Component analysis of multisubject multivariate longitudinal data
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5. Structured latent curve component models for longitudinal three-way data

5.1. Introduction

In Chapter 4, the use of smoothness constraints in the Tucker3 and CP models was discussed. A different approach to constraining the occasion component matrix is to impose a particular functional form. This approach is attractive if theoretical considerations suggest a certain functional relationship. Then, potential advantages of imposing such a functional form are reduction of error fitting and an increase in the interpretability of the estimated model.

Browne and Du Toit (1991) and Browne (1993) proposed structured latent curve models for learning data, in which the occasion component scores are parameterized parsimoniously in terms of a small number of parameters. In the next sections, the principles of this factor analysis approach are elaborated for use with the Tucker3 model for longitudinal three-way data. One could also apply the approach with the usually more restricted CP model and the less restricted Tucker2 and Tucker1 models, as will be explained briefly in the discussion section.

5.2. Structured latent curve two-way component models for growth data

The idea of structured latent curve three-way component models for growth data will be introduced by discussing a structured latent curve two-way component model for growth data. The two-way structured latent curve component model is a straightforward modification of Browne and Du Toit’s (1991) model, which was further elaborated by Browne (1993). In the present model, it is assumed that all individual growth curves are a weighted sum of certain basis functions, while the average growth curve as estimated by the model follows a particular function. As will be shown, the imposed constraints can be generalized fairly easily to component models for longitudinal three-way data.

5.2.1. The SLC two-way component model for data measured at equal time points

Let $X (K \times I)$ denote the matrix of scores of $I$ subjects ($i=1,\ldots,I$) on one variable collected at $K$ occasions. The $k^{th}$ occasion takes place at time point $t_k (k=1,\ldots,K)$, and $t_i$ is usually 0. To facilitate the explanation and notation, it is assumed here that the scores of the subjects are collected at the same time points. At a later stage (Section 5.2.3) the case involving different time points for different subjects is discussed.
The model imposed in structured latent curve (SLC) two-way component analysis is given by

\[ X = CA' + E, \]  

(5.1)

where \( C \) \((K \times R)\) denotes the occasion component scores matrix, \( A \) \((I \times R)\) denotes the subject component scores matrix, and \( E \) \((K \times I)\) denotes the matrix of residuals; the \( R \) components in the occasion component scores matrix represent \( R \) basis functions evaluated at each of the \( K \) time points. The estimated series of a particular subject is thus a weighted sum of the \( R \) basis functions.

A method for specifying the basis curves in structured latent curve analysis, as suggested by Browne and Du Toit (1991) and Browne (1993), is followed broadly here in defining the SLC two-way component model. It is assumed that the average scores across subjects at successive occasions \( (\bar{x}_k, k=1,...,K) \), as estimated by the model, follow a particular target function, which is evaluated in the time points \( t_1,...,t_K \). Browne (1993) discusses the Gompertz function (Richards, 1959), the exponential function and the logistic function as target functions. These are particularly useful functions for modeling growth data. The exponential function has no point of inflection, whereas the logistic and Gompertz functions do have such a point. One should choose the target function on the basis of the characteristics of the data. We will use the Gompertz curve as target function. It is given by

\[ g(t_k, \tau) = \alpha \exp \left\{ \ln \left( \frac{\beta}{\alpha} \right) \exp \left( -\frac{\gamma}{\ln 1} \right) \right\}, \]  

(5.2)

where \( \tau = (\alpha, \beta, \gamma) \), \( \alpha \) is the asymptote and represents potential performance, \( \beta \) is the function value at \( t_0 = 0 \) and represents previously acquired skills, and \( \gamma \) determines the rate of change, which reflects the learning speed. The Gompertz curve is asymmetrical around its point of inflection, which occurs for \( t_k = \frac{1}{\gamma} \ln \left( -\frac{\beta}{\alpha} \right) \), with function value \( g(t_k, \tau) = \frac{\alpha}{e} \).

For any subject \( i \), the scores at successive time points \( t_k = t_1,...,t_K \), as estimated by the model, follow some function that is not necessarily monotonic. The first-order Taylor polynomial about \( \tau \) is used to model the score of subject \( i \) at time point \( t_k \) (Browne & Du Toit, 1991), namely as

\[ x_{ki} = g(t_k, \tau) + a_{i1} g'_1 (t_k, \tau) + a_{i2} g'_2 (t_k, \tau) + a_{i3} g'_3 (t_k, \tau) + e_{ki}, \]  

(5.3)
where $x_{ki}$ denotes the observed score of subject $i$ at time point $t_k$, and $g_m'(t_k, \tau) = (\partial g(t_k, \tau) / \partial \tau_m) g(t_k, \tau)$, with $\tau_1 = \alpha, \tau_2 = \beta, \tau_3 = \gamma$. The first order derivatives $g_m'(t_k, \tau)$ are given in (5.6). As the Gompertz curve (5.2) has the property

$$g(t_k, \tau) = \alpha + \beta \exp(-\ln(\exp(-t_k \gamma)))$$

the observed score of subject $i$ at occasion $k$ (=time point $t_k$) can be written as

$$x_k = a_{i1} g_1'(t_k, \tau) + a_{i2} g_2'(t_k, \tau) + a_{i3} g_3'(t_k, \tau) + e_{ki}, \quad (5.5)$$

where $a_{i1} = (\alpha + \bar{a}_{i1})$, $a_{i2} = (\beta + \bar{a}_{i2})$, and $a_{ir}$ denotes element $(i, r)$ of the subject component scores matrix $A$. To ensure that the average estimated curve follows a Gompertz curve with parameters $\alpha, \beta, \gamma$, the average component score (across subjects) is required to be $\alpha$ and $\beta$ for the first and second components respectively, and 0 for the third component. The elements of the matrix $C$ ($K \times 3$) are now defined as

$$c_{k1} = g_1'(t_k, \tau) = \{1 - \exp(-t_k \gamma)\} \exp\left\{\ln\left(\frac{\beta}{\alpha}\right) \exp(-t_k \gamma)\right\}$$

$$c_{k2} = g_2'(t_k, \tau) = \left(\frac{\alpha}{\beta}\right) \exp\left\{-t_k \gamma + \ln\left(\frac{\beta}{\alpha}\right) \exp(-t_k \gamma)\right\}, \quad (5.6)$$

$$c_{k3} = g_3'(t_k, \tau) = -\alpha \ln\left(\frac{\beta}{\alpha}\right) t_k \exp\left\{-t_k \gamma + \ln\left(\frac{\beta}{\alpha}\right) \exp(-t_k \gamma)\right\},$$

$k = 1, \ldots, K$. The first function in (5.6) is called the ‘asymptote basis function’, the second the ‘initial value basis function’ and the third the ‘learning rate basis function’. Examples of the three basis curves, and the associated target functions are shown in Figure 5.1.
The ‘asymptote basis function’ increases monotonically from zero at time point 0 towards an asymptote of one. A relatively large weight for this function for a particular subject denotes that the subject’s estimated growth curve has a relatively large asymptotic value compared to the other subjects. The ‘initial value basis function’ starts at 1, increases up to the time point where the inflection point of the target function occurs (here at t=3.26), and then decreases towards an asymptote of zero. Unfortunately, this function not only reflects the initial value, but also the fact that the rate of learning first increases and then decreases. However, if the basis functions are evaluated only at time points after the inflection point, because of the particular choice of the measurement occasions, then the function decreases monotonically and the ‘initial value basis function’ can be interpreted well as only reflecting the initial value. If a particular subject has a relatively large weight for this function, this denotes that this subject’s estimated growth curve starts relatively high compared to the other subjects. The third basis function can be viewed as the ‘rate of learning basis function’. A relatively large weight for a particular subject for this function denotes that the subject concerned shows a relatively high rate of learning compared to the other subjects.
5.2. Structured latent curve two-way component models for growth data

5.2.2. Fitting the SLC two-way component model to data with equal measurements

So far we have followed the approach proposed by Browne and Du Toit (1991) and Browne (1993). In their factor model, they make certain assumptions concerning the error structure. To fit the model to data, maximum likelihood estimation is used under the assumption of multivariate normality. Here, however, we use a component approach, and we estimate the parameters of the model via least squares estimation. As a result, to fit the SLC two-way component model to data, we propose to minimize the sum of squared residuals by minimizing

\[ f_i(\tau, A) = \| X - CA \|^2, \]  

where \( \tau = (\alpha, \beta, \gamma) \), the matrix \( C \) (\( K \times 3 \)) consists of elements given by (5.6), and hence is a function of \( \alpha, \beta, \) and \( \gamma \) only, and the matrix \( A \) (\( I \times 3 \)) is restricted so that \( \frac{1}{I}A = [\alpha \beta 0] \). In fitting the SLC two-way component model, using algorithms programmed in MATLAB5 (1998), we found that estimating the parameters of a reparametrization of the Gompertz function given in (5.2), and the corresponding reparametrizations of the basis functions in (5.6), which were given by Browne and Du Toit (1991), led to far fewer computational problems than using the function in (5.2) itself as a target function. Therefore, the (reparametrized) occasion component scores matrix \( \tilde{C} \) (\( K \times R \)) and the subject component scores matrix \( \tilde{A} \) (\( I \times R \)), which can be transformed to \( C \) and \( A \) without altering the model estimates, as already noted by Browne (1993), are estimated. The parametrization of the Gompertz function used by Browne and Du Toit (1991) is

\[ \tilde{g}(t_k, \theta) = \theta_1 \exp \{-\theta_2 \exp(-t_k \theta_3)\}. \]  

(5.8)

The parameters \( \theta_1, \theta_2, \) and \( \theta_3 \) in (5.8) can be transformed into the parameters of the Gompertz function in (5.2) by taking \( \alpha = \theta_1; \beta = \theta_1 \exp(-\theta_2) \) and \( \gamma = \theta_3 \). The associated basis functions and hence the elements of the matrix \( \tilde{C} \) (\( K \times 3 \)) are defined as

\[
\begin{align*}
\tilde{c}_{k1} &= \tilde{g}'(t_k, \theta) = \exp \{-\theta_2 \exp(-t_k \theta_3)\}, \\
\tilde{c}_{k2} &= \tilde{g}'(t_k, \theta) = -\theta_1 \exp(-t_k \theta_3) \tilde{c}_{k1}, \\
\tilde{c}_{k3} &= \tilde{g}'(t_k, \theta) = -t_k \theta_2 \tilde{c}_{k2}.
\end{align*}
\]  

(5.9)

The elements of \( C \) and \( \tilde{C} \) are related to each other as follows:
5. SLC component models for longitudinal three-way data

\begin{align}
  c_{k1} &= \tilde{c}_{k1} + \frac{1}{\theta_1} \tilde{c}_{k2} \\
  c_{k2} &= \frac{-1}{\theta_1 \exp(-\theta_2)} \tilde{c}_{k2} \\
  c_{k3} &= \tilde{c}_{k3}.
\end{align}

(5.10)

Now, the model can be fitted to data by minimizing

\[ \tilde{f}_1(\theta, \tilde{A}) = \|X - \tilde{C}\tilde{A}\|^2, \]

(5.11)

with \( \tilde{A} \) (3x3) restricted so that \( \frac{1}{\theta} \mathbf{1}' \tilde{A} = [\theta_1 \ 0 \ 0] \), because then the average estimated curve follows the Gompertz function as given in (5.8), with parameters \( \theta \). Finding an optimal solution for \( \tilde{A} \) with unconstrained \( \tilde{a}_1 \), which is the first column of \( \tilde{A} \), is easier than finding an optimal solution with \( \frac{1}{\theta} \mathbf{1}' \tilde{a}_1 = \theta_1 \). Because the latter constraint can be satisfied afterwards without affecting the fit, (5.11) can be minimized subject to constraints on the second and third columns of \( \tilde{A} \) only. When (5.11) has been minimized subject to the constraint on the second and third columns of \( \tilde{A} \), \( \mathbf{1}' \tilde{a}_2 = \mathbf{1}' \tilde{a}_3 = 0 \), the constraint on the first column of \( \tilde{A} \), \( \frac{1}{\theta} \mathbf{1}' \tilde{a}_1 = \theta_1 \), can be satisfied as follows: Let \( \theta_1^c \), \( \tilde{a}_2^c \) and \( \tilde{a}_3^c \) be the estimates of \( \theta_1 \), \( \tilde{a}_2 \) and \( \tilde{a}_3 \), respectively, after convergence. Then defining \( \theta_1 = \frac{1}{\theta} \mathbf{1}' \tilde{a}_1 \), rescaling \( \tilde{a}_2 \) and \( \tilde{a}_3 \) as \( \tilde{a}_2 = \frac{\theta_1^c}{\theta_1} \tilde{a}_2^c \) and \( \tilde{a}_3 = \frac{\theta_1^c}{\theta_1} \tilde{a}_3^c \), and recomputing the elements of \( \tilde{C} \) using (5.9), the product \( \tilde{C}\tilde{A}' \) remains equal, and hence the fit is unaffected.

After having estimated the parameters \( \theta_1, \theta_2, \theta_3 \), one can obtain the parameters of the target function in (5.2) by taking \( \alpha = \theta_1; \beta = \theta_1 \exp(-\theta_2) \) and \( \gamma = \theta_3 \). The matrices \( \tilde{C} \) and \( \tilde{A} \) can be obtained from \( \tilde{C} \) and \( \tilde{A} \) as \( \tilde{C} = \tilde{C} \mathbf{T} \), and \( \tilde{A} = \tilde{A} (\mathbf{T}^{-1})' \), where \( \mathbf{T} \) is given by
5.2. Structured latent curve two-way component models for growth data

\[
T = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{\theta_1} & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\] (5.12)

Estimates of $\theta$ and $\tilde{A}$ that minimize (5.11) subject to the constraint $\frac{1}{T}1'\tilde{a}_2 = \frac{1}{T}1'\tilde{a}_3 = 0$ can be obtained via an alternating least squares (ALS) algorithm. The steps in the algorithm are as follows.

**Initial estimates of $\theta$ and $\tilde{A}$ for the ALS algorithm**

To start an ALS algorithm, initial estimates of the parameters are needed. An initial estimate of $\theta$, and hence of $\tilde{C}$ as well, is obtained here by minimizing

\[
f_2(\theta) = \| \bar{x} - \theta_1 \exp\{-\theta_2 \exp(-t \theta_3)\} \|^2,
\] (5.13)

where $\bar{x}$ denotes the $K \times 1$ vector containing the average scores over subjects at time points $t_1, \ldots, t_K$, and $t$ is the vector with time points $t_1, \ldots, t_K$. Least squares estimates of $\theta$ are obtained using the Levenberg-Marquardt algorithm (Seber & Wild, 1989). This algorithm needs starting values. As was discussed in Section 5.2.1, the parameters of the Gompertz curve have a physical interpretation. In the reparametrized form, the parameter $\theta_1$ is the asymptote, $\theta_2$ governs the distance from the asymptote, and $\theta_3$ governs the learning rate. The point of inflection occurs at time point $t_i = \log \theta_2 (\theta_3)^{-1}$. Rational starting values can be obtained from a plot of the averaged scores across time, perhaps with a freehand smooth curve added (Seber & Wild, 1989). One should guess the asymptote value (denoted by $\bar{x}_{max}$), the value at time point zero (denoted by $\bar{x}_0$) and the time point at which the inflection point occurs (denoted by $t_{inf}$). Then, as starting values, one takes $\bar{x}_{max}$ for $\theta_1$, $-\log(\bar{x}_0 / \bar{x}_{max})$ for $\theta_2$, and subsequently $(\log \theta_2 / t_{inf})$ for $\theta_3$.

As an initial component scores matrix $\tilde{A}$, an unconstrained least squares estimate of $\tilde{A}$, considering $\tilde{C}$ fixed, is taken as $\tilde{A} = X'\tilde{C}(\tilde{C}'\tilde{C})^{-1}$. To satisfy the constraint $\frac{1}{T}1'\tilde{a}_2 = \frac{1}{T}1'\tilde{a}_3 = 0$, the second and third columns of the estimate of $\tilde{A}$ are centered.
Finding an update of $\tilde{A}$

To find an update of $\tilde{A}$, it is proposed that the first, and the second and third columns of $\tilde{A}$ be updated separately. To update the first column of $\tilde{A}$, we minimize (5.11) considering $\tilde{C}$, and $\tilde{a}_2$ and $\tilde{a}_3$ fixed. The latter is equivalent to minimizing

$$f_3(\tilde{a}_1) = \| \left( X - \tilde{c}_2 \tilde{a}_2' - \tilde{c}_3 \tilde{a}_3' \right) - \tilde{c}_1 \tilde{a}_1' \|_2^2 = \| X_{-23} - \tilde{c}_1 \tilde{a}_1' \|_2^2,$$

(5.14)

where $X_{-23} = X - \tilde{c}_2 \tilde{a}_2' - \tilde{c}_3 \tilde{a}_3'$. An update of $\tilde{a}_1$ can be obtained by taking $\tilde{a}_1 = X'_{-23} \tilde{c}_1 \left( \tilde{c}_1' \tilde{c}_1 \right)^{-1}$. An update for $\tilde{a}_2$ and $\tilde{a}_3$ subject to the constraint $\tilde{1}'_f \tilde{a}_2 = \tilde{1}'_f \tilde{a}_3 = 0$ can be obtained by minimizing

$$f_3(\tilde{a}_2, \tilde{a}_3) = \| \left( X - \tilde{c}_1 \tilde{a}_1' \right) - \left( \tilde{c}_2 \tilde{a}_2' + \tilde{c}_3 \tilde{a}_3' \right) \|_2^2,$$

subject to $\tilde{1}'_f \tilde{a}_2 = \tilde{1}'_f \tilde{a}_3 = 0$. This is equivalent to minimizing

$$f_3(\tilde{A}) = \| X_{-1} - \tilde{C} \tilde{A}' N' \|_2^2,$$

(5.16)

over arbitrary $\tilde{A}$ ($(I-1)\times 2$), where $X_{-1} = X - \tilde{c}_1 \tilde{a}_1'$, $\tilde{C}$ ($K\times 2$) denotes a matrix containing the second and third columns of $\tilde{C}$, $N$ ($I\times (I-1)$) is a basis for the null space of $\tilde{1}'_f$, and $[\tilde{a}_2 | \tilde{a}_3] = N\tilde{A}$. A least squares update for $\tilde{A}$ can be obtained by taking $\tilde{A} = (N'N)^{-1}N'X_{-1}\tilde{C}(\tilde{C}'\tilde{C})^{-1}$ (Penrose, 1956). Hence, $[\tilde{a}_2 | \tilde{a}_3] = N\tilde{A} = JX_{-1}'\tilde{C}(\tilde{C}'\tilde{C})^{-1}$, where $J$ is the centering operator $N(N'N)^{-1}N'$.

Finding an update of $\theta$

To update $\theta$, we minimize (5.11) considering $\tilde{A}$ fixed. Least squares estimates of $\theta_1$, $\theta_2$, and $\theta_3$ can be obtained using the Levenberg-Marquardt algorithm (Seber & Wild, 1989). Note that after convergence the elements of $\tilde{a}_2, \tilde{a}_3, \tilde{c}_2$, and $\tilde{c}_3$ have to be rescaled to satisfy the constraint $\frac{1}{3} \tilde{1}'_f \tilde{a}_i = \theta_i$ as described above. Furthermore, the solutions for $\tilde{C}$ and $\tilde{A}$ have to be transformed to $C$ and $A$ as described above.
5.2.3. The SLC two-way component model for growth data measured at unequal time points

The method discussed in Section 5.2.1 aims at decomposing a two-way matrix consisting of the scores of \( I \) subjects on one variable measured at \( K \) time points into an occasion component matrix \( C \), and a subject component matrix \( A \). The three components in the occasion component matrix represent three basis functions evaluated at each of the occasions \( t_1, \ldots, t_K \). In practice, the measurement time points are not necessarily equal for the different subjects. This can be easily covered in the model by allowing the evaluated time points of the basis functions to differ across subjects. To explore this idea further, the time point of the \( k_i^{th} \) measurement of subject \( i \) is indicated by \( t_{ik_i} \), where \( k_i=1, \ldots, K_i \) denotes the sequence number of measurements of subject \( i \), \( i=1, \ldots, I \), and the time points at which the scores of any subject are collected are denoted by \( t_1, \ldots, t_K \), where \( K \) is the total number of different time points at which measurements are available. A two-way data matrix \( X^F \) \((K \times I)\) is constructed with rows corresponding to all time points for which measurements are encountered. The scores of subject \( i \), \( i=1, \ldots, I \), are positioned in the rows of \( X^F \) that correspond to the time points at which the scores of subject \( i \) are collected \((t_{i1}, \ldots, t_{ik_i})\); the remaining \( K-K_i \) values are missing. A binary indicator matrix \( W \) \((K \times I)\) is constructed, with zeros indicating missing values in \( X^F \), \( k=1, \ldots, K \), \( i=1, \ldots, I \). The SLC two-way component model for growth data can be fitted to the observed data by minimizing

\[
\sum(I,J) \left\| W_{ij} (X^F_{ij} - CA') \right\|^2, \tag{5.17}
\]

where * denotes the Hadamard (or elementwise) product, \( C \) is restricted to be a function of \( \alpha, \beta \) and \( \gamma \) according to (5.6), and \( \frac{1}{I} 1' A = [\alpha, \beta, 0] \). Analogously to the unweighted case (see Section 5.2.2), the minimization problem in (5.17) is treated using a reparametrized Gompertz function (see (5.8)) as target function, and associated reparametrized matrices \( \tilde{C} \) and \( \tilde{A} \), with \( \tilde{A} \) constrained so that \( \frac{1}{I} 1' \tilde{A} = [\theta_1, 0, 0] \). Again, the constraint on the first column of \( \tilde{A} \) can be satisfied by proper rescaling after minimizing (5.17), and hence we will only discuss an ALS algorithm to minimize the weighted least squares loss function in (5.17) subject to the constraint \( 1' \tilde{a}_2 = 1' \tilde{a}_3 = 0 \). Weighted least squares estimates of \( \theta_1, \theta_2, \) and \( \theta_3 \), and hence of \( \tilde{C} \), can be obtained using the Levenberg-Marquardt algorithm (Seber & Wild, 1989). The first column of \( \tilde{A} \) can be updated by means of row-wise weighted least squares regression (Gabriel & Zamir, 1979). The second and third columns of \( \tilde{A}, \tilde{a}_2 \) and \( \tilde{a}_3 \), can be estimated using the procedure for weighted least squares fitting by Kiers (1997b). The latter method boils down to missing data imputation in \( X^F \), where the optimal least squares estimates of the missing elements are imputed at
each step, followed by performing the OLS step to find updates for \( \tilde{a}_2 \) and \( \tilde{a}_3 \), which is discussed in Section 5.2.2.

The above approach is useful if the missing data are missing completely at random (Little & Rubin, 1987; see also Chapter 3). The degree of reliability of the estimates is influenced by the number of missing data as well as by the time points at which missing data occur. Generally, a large amount of missing data restricted to certain small time periods will decrease the reliability greatly. However, this also depends on the functional form of the true scores. For example, the occurrence of missing data at a certain time interval decreases the reliability more when the true scores fluctuate greatly, than when they fluctuate only slightly.

5.3. The SLC Tucker3 model for longitudinal three-way data

In this section, a three-way generalization of the SLC two-way component model, namely the SLC Tucker3 model, is elaborated. In the Tucker3 model, three-way data are decomposed into three component matrices (see Section 2.3). The components for the three modes are weighted via the core array. The SLC Tucker3 model, as elaborated here, is particularly useful if the longitudinal three-way data consist of scores on variables that are intended to measure a certain growth process. In the SLC two-way component model, it is assumed that the estimated average scores across subjects follow a Gompertz curve. In the SLC Tucker3 model, it is assumed that the estimated average scores across subjects and variables follow a Gompertz curve. Furthermore, in the models, the estimated score of a subject on a variable at a particular time point is a weighted sum of basis functions that are evaluated in that particular time point. Just as in the SLC two-way component model, the measurement time points are not necessarily identical for all subjects, and for all variables. However, to facilitate description and notation, the model is described here as if the measurements are collected at the same time points for all subjects and all variables.

The SLC Tucker3 model is given by

\[
X_c = C G_c (B \otimes A') + E_c, \quad (5.18)
\]

where \( X_c \) (\( K \times IJ \)) denotes the matricized three-way array \( X \) (\( I \times J \times K \)), \( C \) (\( K \times 3 \)) the occasion component scores matrix, \( A \) (\( I \times P \)) the subject component matrix, \( B \) (\( J \times Q \)) the variable component matrix, \( G_c \) (\( 3 \times PQ \)) the matricized core array \( G \) (\( 3 \times P \times Q \)), and \( E_c \) (\( K \times IJ \)) the matricized error array \( E \); the three components in the occasion component matrix \( C \) represent three basis functions following (5.6), that are evaluated at each of the time points \( t_1, \ldots, t_K \); the matrix \( G_c (B \otimes A') \) is restricted so that \( \frac{1}{IJ} [ \alpha ~ \beta ~ 0 ] (B \otimes A) G_c = [ \alpha ~ \beta ~ 0 ] \). Because of the nature of \( C \), and the restriction on the weights for the basis functions, the estimated average scores across variables and subjects follow a Gompertz curve.
5.3. The SLC Tucker3 model for longitudinal three-way data

The parameters $\alpha$, $\beta$, and $\gamma$ govern the estimated average scores across subjects and variables, and they are interpreted in the same way as in the two-way SLC model (see Section 5.2.1). The occasion component matrix consists of evaluations of three basis functions, which can be interpreted as the asymptote basis function, the initial basis function, and the rate of learning basis function. The subject and variable component matrices and the core array are interpreted just like their counterparts in the unconstrained Tucker3 model (see Section 2.7).

5.3.1. Fitting the SLC Tucker3 model to data

In this section, an algorithm to fit the SLC Tucker3 model to data will be discussed. The function to be minimized is the least squares loss function, for which an ALS algorithm is proposed.

The SLC Tucker3 algorithm aims at minimizing

$$f_7(\tau, A, B, G) = \|X_c - CG_e (B \otimes A')\|^2,$$

subject to the constraint $\frac{1}{I_I} (B \otimes A) G_e' = [\alpha \beta 0]$, and where $C$ consists of elements given by (5.6). Just as in the two-way case (see Section 5.2.2), a reparametrized version of the Gompertz function, given by (5.8), is used to arrive at an estimate of the parameters $\alpha$, $\beta$ and $\gamma$. Instead of $C$ and $G_e$, the (reparametrized) occasion component matrix $\tilde{C}$ and the core array $\tilde{G}_e$ which can be transformed to $C$ and $G_e$ without altering the model estimates, are estimated, analogously to the two-way case.

The elements of $\tilde{C}$ are given by (5.9), and thus are a function of $\theta_1$, $\theta_2$ and $\theta_3$ only. The model can be fitted to data by minimizing

$$f_8(\theta, A, B, \tilde{G}_e) = \|X_c - \tilde{C}G_e (B \otimes A')\|^2,$$

with $\tilde{C}$ given by (5.9), subject to the constraint $\frac{1}{I_I} (B \otimes A) \tilde{G}_e' = [\theta_1 0 0]$, to achieve the average estimated curve to follow (5.8), with parameters $\theta_1$, $\theta_2$ and $\theta_3$. Analogously to the algorithm to fit the SLC two-way component model (see Section 5.2.2), the constraint on the first column of $(B \otimes A) \tilde{G}_e'$ can be satisfied by proper rescaling of $\theta_1$ (and hence of $\tilde{C}$) after minimizing (5.20), subject to the constraints on the second and third columns of $(B \otimes A) \tilde{G}_e'$ only. In the SLC Tucker3 model, the rescaling should be compensated in the core matrix $\tilde{G}_e$. Estimates of $A$, $B$, $\tilde{G}_e$, and $\theta$ can be obtained by alternating least squares, as follows.
Initial estimates of $\theta$ and $A$ and $B$ for the ALS algorithm

To start the ALS algorithm, initial estimates of the parameters are needed. Analogously to the two-way case, initial values of $\theta$ can be chosen on the basis of a plot of the average scores across subjects and variables (see Section 5.2.2). Initial estimates of $A$ and $B$ can be obtained by taking the first $P$ and $Q$ eigenvectors of $X_aX_a'$ and $X_bX_b'$, respectively. Subsequently, an initial (unconstrained) estimate of $\tilde{G}_e$ is obtained by taking $\tilde{G}_e=(\tilde{C}'\tilde{C})^{-1}\tilde{C}'X_e(B \otimes A)(B'B \otimes A'A)^{-1}$ (Penrose, 1956). Those initial values for $\theta$, $A$, $B$ and $\tilde{G}_e$ suffice to start the algorithm by finding an estimate for the second and third columns of $\tilde{G}_e$.

Finding an update of $\theta$

An update of $\theta$, considering $A$ and $B$ fixed, can be obtained the same way as finding an update of $\theta$ is obtained in the SLC two-way component model, as discussed in Section 5.2.2.

Finding an update of the core array

To find an update for $\tilde{G}_e$, we propose updating the rows of $\tilde{G}_e$ successively. An update of the first row of $\tilde{G}_e$ can be obtained by minimizing

$$f_9(\tilde{g}_{e,1}) = \left\|X_e - \tilde{c}_{1\tilde{g}_{e,1}'} (B \otimes A') - \tilde{c}_3\tilde{g}_{e,3}' (B' \otimes A') - \tilde{c}_{1\tilde{g}_{e,1}'} (B \otimes A')\right\|^2$$

$$= \left\|X_{e,-23} - \tilde{c}_{1\tilde{g}_{e,1}'} (B \otimes A')\right\|^2,$$

(5.21)

where $\tilde{g}_{e,m}'$ $(1 \times QP)$ denotes the $m$th row of $\tilde{G}_e$, and $X_{e,-23} = (X_e - \tilde{c}_2\tilde{g}_{e,2}' (B' \otimes A') - \tilde{c}_3\tilde{g}_{e,3}' (B' \otimes A') \otimes (B' \otimes A')^{-1} (Penrose, 1956)$.

An update of the second row of $\tilde{G}_e$ can be obtained by minimizing

$$f_{10}(\tilde{g}_{e,2}) = \left\|X_e - \tilde{c}_{1\tilde{g}_{e,1}'} (B \otimes A') - \tilde{c}_3\tilde{g}_{e,3}' (B' \otimes A') - \tilde{c}_2\tilde{g}_{e,2}' (B' \otimes A')\right\|^2$$

$$= \left\|X_{e,-13} - \tilde{c}_{2\tilde{g}_{e,2}'} (B' \otimes A')\right\|^2,$$

subject to the constraint $1'(B \otimes A)\tilde{g}_{e,2} = 0$, where $X_{e,-13} = (X_e - \tilde{c}_1\tilde{g}_{e,1}' (B' \otimes A') - \tilde{c}_3\tilde{g}_{e,3}' (B' \otimes A') \otimes (B' \otimes A')^{-1})$. By requiring $\tilde{g}_{e,2} = N\tilde{g}_{e,2}$, where $N(QP \times (QP-1))$ is a basis for the nullspace of $1_{Ul}(B \otimes A)$, and minimizing (5.22) over
unconstrained $\tilde{g}_{c,2} ((QP-1)\times 1)$, a least squares estimate of $\tilde{g}_{c,2} = N \tilde{g}_{c,2}$, subject to the constraint $1'(B \otimes A)\tilde{g}_{c,2} = 0$, can be obtained. An update of the unconstrained vector $\tilde{g}_{c,2}$ can be obtained as $\tilde{g}_{c,2}' = (\tilde{c}_2' \tilde{c}_2)^{-1} \tilde{c}_2' X_{c,13} (B \otimes A) N (N'(B'B \otimes A'A)N)^{-1}'.

An update of the third row of $G_e$, subject to the constraint $1'(B \otimes A)\tilde{g}_{c,3} = 0$, can be obtained the same way as an update of the second row of $G_e$ was obtained.

Finding an update of $A$ and $B$

An update of $A$ can be obtained by minimizing

$$f_{11}(A) = \left\| X_c - \tilde{C} G_e (B' \otimes A') \right\|^2,$$

subject to the constraint $1'_f (B \otimes A) G_{23} = 0'$, where $G_{23} = [\tilde{g}_{c,2} | \tilde{g}_{c,3}]$. Upon defining $V$ as

$$V = \begin{bmatrix} 1'_f B \tilde{G}_2' \otimes 1'_f \\ 1'_f B \tilde{G}_3' \otimes 1'_f \end{bmatrix},$$

where $\tilde{G}_m (P \times Q)$ denotes the $m^{th}$ horizontal slab of the core array $\tilde{G}$ $(3 \times P \times Q)$, the latter constraints are equivalent to requiring $VV vec(A) = 0$, as is shown in Appendix 5.1. Minimizing (5.23) over $A$ is equivalent to minimizing

$$f_{12}(A) = \left\| X_a - A \tilde{G}_a (\tilde{C}' \otimes B') \right\|^2.$$  

(5.24)

An update for $A$ subject to the constraint $VV vec(A) = 0$ can be obtained by minimizing

$$f_{13}(\tilde{a}) = \left\| vec(X_a) - ((\tilde{C} \otimes B)\tilde{G}_a' \otimes I_f)N\tilde{a} \right\|^2,$$

(5.25)

where $N$ is a basis for the nullspace of $V$, and $vec(A) = N\tilde{a}$. By defining $Z = ((\tilde{C} \otimes B)\tilde{G}_a' \otimes I_f)N$, an update for $vec(A)$ can be obtained by taking $vec(A) = N(Z'Z)^{-1}Z' vec(X_a)$. Rearranging the elements of $vec(A)$ into an $I \times P$ matrix yields $A$.

Upon interchanging the role of the $A$ and $B$ mode, an update of $B$ can be obtained the same way an update for $A$ was obtained in the SLC Tucker3 model, as was discussed above.
Rescaling of the estimates after convergence of the ALS algorithm

After convergence, the rescaling of $\theta_1$ so that $\frac{1}{\theta_1} \textbf{1}'_\mu (\textbf{B} \otimes \textbf{A}) \tilde{\textbf{g}}_{c,1} = \theta_1$ can be obtained as follows: Let $\theta_1^c$, $\tilde{\textbf{g}}_{c,2}$ and $\tilde{\textbf{g}}_{c,3}$ be the estimates of $\theta_1$, $\tilde{\textbf{g}}_{c,2}$ and $\tilde{\textbf{g}}_{c,3}$, respectively, after convergence. After defining $\theta_1 = \frac{1}{\theta_1} \textbf{1}'_\mu (\textbf{B} \otimes \textbf{A}) \tilde{\textbf{g}}_{c,1}$, the second and third rows of $\tilde{\textbf{G}}_c$ are defined as $\tilde{\textbf{g}}_{c,2} = \frac{\theta_1^c}{\theta_1} \textbf{g}_{c,2}$ and $\tilde{\textbf{g}}_{c,3} = \frac{\theta_1^c}{\theta_1} \textbf{g}_{c,3}$, respectively. The rescaled estimates of the elements of $\tilde{\textbf{C}}$ are obtained using (5.9).

Subsequently, the parameters $\alpha$, $\beta$, and $\gamma$ can be obtained by taking $\alpha = \theta_1$, $\beta = \theta_1 \exp(-\theta_1)$, and $\gamma = \theta_3$. Then, the matrix $\textbf{C}$ is computed as $\textbf{C} = \tilde{\textbf{C}} \textbf{T}$, and the matrix $\textbf{G}_c$ as $\textbf{G}_c = (\textbf{T})^{-1} \tilde{\textbf{G}}_c$, where $\textbf{T}$ is given by (5.12).

5.3.2. Transformational freedom and interpretation in the SLC Tucker3 model

The SLC Tucker3 model has transformational freedom, as the subject and variable component matrices can be transformed provided that such transformations are compensated in the core array $\textbf{G}$. The occasion component matrix $\textbf{C}$ is given by (5.6), and may therefore not be transformed.

A rescaling of the occasion component matrix in the SLC Tucker3 model would imply that the elements of $\textbf{C}$ no longer satisfy (5.6). However, in practice, the interpretation of the model may be facilitated by rescaling the component matrix $\textbf{C}$ column-wise. Then the columns of the occasion component matrix follow (5.6) up to a multiplication by a constant. The rescaling should be compensated in the subject and/or variable component matrices, or in the core array, to preserve the fit. Note that after rescaling the first column of $\textbf{C}$, the constraint on the first column of $\textbf{1}' (\textbf{B} \otimes \textbf{A}) \textbf{G}_c^\prime$ in the SLC Tucker3 model is no longer satisfied. However, this does not pose a problem as the estimated average curve across subjects and variables, as given by $\frac{1}{\theta_1} \textbf{C} \textbf{G}_c (\textbf{B} \otimes \textbf{A}) \textbf{1}_\mu$ in the SLC Tucker3 model, is still a Gompertz curve with parameters $\alpha$, $\beta$, and $\gamma$. Such a rescaling of the occasion component matrix is illustrated in the empirical example (Section 5.4).

5.4. Empirical example: Learning to read study (II)

In this section, the results of an SLC Tucker3 analysis of the data of the Learning to Read study (Jansen & Bus, 1982; Bus & Kroonenberg, 1982) are discussed. The learning to read study was discussed in Section 4.5. The data contain the scores of seven subjects on five reading tests, collected at 37 time points during process of learning to read. This data set was analyzed by the unrestricted Tucker3 model, and two smoothness constrained Tucker3 models. The results are discussed in Section 4.5.
In this section, we present the results of an SLC Tucker3 analysis and compare them to the results of the previous analyses.

The raw scores were rescaled in the same way as in Section 4.5, that is, so that the scores on all variables ranged from 0 to 1. The SLC Tucker3 model (see (5.18)) was fitted to the rescaled data. Analogously to the unconstrained Tucker3 analysis (see Section 4.5), we chose two components for the subject mode, and one component for the variable mode.

The parameters $\alpha$, $\beta$, and $\gamma$ were estimated as 0.92, 0.06, and 0.13, respectively. Hence, the asymptote of the mean curve (over all subjects and variables) is 0.92, and the estimated mean score at time point (=week) 0 is 0.06. The inflection point is estimated to occur at week $7.7 = \frac{1}{\gamma} \ln \left( -\ln \frac{\beta}{\alpha} \right)$. The observed mean scores across variables and subjects, and the mean curve as estimated by the SLC Tucker3 model are plotted in Figure 5.2.

The fit of the SLC Tucker3 model is 96.13%, which is 0.13% lower than the unconstrained Tucker3 model. Note that, although the unconstrained Tucker3 model has two occasion components, whereas the SLC Tucker3 model has three, the latter model uses fewer parameters: the Tucker3 model requires $(I \times P + J \times Q + K \times R + P \times Q \times R - P^2 - Q^2 - R^2) - (I \times P + J \times Q + 2 + P \times Q \times 3 - P^2 - Q^2 - 2) = 68$ parameters more (see also Weesie & Van Houwelingen, 1983) than the SLC Tucker3 model in the current example.

The estimated subject and variable component matrices of the SLC Tucker3 model were regressed onto their counterparts found in the unconstrained Tucker3 analysis. This transformation was compensated in the core array. The transformed component matrices were compared to the solutions of the unconstrained Tucker3 model by computing the coefficient of congruence between the pairs of components.
concerned. The large coefficients (minimally 0.9997) indicate that those pairs resembled each other strongly. The subject component matrix and the variable component matrix are presented in Tables 5.1 and 5.2, respectively.

Table 5.1. Subject component scores of the SLC Tucker3 solution of the Learning to Read study.

<table>
<thead>
<tr>
<th>A (subjects)</th>
<th>1\textsuperscript{st} component</th>
<th>2\textsuperscript{nd} component</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.06</td>
<td>−0.41</td>
</tr>
<tr>
<td>2</td>
<td>0.96</td>
<td>−0.30</td>
</tr>
<tr>
<td>3</td>
<td>0.99</td>
<td>−0.37</td>
</tr>
<tr>
<td>4</td>
<td>1.28</td>
<td>1.01</td>
</tr>
<tr>
<td>5</td>
<td>1.16</td>
<td>0.17</td>
</tr>
<tr>
<td>6</td>
<td>1.09</td>
<td>−0.01</td>
</tr>
<tr>
<td>7</td>
<td>0.89</td>
<td>−0.43</td>
</tr>
</tbody>
</table>

Table 5.2. Variable component scores of the SLC Tucker3 solution of the Learning to Read study.

<table>
<thead>
<tr>
<th>B (variables)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Letter Knowledge</td>
<td>0.91</td>
</tr>
<tr>
<td>Regular Orthographic Short Words</td>
<td>1.00</td>
</tr>
<tr>
<td>Regular Orthographic Long Words</td>
<td>0.87</td>
</tr>
<tr>
<td>Regular Orthographic Long and Short Words within Context</td>
<td>0.99</td>
</tr>
<tr>
<td>Irregular Orthographic Long and Short Words</td>
<td>0.58</td>
</tr>
</tbody>
</table>

To facilitate the interpretation, the slabs of the core array pertaining to the three basis functions were rescaled so that the maximal weight for each of the three basis functions equals one, and this rescaling was compensated in the basis functions. As explained earlier, the estimated mean curve still is a Gompertz function with parameters $\alpha$, $\beta$, and $\gamma$. The core array is presented in Table 5.3. The estimated and rescaled occasion basis functions are plotted in Figure 5.3.

Table 5.3. Core of the SLC Tucker3 model of the Learning to Read data.

<table>
<thead>
<tr>
<th>G (core)</th>
<th>$b_1 c_1$</th>
<th>$b_1 c_2$</th>
<th>$b_1 c_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1.00</td>
<td>0.33</td>
<td>0.05</td>
</tr>
<tr>
<td>$a_2$</td>
<td>−0.18</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>
The sizes of the core elements indicate that the combinations of the first subject component and the first basis function, and the second subject component and the second and third basis functions weigh heavily in the final solution. However, the learning rate basis function (the third basis function) hardly influences the estimated solution, as the scores are almost zero, as can be seen in Figure 5.3. The initial value basis function reflects not only the initial value, but also the fact that the acceleration first increases and then decreases. In fact, one could interpret the initial value basis function as a second learning rate basis function, with its maximum occurring at the inflection point of the Gompertz curve.

The ordering in the size of the variable component scores indicates the relative difficulty of the variable, with a large score indicating that the skill measured by the variable concerned is mastered relatively quickly. As only one component is used for the variables, the ordering in difficulty among variables is equal across time for the subjects.

The subject component scores can be interpreted as follows: The subject component scores on the first component indicate the weight of the asymptote basis function, whereas the score on the second subject component indicates the weight of the initial value basis function and the learning rate basis function.

As the learning rate and initial basis functions approach zero at the last measurement occasion, the weight of the first subject component scores indicates the relative size of the maximal score of the subject. Thus, Subject 4 ends highest, successively followed by Subjects 5, 6, 1, 3, 2, 7.

The evaluated values of the learning rate basis function are close to zero, and therefore hardly play a role in the model. On the contrary, the initial value basis function influences the solution greatly. It reflects the initial score as well as the growth rate. Thus, on the basis of the ordering of the second subject component scores, the subjects can be ordered from fast growth rate combined with relatively high initial values, to slow learning rate combined with low initial values. Note that a subject can also start and end high, and show relatively slow learning rate due to
ceiling effects. This phenomenon is reflected in the subject component scores by a relatively high score on the first component and a relatively low score on the second component. This can be seen in Subject 6.

In order to compare subjects, it can be useful to estimate the growth curve(s) per subject. This can be done by computing $G_b(A' \otimes C')$: the rows of this matrix refer to the weights for the $Q$ variable components; columns 1 through $K$ are the evaluations of the estimated growth curve at the $k=1,\ldots,K$ measurement occasions for the first subject, $K+1$ through $2K$ for the second subject, and so forth. Here, we have only one variable component, and thus each subject has only one estimated curve, which could be called the ‘general growth curve’. Per variable, the estimated curve per subject is just proportional to the general growth curve of the subject concerned. For illustration, the curves for Subjects 1, 4 and 6 are plotted in Figure 5.4. It can be seen in this figure that Subject 4 starts high, and approaches the asymptote quickly. Subject 6 starts somewhat higher than Subject 1, who shows a somewhat higher growth rate than Subject 6. Subjects 1, 4 and 6 end up at about the same asymptote level.

![Figure 5.4. Estimated ‘general growth curves’ of Subjects 1, 4 and 6, of the SLC Tucker3 model of the learning to read data.](image-url)

### 5.5. Discussion and conclusion

In this chapter, the Structured Latent Curve (SLC) two-way component and the SLC Tucker3 models were discussed. The empirical example showed the use of the SLC Tucker3 model in practice. In the SLC Tucker3 model of the ‘Learning to read’ data, it can be observed that the asymptote basis function and the initial value basis function (see Figure 5.3) resemble the two respective occasion component scores of the T3-Bs model (see Figure 4.5) closely. However, the initial value basis function can be interpreted somewhat better than the second occasion component of the T3-Bs analysis. In the latter analysis, the scores are negative after week 22, which may suggest decreasing scores on the variables after week 22. This is not the case, for
either the observed scores or the model, but this is difficult to see at once. (In problematic cases, plots of the estimated scores per subject per variable can be helpful.) The interpretation of the SLC Tucker3 model is therefore somewhat easier than the interpretation of the T3-Bs model.

In general, the SLC Tucker3 model is more parsimonious than its smoothness constrained counterpart. On the other hand, smoothing the component matrix is to be preferred over imposing a functional form if knowledge about the functional form is lacking. Moreover, if the functional form of the mean curve is intricate, the smoothness constrained model offers a simple approach to restricted modeling of the data.

The principles of the SLC Tucker3 model could also be applied to the CP model, which is more restricted, and to the Tucker2 and Tucker1 models that are less restricted than the Tucker3 model. That is, SLC CP, SLC Tucker 2 and SLC Tucker1 models can be defined completely analogously to the SLC Tucker3 model. An SLC CP model is usually heavily constrained, because the latent curves, as well as the CP model itself, are restricted. Note that, in a CP model, the number of components is equal for all three modes. As a result, when, for example, a Gompertz function is used as the target function, the number of basis functions is three, and hence the number of subject components and the number of variable components must always be three. In general, a CP model implies that the scores of the entities of one mode are proportional to each other, and hence no interactions across the modes are allowed for. To put it differently in the current context, the weights for the variables for each basis function are equal for all subjects, and per variable the subjects’ curves over time are proportional to each other. Hence, the SLC CP model requires the data to have a rather special structure. Moreover, it is difficult to fit the model to data. For these two reasons, we did not elaborate the SLC CP model further.

The SLC Tucker2 model and SLC Tucker1 model are less restricted than the SLC Tucker3 model. The SLC Tucker2 model can be fitted to data using the SLC Tucker3 fitting procedure by taking the number of observed entities of the unreduced mode as the number of components for the unreduced mode in the SLC Tucker3 fitting procedure. Then, the extended core array is computed by multiplying the component matrix of the unreduced mode by the (appropriately reordered) estimated core array. The SLC Tucker1 model can be fitted to data by applying the fitting procedure for the SLC two-way component model to the matricized data array $X_c (K \times IJ)$.

The SLC Tucker3 model was discussed for the case where the mean curve (across subjects and variables) could be described well by a Gompertz curve. The approach can also be used if the mean curve follows another target function. For an example, in the three-way SLC factor analysis context, we refer to Oort (2001), who applied a linear target function in an empirical example. Of course, one may think of more intricate target functions as well.
Appendix 5.1.
The constraint \( 1'_{ij} (B \otimes A) \tilde{G}_{j23} = 0' \) is equivalent to the constraint \( V \text{Vec}(A) = 0 \), where A \((I \times P)\) and B \((J \times Q)\) are component matrices, \( \tilde{G}_{j23}' = [\tilde{g}_{c,2} | \tilde{g}_{c,3}] \), with \( \tilde{g}_{c,m} \) \((1 \times P)\) denoting the \( m \)th row of \( \tilde{G}_c \), where \( \tilde{G}_c \) \((3 \times P \times Q)\) denotes the matricized core array \( \tilde{G} \) \((3 \times P \times Q)\). Vec(A) \((IP \times 1)\) denotes the vectorized version of matrix A, and V is defined as

\[
V = \begin{bmatrix}
1'_{ij} B \tilde{G}_2' \otimes 1'_{ij} \\
1'_{ij} B \tilde{G}_3' \otimes 1'_{ij}
\end{bmatrix}.
\]

This can be seen as follows. The constraint \( 1'_{ij} (B \otimes A) \tilde{G}_{j23} = 0' \) is equivalent to the constraints \( 1'_{ij} (B \otimes A) \tilde{g}_{c,2} = 0 \), The constraint on the scalar \( 1'_{ij} (B \otimes A) \tilde{g}_{c,2} \) can be written as \( 1'_{ij} (B \otimes A) \tilde{g}_{c,2} = (1'_{ij} \otimes 1'_{ij}) (B \otimes A) \tilde{g}_{c,2} = ((1'_{ij} B) \otimes (1'_{ij} A)) \tilde{g}_{c,2} = \text{Vec}(1'_{ij} A \tilde{G}_2 B'1'_{ij}) = (1'_{ij} B \tilde{G}_2') \otimes 1'_{ij} \text{Vec}(A) = 0 \), where \( \tilde{G}_2 \) \((P \times Q)\) denotes the 2nd horizontal slab of the core array \( \tilde{G} \) \((3 \times P \times Q)\). Analogously, the constraint on the scalar \( 1'_{ij} (B \otimes A) \tilde{g}_{c,3} \) can be written as \( (1'_{ij} B \tilde{G}_3') \otimes 1'_{ij} \text{Vec}(A) = 0 \). When V is defined as above, the constraints \( (1'_{ij} B \tilde{G}_2') \otimes 1'_{ij} \text{Vec}(A) = 0 \) can be rewritten as \( V \text{Vec}(A) = 0 \).