The parametrically forced pendulum
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Chapter 1

Introduction

Nonlinear dynamical systems, depending on parameters, often occur in the modeling of physical, chemical, biological and economical phenomena. A general goal is to understand the qualitative, and possibly also quantitative behavior of such systems, in particular the changes in this behavior due to variations of certain parameters. Qualitative changes usually are called bifurcations. The dynamical behavior is organized in terms of equilibrium, periodic and quasi-periodic solutions, and in general invariant manifolds, as well as corresponding homo- and heteroclinic cycles, which are often accompanied by chaos. Bifurcations can occur when certain (sub-) harmonic solutions branch off, heteroclinic connections break, etc.

We consider the quite general situation where the stability of some of these organizing centers (periodic solutions) depends non-trivially on parameters. For the investigation of the dynamical regimes near the organizing centers we use information, obtained by averaging associated to the periods at hand. Also for nearby parameter values, a certain form of averaged information can be used. This so-called method of normal forms goes back to Poincaré [Poi92] and Lie and has some reminiscence to Lindstedt series.

The present research aims at developing methods to obtain such information in certain cases where the equations of motion are explicitly given. In such cases the information usually can be obtained only in an approximate way, partly employing numerical methods. We intend to deal with this uncertainty by analyzing the persistence of the results under small perturbations.

This brings us to the realm of perturbation theory, for classical reference again see Poincaré [Poi92] and Kolmogorov [Kol54]. During the last century this area of research expanded a lot in the direction of bifurcation theory, often involving the application of singularity theory to dynamical systems, and in the direction of Kolmogorov-Arnol'd-Moser theory, which concerns persistence of quasi-periodic invariant tori. For more details and references, see below.

The results of perturbative methods, in turn, can be used as input for computer assisted methods, which widely opens the scope of systems that can be explored in this way. Apart from directly computing individual solutions or phase portraits, more sophisticated methods exist, based on numerical continuation of the information obtained so far. This approach enables a systematic bifurcation analysis of families of dynamical
systems, also for moderate values of the parameters, compare [HH64, Doe81, Sim89, DKK91a, DKK91b, BGM+92, Kuz95]. Here the interpretation of the numerical data has to be guided by the theory of dynamical systems. For a case study in the dissipative setting, see [BST98].

The present work presents a case study in the conservative setting, namely the parametrically forced pendulum. This classical example is representative for the class of nonlinear mechanical devices without friction and few degrees of freedom. Insight in such model examples is of great importance for the understanding of more complicated systems.

We conclude with a brief outline of the introduction. The model of the parametrically forced pendulum is described in the next section, where also some historical background is given. Subsequently, section 1.2 summarizes the methods and results of the following chapters, while a brief overview of these chapters is given in section 1.3.

1.1 Setting of the problem

The parametrically forced pendulum is a planar pendulum with a vertically oscillating suspension point. It is given by the equation of motion

\[ \ddot{x} + (\alpha + \beta \rho(t)) V'(x) = 0, \text{ with } i = 1, \]

where \( \rho(t) = \cos t \) is the forcing function and \( V(x) = \cos x - 1 \) is the potential function. The deviation from the upper equilibrium is denoted by \( x \in S^1 \), where \( S^1 = \mathbb{R}/(2\pi\mathbb{Z}) \) is the unit circle. This means that \( x = 0 \) corresponds to the upper equilibrium, where the pendulum stands upright, while \( x = \pi \) is the location of the lower equilibrium, where it hangs down. The independent variable is time, while \( t \in S^1 \) denotes the phase of the forcing. In the sequel we will be sloppy on this point, and call \( t \) the time variable.

The parameters \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R} \) correspond to the square of the eigenfrequency of the free pendulum at the lower equilibrium, and the amplitude of the forcing, respectively. For \( \beta = 0 \) we retrieve the free planar pendulum. Due to symmetries of the system one can restrict to the quadrant of positive \( \alpha \) and \( \beta \) in the parameter plane without loss of generality.

Introducing the velocity \( y = \dot{x} \), the forced pendulum is given by the system

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -(\alpha + \beta \rho(t)) V'(x), \\
i &= 1.
\end{align*}
\]

The corresponding vector field \( X = X(x, y, t; \alpha, \beta) \) is given by

\[ X = y \frac{\partial}{\partial x} - (\alpha + \beta \rho(t)) V'(x) \frac{\partial}{\partial y} + \frac{\partial}{\partial t}, \tag{1.1} \]

with \( (x, y, t) \in S^1 \times \mathbb{R} \times S^1 \). This is a Hamiltonian vector field in \( 1 \frac{1}{2} \) degree of freedom, with ‘time-dependent’ Hamilton function

\[ H(x, y, t; \alpha, \beta) = \frac{1}{2} y^2 + (\alpha + \beta \rho(t)) V(x). \]
It is natural to work with its Poincaré map \( P : \mathbb{S}^1 \times \mathbb{R} \rightarrow \mathbb{S}^1 \times \mathbb{R} \), defined on the section \( t = 0 \) in phase space. The map corresponds to the flow of \( X \) over time \( 2\pi \), and is orientation- and area-preserving.

The goal of this thesis is to give a description of the global coherent (i.e., non-chaotic) dynamics of the map \( P \). Sometimes we extend our scope to a more general system, where \( \rho \) and \( V \) are arbitrary \( 2\pi \)-periodic, smooth functions. We refer to this as the \textit{general} setting. Depending on \( V \) and \( \rho \), the vector field \( X \) can have symmetries, that directly translate to symmetries of \( P \). Indeed, if \( V \) is even, then \( P \) is \( S \)-equivariant, meaning that \( S \circ P \circ S = P \), where \( S \) is the involution given by \( S : (x, y) \mapsto (-x, -y) \). If \( \rho \) is even, then \( P \) is \( R \)-reversible, with \( R : (x, y) \mapsto (x, -y) \), meaning that \( R \circ P \circ R = P^{-1} \).

The special case of the forced pendulum, with \( \rho(t) = \cos t \) and \( V(x) = \cos x - 1 \), has both symmetries.

In the general setting, the system (1.1) is widely studied as a classical perturbation problem, and also numerically. In the remainder of this section we present an overview of the literature. To start with, singularities of \( V \) correspond to equilibria of the vector field, and the stability type at such an equilibrium is determined by the linearized system (at the equilibrium), given by

\[
\dot{x} = \pm (\alpha + \beta \rho(t))x,
\]

where for simplicity we assumed that \( V''(x_0) = \pm 1 \) at the equilibrium \( x = x_0 \) — this can always be achieved by rescaling \( \alpha \) and \( \beta \). Then a minus sign corresponds to a local minimum of \( V \), and a plus sign to a local maximum. In the pendulum case this corresponds to the lower and upper equilibrium, respectively. The linear equation is known as Hill’s equation or the periodic Schrödinger equation. In case \( \rho(t) = \cos t \) it is called the Mathieu equation. Hill’s equation shows that the lower equilibrium of the forced pendulum destabilizes when its eigenfrequency resonates with the frequency of the forcing, while the upper equilibrium becomes stable in several ranges in the parameter plane. Figure 1.1 shows a stability diagram of the Mathieu equation illustrating this.

The linear equation is extensively studied, especially in the near-integrable case of small \( \beta \), and for \( \alpha, \beta \rightarrow \infty \), cf., e.g., Mathieu [Mat68], van der Pol and Strutt [PS28], Stoker [Sto50], Meixner and Schäffer [MS54], Hale [Hal63, Hal69], Harell [Har79], Arnołd’ [Arn83], Weinstein and Keller [WK85, WK87], Levi et al. [LW95], and Broer et al. [BL95, BS98, BS00]. Levi [Lev88, Lev99] gives a physical explanation of the stabilization of the upper equilibrium of the pendulum. Arnołd’ [Arn89] discusses the special case \( \rho(t) = \text{sgn}(\sin t) \), where the system can be integrated explicitly.

Let us return to the nonlinear dynamics. For \( \beta = 0 \) the system is integrable, and thus for small \( \beta \) an integrable approximation can be constructed. Using such an approximation, Broer and Vegter [BV92] present a bifurcation and persistence analysis of \( P \) for general \( V \) and \( \rho \), valid near local minima of \( V \), and near resonance points of the corresponding fixed point of \( P \) on the \( \alpha \)-axis. In particular this analysis applies to resonances of the lower equilibrium of the forced pendulum.

The nonlinear system can also be analyzed for parameters near an organizing center at a point ‘at infinity’ in the parameter plane. Consider the rescaling in phase and
Figure 1.1: Numerically computed stability diagram of the Mathieu equation $\ddot{x} = (\alpha + \beta \rho(t))x$, with $\rho(t) = \cos t$. Shading indicates stability of the equilibrium at $(x, y) = (0, 0)$. The diagram is symmetric in the $\alpha$-axis. The lower equilibrium of the forced pendulum corresponds to $\alpha < 0$, the upper to $\alpha > 0$.

parameter space given by

$$(x, y, t) \mapsto (\tilde{x}, \tilde{y}, \tilde{t}) := (x, \varepsilon y, t), \ (\alpha, \beta) \mapsto (\tilde{\alpha}, \tilde{\beta}) := (\varepsilon^2 \alpha, \varepsilon^2 \beta), \text{ and } X \mapsto \tilde{X} := \varepsilon X,$$

(1.2)

where $\varepsilon$ is a small parameter. Dropping the tildes for simplicity, the rescaled vector field is given by

$$X = y \frac{\partial}{\partial x} - (\alpha + \beta \rho(t)) V'(x) \frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial t}.$$ 

This is a planar system depending on a slowly varying parameter $t$, that can be seen as a perturbation of a ‘frozen’ system at the organizing center, i.e., where this parameter is fixed. Its analysis involves adiabatic invariants, where orbits crossing the ‘frozen’ separatrices, and separatrices shifting from one equilibrium to another play an important role. For background on adiabatic theory and separatrix crossing, compare Neishtadt et al. [Nei75, LN94, NST97], Tennyson et al. [TCE86], Arnol’d [Arn89], Cary and Skodje [CS89], Elskens and Escande [EE91], and Henrard [Hen93]. A complete analysis of the coherent dynamics near the organizing center remains an open problem up to now.

In many regions in phase and parameter space KAM theory can be applied to establish the existence of quasi-periodic invariant circles of $P$. For example, many invariant circles of the integrable system at $\beta = 0$ persist for small $\beta$. One can also show the existence of invariant circles winding around the $(x, y)$-cylinder for large $y$, using the rescaling (1.2) defined above. For more details on KAM theory we refer to Kolmogorov [Kol54], Arnol’d [Arn63], and Moser [Mos62]. It is applied to the forced pendulum or closely related Hamiltonian systems by, e.g., Moser [Mos67, Mos89a, Mos89b], Chierchia and Zehnder [CZ89], You [You90], Levi et al. [Lev90, Lev91, LL91], and Huang [Hua98].
Finally, there are experimental results. Physical experiments, showing the stabilization of the upper equilibrium, have been conducted by, e.g., Leven et al. [LPWK85], Smith and Blackburn [SB92, BSGJ92], and Acheson and Mullin [AM93]. For numerical experiments we mention Acheson [Ach95], Bishop and Sudor [BS99], and references therein.

We conclude this section with the remark that the forced pendulum system has many applications in other fields, for example engineering, wave propagation, and molecular dynamics, see, e.g., van der Pol and Strutt [PS28], Bishop and Sudor [BS99], Basov and Pavlichenkov [BP94], and references in these papers.

1.2 Method and results

Recall that we are after the understanding of the coherent dynamics of the parametrically forced pendulum. Our investigation starts with a perturbation analysis in several limiting cases in phase-parameter space, in particular for small $\beta$, and for large $y$. In both cases the system is near an integrable one.

In the first case, the dynamics for small $\beta$, thus obtained, is continued by numerical means to moderate parameter values. The numerical investigation is guided by Hamiltonian bifurcation theory. In this section we present the perturbative and numerical methods involved in this analysis, as well as a summary of the results. This is worked out in more detail in the following chapters.

Using normal form theory an integrable approximation to the map $P$ is obtained in the general setting, valid near the origin $(\alpha, \beta) = (0, 0)$ of the parameter plane, where $P$ is degenerate. The approximation has the same dynamics as $P$ in some qualitative sense, made more precise below. The bifurcations of the approximation and their persistence in a restricted class of systems are discussed, in the spirit of [BV92]. The normal form analysis is carried out for a general system of the form (1.1), with arbitrary $V$ and $\rho$, and we look for persistence in this class of systems, where the symmetries $S$ and $R$ are optional.

Next we turn to the forced pendulum, with $V(x) = \cos x - 1$ and $\rho(t) = \cos t$. Integrable approximations, valid for small $\beta$, are obtained near the lower equilibrium by [BV92], and near the upper by the analysis mentioned above. Their dynamical properties are continued to a larger part of phase and parameter space by numerical methods. The linearized system, given by Mathieu’s equation, induces a division of the parameter plane into bounded regions, so-called fundamental domains, suggesting that we can restrict our study to a few of these domains.

For large $y$, after the rescaling (1.2), the forced pendulum is close to the ‘frozen’ system at the organizing center at infinity in the parameter plane. By KAM theory, applied to the rescaled system, at any parameter point a restriction to a bounded region of interest in phase space is possible. Thus we restrict our attention to a bounded region in phase-parameter space, and use numerical methods to investigate aspects of the dynamics not covered by the integrable approximations.

Below, we first discuss the integrable approximation near $(\alpha, \beta) = (0, 0)$, its bifurca-
tions and their persistence, then the region of interest and the fundamental domains, and finally the numerical methods and their results.

1.2.1 Integrable approximation near \((\alpha, \beta) = (0, 0)\)

As noted before, the unstable upper equilibrium of the free pendulum can be stabilized by a vertical oscillation of its suspension point in a specific frequency and amplitude domain. (The system near the upper equilibrium is also known as the inverted pendulum.) Figure 1.1 shows that for small \(\beta\) the only stability change occurs on a curve in the parameter plane passing through the origin \((\alpha, \beta) = (0, 0)\). The Poincaré map \(P = P_{\alpha,\beta} : \mathbb{S}^1 \times \mathbb{R} \rightarrow \mathbb{S}^1 \times \mathbb{R}\) is highly degenerate at this parameter point, even in the general setting. Indeed, at \((\alpha, \beta) = (0, 0)\) it is given by

\[
P(x, y) = (x + 2\pi y, y).
\]

We therefore compute, in the general setting, an integrable approximation to the Poincaré map \(P\), valid for global \(x \in \mathbb{S}^1\), and for small \(\alpha, \beta, \text{and} y\). The integrable approximation, obtained by normal form (or averaging) techniques, is the flow \(\dot{X}^{2\pi}\) of a planar Hamiltonian vector field \(\dot{X}\) over time \(2\pi\), that preserves the symmetries \(S\) and \(R\) of \(P\), if present. Of course the approximation does not capture the full dynamics of the original problem (see below for more details), but it is sufficiently good to show bifurcations of \(P\) at the upper equilibrium.

The approximation permits the analysis of codimension one bifurcations taking place on curves in the parameter plane passing through the origin \((\alpha, \beta) = (0, 0)\), and on the \(x\)-axis in the cylindrical phase space.

We are also interested in persistence of the bifurcations under small perturbations corresponding to small perturbations of \(V\) and \(\rho\), i.e., such that the perturbed system (in the non-integrable setting) is of the form (1.1).

The bifurcations and their persistence depend on the symmetries of the unperturbed system, and whether or not they are preserved by the perturbation. It turns out that without loss of generality we can take the system to be \(R\)-reversible, since the integrable approximation is always \(R\)-reversible, independent of the presence of this symmetry in the original map \(P\). Thus we always assume \(R\)-reversibility, which leaves us with three symmetry contexts:

1. The spatio-temporally symmetric case, where \(P\) and its perturbations are \(R\)-reversible and \(S\)-equivariant.

2. The temporally symmetric case, where \(P\) and its perturbations are only \(R\)-reversible.

3. The perturbative temporally symmetric case, where \(P\) is \(R\)-reversible and \(S\)-equivariant, but its perturbations are only \(R\)-reversible.

To discuss persistence we can again make use of the integrable system \(\dot{X}^{2\pi}\). In each symmetry context, local models are constructed at bifurcation points, using (equivariant)
singularity theory in the setting of planar Hamiltonian systems corresponding to perturbations of $V$ an $\rho$ in this symmetry context, see, e.g., Poénaru [Poé76], Gibson [Gib79], Martinet [Mar82], Damon [Dam84, Dam88, Dam95], Golubitsky et al. [GS85, GSS88], Wassermann [Was88], and Broer et al. [BLV98, BHLV98]. A local model incorporates all qualitative dynamics of the integrable approximation $\tilde{X}^2\epsilon$, and is structurally stable in the symmetry context at hand. In many cases the integrable approximation itself is a local model. An approximation to the dynamics of $P$ valid for all $x \in S^1$, and small $y$, $\alpha$, and $\beta$, then can be obtained by ‘gluing’ together the local models at bifurcation points, using a standard homotopy method, cf. [BV92].

![Poincaré map](image)

Figure 1.2: Top: Poincaré map $P$ of the forced pendulum, with $\rho(t) = \cos t$, for one parameter point $(\alpha, \beta)$ on each side of the stability boundary. The upper equilibrium is at the center of each phase portrait, the lower is near the left and right boundaries. Bottom center: sketch of the local stability diagram, shading indicates stability of the upper equilibrium. Bottom left and right: corresponding normal form phase portraits. The stability boundary is a line of Hamiltonian pitchfork bifurcations.

Let us summarize the results. A normal form computation yields a planar vector field $\tilde{X}$, such that $P$ is approximated by the flow of $\tilde{X}$:

$$P \simeq \tilde{X}^2\epsilon + p,$$

compare Chen [Che63], Gustavson [Gus66], Takens [Tak74c], Broer et al. [Bro81, BT89, BV92], van der Meer [Mee85], Sanders and Verhulst [SV85], and Hoveijn [Hov92]. Here
\( \dot{X}^t \) is the flow of \( X \) over time \( t \), and \( p \) is a perturbation, infinitely flat as \( (y, \alpha, \beta) \to 0 \). The map \( \dot{X}^{2\pi} \) is orientation and area preserving since \( X \) is Hamiltonian (and hence divergence free), and preserves the symmetries \( S \) and \( R \) of \( P \), if present.

Instead of the map \( P \), we consider the integrable approximation \( \dot{X}^{2\pi} \). As to the effect of the perturbation \( p \), we restrict to a few general remarks, foregoing a detailed analysis. The bifurcations of \( \dot{X}^{2\pi} \) at the upper equilibrium, as well as the fixed and period two points involved, persist under the perturbation \( p \). By the implicit function theorem, other non-degenerate periodic points also persist. Generically, homoclinic and heteroclinic connections will be broken, leading to transversal homoclinic intersections. By KAM theory, invariant circles of \( X^{2\pi} \) with Diophantine rotation number persist, while others break, creating strings of islands.

![Sketches of the local stability diagram in the spatio-temporally symmetric (left) and perturbative temporally symmetric (right) contexts. Stability of the upper equilibrium again is indicated by shading. The coding is explained in figure 1.4.](image)

The qualitative dynamics of the integrable approximation \( \dot{X}^{2\pi} \) is completely determined by the level curves of the corresponding Hamiltonian \( \dot{H} \). Thus from now on non-symplectic transformations are allowed, that change the time parameterization, but not the configuration of level curves.

We use such a transformation, and a further rescaling of phase variables and parameters, to simplify the planar Hamiltonian, and remove the degeneracy at \( (\alpha, \beta) = (0,0) \). Since the rescaling is not defined for \( \beta = 0 \), the connection between the dynamics of \( P \) and \( \dot{H} \), as sketched above, only holds for small \( (\alpha, \beta) \) in the complement of the \( \alpha \)-axis. In particular, it does not hold at the degenerate point \( (\alpha, \beta) = (0,0) \).

In the pendulum case, with \( \rho(t) = \cos t \) and \( V(x) = \cos x - 1 \), the map \( P \) is \( S \)-equivariant and \( R \)-reversible. Phase portraits of the planar Hamiltonian and of the original Poincaré map are shown in figure 1.2, revealing that the bifurcation is of pitchfork type. The bifurcation is persistent under \( S \)-equivariant perturbations, i.e., in the spatio-temporally symmetric case. In the perturbative temporally symmetric case, where non-\( S \)-equivariant perturbations are allowed, it breaks up into a transcritical, a saddle-center and a heteroclinic bifurcation, compare figure 1.3.
\textbf{PF}: pitchfork bifurcation \hspace{1cm} \diamond \hspace{1cm} \text{unstable period 2 points}

\textbf{PD}: period doubling bifurcation \hspace{1cm} \bullet \hspace{1cm} \text{stable period 2 points}

\textbf{SC}: saddle-center bifurcation \hspace{1cm} \blacktriangledown \hspace{1cm} \text{unstable fixed points}

\textbf{TC}: transcritical bifurcation \hspace{1cm} \triangle \hspace{1cm} \text{stable fixed points}

\textbf{HC}: heteroclinic bifurcation

--- stable manifold upper equilibrium

- - - - unstable manifold upper equilibrium

------ stable manifold lower equilibrium

----- unstable manifold lower equilibrium

Figure 1.4: \textit{Coding of periodic points, invariant manifolds of equilibria, and bifurcations of the Poincaré map, used in phase portraits, bifurcation and stability diagrams.}
1.2.2 Region of interest

From now on we restrict to the case of the pendulum, with \( V(x) = \cos x - 1 \) and \( \rho(t) = \cos t \). For any given \((\alpha, \beta)\), a numerical study of this system can be restricted to a bounded region in phase space, called the region of interest, by excluding regions of periodic and quasi-periodic motion at large \(|y|\).

As said before, KAM theory implies that, at any parameter point \((\alpha, \beta)\), invariant circles of \( P \) with Diophantine rotation number exist for \(|y|\) sufficiently large. Moreover, their relative measure tends to 1 as \( y \to \infty \). Here we use again the rescaling (1.2) and subsequently pass to suitable action-angle coordinates to obtain a well-defined perturbation problem, where the fast pendulum, i.e., at unbounded \( y \), \( \alpha \) and \( \beta \), is the unperturbed system.

The invariant circles are interlaced with resonant dynamics, by the Poincaré-Birkhoff theorem [Poi12, Bir13, Bir25]. By excluding this region of periodic and quasi-periodic dynamics, we obtain a bounded ‘region of interest’ in phase space.

A numerical computation, and some results from adiabatic theory in several regions in the parameter plane, suggest that a good estimate for the bound on the region of interest, in particular at large \( \alpha \) and \( \beta \), is

\[
|y| \approx 2\sqrt{\alpha + \beta}, \quad \alpha \geq 0, \quad \beta \geq 0.
\]

In chapter 3 we discuss how a rigorous bound can be obtained for all parameter values by a quantitative KAM theorem, referring for concrete results to [BNS]. Moreover, it is shown that the relative measure of invariant circles of \( P \) is given by \( 1 - O(1/y) \) as \( y \to +\infty \), for all \((\alpha, \beta)\).

1.2.3 Fundamental domains

![Diagram of stability diagrams](image)

Figure 1.5: Part of the stability diagrams of the equilibria of the forced pendulum, first separate, then combined. Shading indicates stability. The diagrams are symmetric around the \( \alpha \)-axis. The coding is explained in figure 1.4. In the rightmost diagram, thickened curves correspond to boundaries of fundamental domains.

We systematically explore the dynamics in the region of interest by dividing the parameter plane into bounded sets, so-called fundamental domains, based on the linearized
dynamics at the equilibria. The linearized dynamics at the upper equilibrium is given by the vector field

$$X_{\text{lin}} = \frac{\partial}{\partial x} y + (\alpha + \beta \rho(t)) \frac{\partial}{\partial y} x + \frac{\partial}{\partial t},$$

where $\rho(t) = \cos t$,

and the corresponding stability diagram is shown in figure 1.1. By a symmetry of the nonlinear system, involving a transformation of the parameters, the stability diagram of the lower equilibrium is obtained by reflecting figure 1.1 in the $\beta$-axis, see figure 1.5. Figure 1.6 displays the two stability diagrams combined, showing that the parameter plane is divided into stability regions, bounded by curves of bifurcation points of the two equilibria. These regions are of four different types, depending on the stability types of the equilibria.

![Stability diagram of the forced pendulum](image)

Figure 1.6: Stability curves of both equilibria of the forced pendulum (with $\rho(t) = \cos t$) combined in one plot. The thicker curves are the boundaries of fundamental domains.

Fundamental domains are bounded regions in the parameter plane. Except near the $\beta$ axis they consist of four stability regions, one of each type. In figures 1.5 and 1.6 the fundamental domain boundaries are indicated by thick curves. Near the $\beta$-axis the fundamental domains have two or three stability regions.

There are four different kinds of fundamental domains, depending on the equilibrium bifurcations that occur on its boundaries. Indeed, the curves of bifurcation points are of pitchfork or period doubling type, and the fundamental domains (except those near the $\beta$ axis) have four sides, with opposite sides corresponding to different bifurcation types.

If we traverse a fundamental domain from left to right, we pass through two bifurcation curves of the lower equilibrium (including the right hand boundary of the fundamental domain, but not the left one), where fixed or period two points are born in pitchfork or period doubling bifurcations. If the fundamental domain is traversed from bottom to top, the upper equilibrium undergoes two bifurcations. Here also new fixed or periodic points are created. As an example, figure 1.7 shows a sketch of the dynamics near the
Figure 1.7: Sketch of one fundamental domain, with an indication of the stability types and bifurcations of the two equilibria, cf. figure 1.4 for the coding. The bifurcations on the stability boundaries may differ from one fundamental domain to the other. In each stability region, the dynamics of the Poincaré map is sketched near the upper equilibrium in the top frame, and near the lower in the bottom one. In each frame the equilibrium is at the center.

equilibria in one fundamental domain. We note that the presented phase portraits are only correct regarding the stability types and bifurcations of the equilibria; for example, the real Poincaré map generically has no heteroclinic connections.

In domains of the same type the linearized dynamics at the two equilibria is similar. We conjecture that the same is true to some extent for the nonlinear coherent dynamics. By this conjecture our numerical study can be restricted to a few fundamental domains. Below we discuss the results of explorations in some domains near the origin \((\alpha, \beta) = (0, 0)\) of the parameter plane, and some farther away.

1.2.4 Coherent dynamics near \((\alpha, \beta) = (0, 0)\)

Integrable approximations for small \(\beta\) at the bifurcation points of the upper and lower equilibrium are used as a starting point for a numerical investigation of the dynamics. Local phase portraits of the approximation at the degenerate point \((\alpha, \beta) = (0, 0)\) of the upper equilibrium, discussed above, are shown in figure 1.8, together with a stability diagram in the parameter plane, compare [BHN98, BHNV99]. Figure 1.9 displays similar phase portraits at the resonance points on the \(\alpha\)-axis of the lower equilibrium, as obtained
Figure 1.8: Stability diagram and phase portraits of an integrable approximation to $P$ at the degenerate point $(\alpha, \beta) = (0, 0)$ of the upper equilibrium. The stability boundary consists of pitchfork bifurcations.

by [BV92]. These resonance points are given by

$$(\alpha, \beta) = \left( \frac{1}{4} k^2, 0 \right), \text{ with } k = 1, 2, \ldots.$$  

The curves of bifurcation points passing through the resonance points are of pitchfork type if $k$ is even, and of period doubling type otherwise. At both equilibria the non-integrable Poincaré map $P$ is an infinitely flat perturbation of the integrable approximation as the phase variables go to the equilibrium and the parameters to the degenerate or resonance point.

The integrable approximations provide a skeleton for the full dynamics on the whole parameter plane, in the sense that all the stability-changing bifurcations of the equilibria are present in the approximations. Moreover, the bifurcations that we find in the approximations, and the periodic points involved, persist for all higher $\beta$. In fact, these bifurcations occur on curves in the parameter plane of the form $\alpha = f(\beta)$, i.e., on graphs over $\beta$, and hence the curves have to intersect the $\alpha$-axis.

More detailed information can be obtained by numerical methods. We compute phase portraits and bifurcation diagrams, by straightforward integration and numerical continuation, using the Taylor series method as numerical integrator, see appendix 3.B for more details. As an example, figure 1.10 shows some phase portraits at several parameter points on the line $\alpha = 0.079$, while figure 1.11 displays bifurcation diagrams of the two equilibria on the same line, with $\beta$ as bifurcation parameter.

Due to the symmetries $S$ and $R$ of $P$, the periodic points bifurcating from the equilibria are restricted to the lines $x = 0$, $x = \pi$ and $y = 0$. Numerical continuation shows that, with increasing $\beta$, they either go towards the upper equilibrium, or escape from the equilibria in $y$-direction (but do not leave the region of interest). Sometimes these points undergo further pitchfork or period doubling bifurcations, see figure 1.11. This does not
show up in the integrable approximations, since these are only valid in a small neighborhood in phase-parameter space of the equilibria at a resonance or degenerate point. It seems that only stable periodic points bifurcate. There is some numerical evidence that these bifurcations are the start of period doubling cascades, see [KH98, McL81].

We mention two further phenomena that influence the coherent dynamics to a large extent. The integrable Poincaré map on the $\alpha$-axis has an invariant circle winding around phase space, consisting entirely of fixed points. For $\beta \neq 0$ this invariant circle is generically destroyed, creating a stable and an unstable fixed point, see figure 1.10. The stable one of these is located above the lower equilibrium, in a stable region that is large for small $(\alpha, \beta)$. It undergoes a period doubling bifurcation, and other $p: q$ subharmonic bifurcations.

More large stable regions can be found around the $x$-axis. These correspond to a stable period two orbit born at a saddle-center bifurcation, see the diagram with $\beta = 2$ in figure 1.10.

Figure 1.12 shows the location of these bifurcations in the parameter plane. Except for the bifurcations of the equilibria (i.e., the stability boundaries), all lie on curves that for large $\alpha$ seem to tend either to the $\alpha$-axis or to one of the stability curves of the upper equilibrium.

### 1.2.5 Coherent dynamics for large $(\alpha, \beta)$

For large $(\alpha, \beta)$, the ‘region of interest’ is almost completely filled with chaos, compare the last phase portrait in figure 1.10. On the other hand, at any parameter point there is coherent dynamics of relatively small but positive measure, see Neishtadt et al. [NST97],
Figure 1.10: Phase portraits of the Poincaré map at $\alpha = 0.079$ and various $\beta$, as indicated. Outside a ‘region of interest’, the dynamics consists mainly of invariant circles winding around the cylinder. Inside, chaos increases with $\beta$, but coherent dynamics remains present. Both equilibria undergo period doubling and pitchfork bifurcations. Parts of invariant manifolds of (unstable) equilibria are also plotted. Periodic points and invariant manifolds are marked according to the coding in figure 1.4.
Figure 1.11: Bifurcation diagrams on the parameter line $\alpha = 0.079$ in $(\beta, x, y)$-space, and projected onto the $(\beta, x)$- and $(\beta, y)$-planes. They display period one and two points of $P$ bifurcating from the upper and lower equilibria. The lines parallel to the $\beta$-axis correspond to the positions of the equilibria. For simplicity only the branches in one quadrant of $(\beta, x, y)$-space are displayed, the others are their symmetric counterpart under reflection in the $(\beta, x)$- or $(\beta, y)$-plane.
Figure 1.12: Curves of codimension one bifurcations in the parameter plane. The curves correspond to stability changes of the equilibria, a period doubling bifurcation of a broken invariant circle, bifurcations of the periodic points bifurcated from the equilibria (marked as secondary bifurcations), and a saddle-center bifurcation, as indicated.
and Broer et al. [BLS]. In a qualitative sense the dynamics does not seem to vary much with the parameters.

1.3 Overview

In chapter 2 the integrable approximation to the Poincaré map near the origin of the parameter plane is computed, and its bifurcations and their persistence are analyzed in the general setting of arbitrary $V$ and $\rho$, in the three aforementioned symmetry contexts. Chapter 3 extends these results in the case of the forced pendulum to a larger part of phase-parameter space, dealing with fundamental domains, the region of interest, and numerical continuation.

The main part of chapter 2 is published as [BHNV99], while a few paragraphs and the appendix on normal form theory are borrowed from [BHN98]. Chapter 3 is submitted for publication as [BHN+]. The chapters are self-contained and can be read independently.