of the points in the sorted list is not invariant under the map $P$, the orbit is discarded. This means the following. Let $z_0, z_1, \ldots, z_n$ be the unsorted finite orbit, with $z_j = (x_j, y_j)$, and $z_j = P^j(z_0)$. Suppose that $z_k$ and $z_{k+m}$, for some $k, k+m \in \{0, 1, \ldots, n\}$, are two consecutive points in the sorted list. If the orbit corresponds to an invariant circle, then the map $P^m$ maps the part of the curve between $z_k$ and $z_{k+m}$ to the part between $z_{k+m}$ and $z_{k+2m}$. Hence the next point in the ordered list has to be $z_{k+2m}$, except in the case that $k + 2m$ falls outside
the range 0, 1, . . . , n, or there is a point between \( z_{k+m} \) and \( z_{k+2m} \) whose pre-image under \( P^m \) falls outside the range. If the orbit is not an invariant circle, but for example chaotic, then the ordering is generically not preserved.

- We compute an upper and lower bound on the rotation number. If the lower bound is larger than the upper bound, the orbit is discarded. The bounds are computed as follows. Using the notation of the previous test, let \( n_k \) be the number of ‘turns’ around the cylinder of the map \( P \) between \( z_0 \) and \( z_k \) (i.e., \( n_k \) is the number of times that \( x_{j+1} < x_j \) for \( 0 \leq j < k \)). Let \( z_k \) and \( z_{k+m} \) be two consecutive points in the sorted list, with \( x_k < x_{k+m} \). If \( m > 0 \), then the rotation number is at least \( \frac{n_{k+m} - n_k}{m} \). Otherwise, it is at most \( \frac{n_{k+m} - n_k}{m} \).

Again, if the orbit is discarded, we proceed with the next \( y \). Otherwise it is accepted as an invariant circle, and the estimate for \( y \) is refined as described above. To obtain an estimate for the corresponding \( y \)-value at \( x = \pi \) (this is the desired location of the ‘lowest’ invariant circle), we take the 20 points on the orbit nearest to \( x = \pi \), and fit a parabola through them. A numerical test, recomputing the orbit with many more points (e.g., 10000), shows that this is a good approximation of the desired value, with an error of at most \( 10^{-6} \).

In this appendix we discuss the numerical integrator used to compute phase portraits and bifurcation and stability diagrams, called the Taylor series method, compare, e.g., [BWZ71, BS98]. For phase portraits one simply needs to compute the Poincaré map $P$. Stability and bifurcation diagrams are obtained by numerical continuation, which also requires the first and (at bifurcation points) second variationals of $P$. We concentrate here on integration of $P$ only; the variationals can be treated similarly.

Let us first explain why we use the Taylor series method. For long time integrations of a Hamiltonian vector field, a symplectic method is more efficient than a non-symplectic one, in terms of required CPU time for a given global error tolerance, because its propagated error grows linear with the integration time, compared to quadratic growth for non-symplectic methods, see [CSS93, CSS97, SS92b, SSC94]. Although formally the Taylor series method is non-symplectic, numerical experiments show that it preserves the symplectic form very well, because of its large time step, compared with a high order approximation of the orbit. Moreover, the integrator preserves the equivariant and reversible symmetries $S$, $\mathcal{R}$, $\mathcal{T}$ and $\mathcal{U}$ of the forced pendulum. For more background on symplectic and non-symplectic integrators we refer to [CSS92, HNW93, MQR99, SS92a, Ske99].

We first describe the Taylor series method. As before, let the vector field $X$ of the forced pendulum be given by the differential equations

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -(\alpha + \beta \rho(t)) V'(x) \\
\dot{t} &= 1,
\end{align*}
\]

(3.7)

where $\rho(t) = \cos t$ and $V(x) = \cos x - 1$. Let $(x(t), y(t), t)$ be a solution of this system, depending on the parameters $\alpha, \beta$, with initial condition $(x(t_0), y(t_0)) = (x_0, y_0)$. Since $x(t)$ and $y(t)$ are analytic functions of $t$, they satisfy

\[
(x(t+h), y(t+h)) = \sum_{k \geq 0} \frac{h^k}{k!} \left( \frac{\partial}{\partial t} \right)^k (x(t), y(t)),
\]

(3.8)

for $|h|$ sufficiently small. The Taylor series method approximates $x(t+h)$ and $y(t+h)$ by $\hat{x}(t+h)$ and $\hat{y}(t+h)$, given by

\[
\hat{x}(t+h) = \sum_{k=0}^{N} \hat{x}_k h^k, \quad \hat{y}(t+h) \text{ analogous},
\]

(3.9)

for some $N \in \mathbb{N}$, and

\[
\hat{x}_k = \frac{1}{k!} \left( \frac{\partial}{\partial t} \right)^k \hat{x}(t).
\]

Here the dependence of $\hat{x}_k$ (and $\hat{y}_k$) on $t$ is suppressed. Given $\hat{x}$ and $\hat{y}$, all coefficients $\hat{x}_k$ and $\hat{y}_k$ can be computed using the differential equations (3.7).
Remark 39: The coefficients $\hat{x}_k$ and $\hat{y}_k$ are efficiently computed by recursion. Abbreviate $q(t) = \alpha + \beta \cos t$, $u(\hat{x}(t)) = \sin (\hat{x}(t))$ and $v(\hat{x}(t)) = \cos (\hat{x}(t))$, and define $q_k$, $u_k$ and $v_k$ analogous to $\hat{x}_k$. Then $\hat{x}_0 = x(t)$, $\hat{y}_0 = y(t)$, and for $k > 0$, by (3.7) and Leibniz’ rule

$$\hat{x}_k = \frac{1}{k} \hat{y}_{k-1}$$

$$\hat{y}_k = \frac{1}{k} \sum_{j=0}^{k-1} q_j u_{k-1-j}.$$  

The coefficients $q_k$ and $u_k$ are computed recursively. Indeed, $u_0 = \sin (\hat{x}(t))$, $v_0 = \cos (\hat{x}(t))$, and for $k > 0$, again using Leibniz’ rule

$$u_k = \sum_{j=1}^{k} \frac{j}{k} v_{k-j} \hat{x}_j$$

$$v_k = -\sum_{j=1}^{k} \frac{j}{k} u_{k-j} \hat{x}_j.$$  

Thus each integration step only requires four trigonometric evaluations.

The local (truncation) error $\varepsilon$ in $(\hat{x}(t+h), \hat{y}(t+h))$ is defined by $\varepsilon = ||(x(t+h), y(t+h)) - (\hat{x}(t+h), \hat{y}(t+h))||$, assuming that $(x(t), y(t)) = (\hat{x}(t), \hat{y}(t))$. Let $R$ be the radius of convergence of (3.8), then

$$|| (\hat{x}_k, \hat{y}_k) || \approx AR^{-k}$$

for $k$ sufficiently large, and some constant $A > 0$.

Hence the truncation error is given by

$$\varepsilon \approx \sum_{k>N} \frac{A}{R} \left( \frac{h}{R} \right)^k \approx || (\hat{x}_{N+1}, \hat{y}_{N+1}) || h^{N+1} \frac{1}{1 - h/R},$$

and for small $h/R$ we can approximate

$$\varepsilon \approx || (\hat{x}_{N+1}, \hat{y}_{N+1}) || h^{N+1}.$$  

For given step size $h$ and error tolerance $\varepsilon$ this determines an optimal choice for $N$, namely $N$ is the smallest integer for which $|| (\hat{x}_k, \hat{y}_k) || h^k < \varepsilon$ holds for all $k > N$.

Taking $h = 2\pi / M$ for some $M \in \mathbb{N}$, the Poincaré map $P$ is computed by applying the above algorithm $M$ times to a given point $(x_0, y_0)$. This yields an approximation $(x_1, y_1)$ to $P(x_0, y_0)$. Since the local error in the integrator is of order $h^{N+1}$, and the number of integration steps in one period is of order $1/h$, the symplectic form satisfies

$$dx_1 \wedge dy_1 - dx_0 \wedge dy_0 = O(h^N).$$

We define the error $\varepsilon_k$ in the symplectic form after $k$ iterates by

$$dx_k \wedge dy_k = (1 + \varepsilon_k) dx_0 \wedge dy_0,$$
Table 3.2: Speed and symplecticness of the Taylor integrator, tested at several parameter points, by computing 10000 iterates of the chaotic orbit starting at \((x, y) = (0, 0.1)\). The speed is measured in iterates per second, and the error \(\varepsilon_k\) in the symplectic form is computed after \(k = 2000, 4000, 6000, 8000, 10000\) iterates.

where \((x_k, y_k)\) is the numerical approximation to the \(k\)-th iterate of \((x_0, y_0)\).

Table 3.2 shows the results of a few numerical experiments with this integrator. In each experiment, we computed 10000 iterates \((x_k, y_k)\), \(k = 0, \ldots, 10000\) of the chaotic orbit of \(P\) starting at \((x, y) = (0, 0.1)\). The experiments were done using a Pentium II 400 MHz with Debian Linux 2.2.17. The integrator was written in C++, and compiled using g++ -O (version 2.95.2). The table shows for several parameter points the speed of the integrator and the error \(|\varepsilon_k|\) in the symplectic form after \(k = 2000, 4000, 6000, 8000, 10000\) iterates. We conclude that even at high parameter values the integrator achieves a reasonable speed, and preserves the symplectic form for a large number of iterates.
3. C Proof of lemma 35

In this appendix we prove lemma 35. For convenience it is repeated here.

**Lemma 40** The eigenvalues of the linearized Poincaré map at the upper or lower equilibrium are ±1 on curves in the parameter plane of the form

\[ \mathbb{R} \ni \beta \mapsto (\alpha, \beta) = (f(\beta), \beta) \in \mathbb{R}^2, \]

for some smooth function f. At these curves the stability of the equilibrium changes.

The linearized Poincaré map at the upper or lower equilibrium corresponds to the 2π-flow of the linearized vector field

\[ DX = y \frac{\partial}{\partial x} + (\alpha + \beta \rho(t)) \frac{\partial}{\partial y} + \frac{\partial}{\partial t}, \]

where \( \rho(t) = \cos t \). It needs to be shown that the eigenvalues of the linearized map are ±1 on curves in the parameter plane that can be represented as graphs over \( \beta \). We shall prove this for the more general case that \( \rho \) is a 2π-periodic, even function.

In short, the proof runs as follows. Each stability curve is given by an implicit equation of the form \( g(\alpha, \beta) = 0 \). We show that \( \frac{\partial}{\partial \alpha}g(\alpha, \beta) \neq 0 \) at any zero \( (\alpha, \beta) \) of \( g \). Hence the implicit function theorem implies that at any point \( (\alpha, \beta) \) on the stability curve, this curve can locally be seen as a graph over \( \beta \). Hence the graph can not have ‘endpoints’, and the stability curve is a graph over \( \beta \) defined on the whole of \( \mathbb{R} \), as needs to be shown.

Let us first drive the implicit equation \( g(\alpha, \beta) = 0 \). In general, the linearized Poincaré map \( DP = DP(\alpha, \beta) \) is given by

\[ DP = \begin{pmatrix} a(\alpha, \beta) & b(\alpha, \beta) \\ c(\alpha, \beta) & d(\alpha, \beta) \end{pmatrix}, \]

where \( ad - bc = 1 \), because the map is area-preserving. Since the function \( \rho \) is even, the maps \( P \) and \( DP \) are \( R \)-reversible, for the temporal symmetry \( R : (x, y) \mapsto (x, -y) \) defined in section 3.2. A brief calculation shows that for the linear map \( DP \) this means

\[ a(\alpha, \beta) \equiv d(\alpha, \beta). \]

At the stability curves the eigenvalues of \( DP \) are both +1 or both −1, and hence the trace \( a + d \) equals +2 or −2. Together with \( a = d \) and \( ad - bc = 1 \) this means that \( b = 0 \) or \( c = 0 \). Vice versa, if \( b = 0 \) or \( c = 0 \), then area-preservation and \( a = d \) imply that the eigenvalues are ±1. Thus we have proved the following.

**Lemma 41** Let the coefficients \( b \) and \( c \) of the linear Poincaré map \( DP \) be defined as above. Then the stability curves are given by the equations

\[ b(\alpha, \beta) = 0, \text{ or } c(\alpha, \beta) = 0. \]
Remark 42: Each resonance point of the lower equilibrium on the $\alpha$ axis lies on two stability curves. One is given by $b = 0$, the other by $c = 0$.

Now let $(\alpha^*, \beta^*)$ be such that $c(\alpha^*, \beta^*) = 0$. To prove that $\frac{\partial}{\partial \alpha} c(\alpha^*, \beta^*) \neq 0$ — the proof for $b$ runs similar — we consider the integral curve $t \mapsto (x(t), y(t), t)$ of the linearized vector field with initial condition $(x(0), y(0)) = (0, 1)$. Evaluated at $t = 2\pi$ this solution gives the second column of the matrix $DP$. Thus $x(2\pi) = c(\alpha^*, \beta^*) = 0$ and $y(2\pi) = \pm 1$ equals the eigenvalues. Let $(u(t), v(t)) = \frac{\partial}{\partial \alpha} (x(t), y(t))$, then $(u(0), v(0)) = (0, 0)$ and $t \mapsto (u(t), v(t), t)$ is an integral curve of the first variational of $DX$, given by

$$v \frac{\partial}{\partial u} + ((\alpha^* + \beta^* \rho t)u + x(t)) \frac{\partial}{\partial v} + \frac{\partial}{\partial t},$$

It has to be shown that $u(2\pi) \neq 0$. The vector fields for $(x, y)$ and $(u, v)$ yield

$$x(t)\dot{v}(t) - u(t)\dot{y}(t) = x(t)^2.$$

Integrating this equation from $t = 0$ to $t = 2\pi$, using $\frac{\partial}{\partial t}(xv - uy) = x\dot{v} - u\dot{y}$ and the initial and final conditions $x(2\pi) = u(0) = v(0) = 0$ and $y(2\pi) = \pm 1$, we obtain

$$\mp u(2\pi) = \int_0^{2\pi} x^2(t)dt > 0,$$

and hence $u(2\pi) \neq 0$, as needed.

Thus by the implicit function theorem, locally around any zero $(\alpha^*, \beta^*)$ of $c$ there exists a curve of the form $\beta \mapsto (f(\beta), \beta)$, such that $f$ is smooth, $f(\beta^*) = \alpha^*$, and $c(f(\beta), \beta) = 0$ for $|\beta - \beta^*|$ sufficiently small. This local graph can be extended to all $\beta$. Indeed, suppose that $I$ is the largest interval containing $\beta^*$ on which the stability curve can be defined as a graph over $\beta$. Then $I$ is open since at any point of $I$ the ‘local graph’ construction can be done. Furthermore, $I$ is closed because of continuity of $c$. Thus $I = \mathbb{R}$, and lemma 35 is proved.