Viscous-inviscid interaction with the quasi-simultaneous method for 2D and 3D aerodynamic flow
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Chapter 5

Applications of the quasi-simultaneous method

5.1 Introduction

Although a lot of work has been done on two-dimensional coupling, as discussed in chapter 1, results obtained for the industrially more relevant application of three-dimensional viscous-inviscid interaction are still limited. The numerical modelling of the hyperbolic nature of the system of equations and the lack of suitable empirical closure relations make the three-dimensional coupled problem a more difficult one to solve.

In the preceding chapter, viscous-inviscid interaction with the quasi-simultaneous method for the model problem of two-dimensional flow over a plate has been investigated. In the present chapter the quasi-simultaneous technique is extended to three dimensions and applied to the more practical cases of two-dimensional aerofoil flow and three-dimensional dented plate and wing flow. Furthermore, the solution procedure of the coupled system is considered in more detail.

5.2 Interaction three-dimensional systems

The analysis of interaction algorithms for three-dimensional systems is different from the analysis for the two-dimensional steady flow problems presented in section 4.2. Unlike the two-dimensional steady flow case, where the system of equations is parabolic, the three-dimensional flow case leads to a hyperbolic system of equations.

An analogy exists between the three-dimensional and the unsteady two-dimensional case, and as the unsteady two-dimensional case is easier to understand it is analysed next.

5.2.1 Two-dimensional unsteady interaction

For the two-dimensional unsteady flow problem the boundary-layer equation, given in section 2.4, is written as

$$\frac{\partial u_e}{\partial x} + \frac{b}{a} \frac{\partial \delta^*_x}{\partial x} + \frac{c}{a} \frac{\partial \delta^*_x}{\partial t} = \frac{d}{a}, \quad (5.1)$$
with \( b/a \) changing sign from positive to negative at \( H = H_{sep} \) (4.4) and \( c/a > 0 \) and \( d/a > 0 \).

It is noted that in the preceding chapter the unsteady term in the boundary-layer equation has been treated numerically as a form of relaxation for the steady parabolic problem.

**Direct method**

Applying the classical direct interaction algorithm (4.2) to equation (5.1), with the edge velocity assumed known from the external inviscid flow, the following partial differential equation for \( \delta_x^* \) remains to be solved:

\[
\frac{b}{a} \frac{\partial \delta_x^*}{\partial x} + \frac{c}{a} \frac{\partial \delta_x^*}{\partial t} = \frac{d}{a} - \frac{\partial u_e}{\partial x}.
\]

The characteristic of the above equation is given by

\[
\lambda = \frac{dx}{dt} \implies \lambda = \frac{b/a}{c/a},
\]

where \( \lambda \) represents the tangent of the angle between the characteristic and the \( x \)-axis. If a system of partial differential equations has complex \( \lambda \)'s the equations are called elliptic. If, as is the case here, the \( \lambda \)'s are real and distinct the equations are called totally hyperbolic.

It is seen that for the above scheme the characteristic \( \lambda \) has a positive sign when \( H < H_{sep} \) \((b/a > 0)\), and a negative sign when \( H > H_{sep} \) \((b/a < 0)\). The characteristic passes through zero when \( H = H_{sep} \), and unlike for the two-dimensional steady flow case, this point is generally non-singular for the hyperbolic problem. The passage through zero only implies occurrence of influence from downstream to upstream because the wall streamline becomes orthogonal to the external streamline, which is associated to flow reversal. However, Cousteix and Houdeville [20] have shown that for this characteristic family non-physical singularities can occur from the focusing of characteristic lines.

**Strong interaction methods**

For comparison, boundary-layer equation (5.1) is solved together with a representation of the external inviscid flow. Using again the simple linear relation \( u_e - \kappa_1 \delta_x^* = \kappa_2 \) (4.10), representing either the external inviscid flow equation (fully simultaneous method) or the interaction law equation (quasi-simultaneous method), equation (5.1) is reduced to

\[
\left( \kappa_1 + \frac{b}{a} \right) \frac{\partial \delta_x^*}{\partial x} + \frac{c}{a} \frac{\partial \delta_x^*}{\partial t} = \frac{d}{a},
\]

and \( \lambda \) is found to be

\[
\lambda = \frac{\kappa_1 + b/a}{c/a}.
\]

It is seen that with the use of the outer flow representation (4.10) the characteristic \( \lambda \) does not change sign at \( H = H_{sep} \). Hence, if the flow is not too strongly separated, implying
that the angle between the wall streamline and the external streamline is only slightly larger than $90^\circ$, coefficient $\kappa_1 + b/a$ remains positive, as the value of $b$, although negative, is small.

Cousteix and Hondeville in [20] have shown that with an inverse method the occurrence of any singularity is avoided. Similarly, the strong coupling techniques most probably prevent the occurrence of a discontinuity line.

### 5.2.2 Three-dimensional interaction

The above discussed behaviour of the two-dimensional unsteady hyperbolic problem shows great resemblance with the behaviour of the steady and unsteady three-dimensional problem, when solved with the direct scheme, as described in [18, 20, 94]. When for the steady or unsteady three-dimensional boundary-layer equations coefficient $b$ passes through zero, some characteristics change direction from upwind to downwind and convergence of characteristics might lead to the formation of a non-physical discontinuity line. When solved simultaneously with an interaction law, the characteristics generally originate from the upwind direction, most probably avoiding the occurrence of any singularity.

### 5.3 Newton solution procedure

Before continuing with the formulation of the three-dimensional quasi-simultaneous coupling scheme, the procedure to solve for the boundary-layer and external flow equations which are combined with the quasi-simultaneous method, is explained in more detail. The coupled system being non-linear, an iterative scheme is to be employed to solve the set of equations. As a consequence of the viscous-inviscid interaction technique chosen, the Newton method is to be implemented which is the best-known procedure for solving non-linear problems. For a system of non-linear equations of the form

$$ R(U) = 0, $$

where $U$ is the vector of unknowns, Newton’s method is written

$$ \left[ \frac{\partial R}{\partial U} \right]^{(n)} \left( U^{(n+1)} - U^{(n)} \right) = -R(U^{(n)}). \quad (5.3) $$

The term $\partial R/\partial U$ corresponds with the Jacobian matrix $J$ of the system of equations at some Newton iteration level $n$. Each entry $J_{kl}$ of the Jacobian represents the partial derivative of the $k^{th}$ equation with respect to the $l^{th}$ variable of vector $U$

$$ J_{kl} = \frac{\partial R_k(U)}{\partial U(l)}. \quad (5.4) $$

The Jacobian matrix $J$ can be calculated numerically via

$$ J_{kl} = \frac{R_k(U(l) + \varepsilon) - R_k(U(l) - \varepsilon)}{2\varepsilon}, $$

where $\varepsilon$ is a small perturbation.
or analytically, by making use of the chain-rule. Of the two possibilities the numerical computation is employed for which it is important that \( \varepsilon \) is suitably chosen for \( U(l) \). A setback of the numerical approach is that for each iteration residuals at positions \( l, l - \varepsilon \) and \( l + \varepsilon \) have to be calculated, and hence by implementing the analytical approach Newton’s solution procedure might be faster.

**Present application Newton’s method**

Newton’s method is applied per station for both the two- and three-dimensional flow cases to solve for the unknowns vector \( U \). For the various cases \( U \), containing the discrete viscous and inviscid flow variables, is given by

**Two-dimensional plate/aerofoil case**  **Quasi-three-dimensional swept tapered wing case**

\[
U = \begin{pmatrix}
\delta_s^* \\
H \\
u_{se}
\end{pmatrix}, \quad \quad \quad U = \begin{pmatrix}
\theta_{ss} \\
H \\
\beta \\
\alpha_c \\
u_{se}
\end{pmatrix},
\]

**Three-dimensional dented plate case**  **Three-dimensional wing case**

\[
U = \begin{pmatrix}
\theta_{ss} \\
H \\
\beta \\
u_e \\
v_e
\end{pmatrix}, \quad \quad \quad U = \begin{pmatrix}
\theta_{ss} \\
H \\
\beta \\
U_e \\
V_e \\
W_e
\end{pmatrix},
\]

The coupled system of equations \( \mathbf{R} \) contains the set of discretised boundary-layer equations described for each flow case in chapter 2, and the interaction law formulae, as derived in chapter 4.

For reasons of simplicity the interaction law formulae are linearised by using the velocity of the previous viscous-inviscid iteration, which in two-dimensions results in

\[
I_i = u_{e_i}^{(n)} = \sum_k I_{ik} q_{ek}^{(n-1)} \delta_{ek}^{(n)}.
\]

(5.5)

The interaction law formulae being linear, the number of equations can be reduced by pre-eliminating the velocity unknowns. This has been done for the two-dimensional and quasi-three-dimensional flow case to minimise the computational effort. For the fully three-dimensional flow calculations, this approach requires more rewriting and the pre-elimination of the velocity unknowns has not been implemented.

In the two- and quasi-three-dimensional cases, pointwise Newton iterations are performed until convergence per station is obtained, before Newton iterations start at the
5.4 Three-dimensional quasi-simultaneous scheme

next point. The Newton stopcriterion for these cases with $i$ and $j$ fixed is

$$
\sum_i \frac{|U(l)^{\text{(n+1)}}_{ij} - U(l)^{\text{(n)}}_{ij}|}{|U(l)^{\text{(n+1)}}_{ij}| + 1 \times 10^{-10}} < \epsilon_N,
$$

(5.6)

where $\epsilon_N$ is the allowed error of the Newton iterations.

For the fully three-dimensional case the scheme is somewhat different. Pointwise Newton iterations are performed per spanwise section until convergence is obtained for the spanwise section. For only $i$ fixed

$$
\max_j \left( \sum_i \frac{|U(l)^{\text{(n+1)}}_{ij} - U(l)^{\text{(n)}}_{ij}|}{|U(l)^{\text{(n+1)}}_{ij}| + 1 \times 10^{-10}} \right) < \epsilon_N.
$$

(5.7)

5.3.1 Initial conditions

A good initial guess is required for every grid point to obtain convergence with Newton’s method. For the methods presented in this thesis three possibilities are included, being

- initialisation with the Blasius flat plate solution, which is useful to initialise plate flow or wing/aerofoil flow at low angles of attack;

- initialisation with the solution from a prior calculation, which is useful at higher angles of attack;

- initialisation by setting at the first iteration for each point $(i, j)$ to be calculated the values of all downstream points equal to the values at the previously calculated upstream point

$$
U_{k,j} = U_{k-1,j}, \quad \text{for } k > i - 1.
$$

The last approach can be applied for each flow case. It might be slower than the previous two approaches but does on the other hand not require any knowledge of the specific flow case to be calculated.

5.4 Three-dimensional quasi-simultaneous scheme

The quasi-simultaneous method has a straightforward extension to three dimensions. For the present three-dimensional quasi-simultaneous method this is done by constructing an interaction law consisting of three separate formulae for the global Cartesian edge velocities $U_e$, $V_e$ and $W_e$, termed $I^U$, $I^V$ and $I^W$.

Introducing the operators, $Mx$, $My$ and $En$ for the boundary-layer $x$-momentum, $y$-momentum and entrainment equations, the total system of equations to be solved during
the $n^{th}$ downstream viscous-inviscid march, can be written symbolically as follows:

\[
\begin{align*}
\text{boundary-layer region} & \\
U_{e}^{(n)} - I^{U} [U_{V}^{(n)}] &= U_{e}^{(n-1)} - I^{U} [U_{V}^{(n-1)}], \\
V_{e}^{(n)} - I^{V} [U_{V}^{(n)}] &= V_{e}^{(n-1)} - I^{V} [U_{V}^{(n-1)}], \\
W_{e}^{(n)} - I^{W} [U_{V}^{(n)}] &= W_{e}^{(n-1)} - I^{W} [U_{V}^{(n-1)}], \\
Mx [U_{e}^{(n)}, V_{e}^{(n)}, W_{e}^{(n)}, U_{V}^{(n)}] &= 0, \\
My [U_{e}^{(n)}, V_{e}^{(n)}, W_{e}^{(n)}, U_{V}^{(n)}] &= 0, \\
En [U_{e}^{(n)}, V_{e}^{(n)}, W_{e}^{(n)}, U_{V}^{(n)}] &= 0,
\end{align*}
\]

\[
\begin{align*}
\text{external flow region} & \\
U_{e}^{(n)} &= E^{U} [U_{V}^{(n)}], \\
V_{e}^{(n)} &= E^{V} [U_{V}^{(n)}], \\
W_{e}^{(n)} &= E^{W} [U_{V}^{(n)}],
\end{align*}
\]

where $U_{V}$ represents the vector with the viscous boundary-layer unknowns, which is part of the unknowns vector $U$ (section 5.3), having excluded the velocity components.

The three interaction law equations are solved simultaneously with the three boundary-layer equations, producing three new global velocity components, $U_{e}^{(n)}$, $V_{e}^{(n)}$, $W_{e}^{(n)}$ and a new boundary-layer unknowns vector $U_{V}$ of dimension three. With the new boundary-layer unknowns vector $U_{V}$, a new external inviscid flow calculation can be performed, using the direct scheme, etc.

**Stop criteria**

The viscous-inviscid iterations between the boundary layer and external flow are continued until

\[
\max_{i,j} |U_{e}^{(n)} - U_{e}^{(n-1)}| < \epsilon_{v}^{U}, \quad \max_{i,j} |V_{e}^{(n)} - V_{e}^{(n-1)}| < \epsilon_{v}^{V}, \quad \max_{i,j} |W_{e}^{(n)} - W_{e}^{(n-1)}| < \epsilon_{v}^{W},
\]

where $\epsilon_{v}$ is the allowed viscous-inviscid error. When viscous-inviscid relaxation is applied, $\epsilon_{v}$ is to be multiplied by $\omega_{v}$, the viscous-inviscid relaxation parameter.

The boundary-layer iterations have either been set to the fixed number of three or they are performed until

\[
\max_{i,j} |U_{e}^{(n)} - U_{e}^{(n-1)}| < \epsilon_{bl}^{U}, \quad \max_{i,j} |V_{e}^{(n)} - V_{e}^{(n-1)}| < \epsilon_{bl}^{V}, \quad \max_{i,j} |W_{e}^{(n)} - W_{e}^{(n-1)}| < \epsilon_{bl}^{W},
\]

where $\tilde{n}$ indicates the $\tilde{n}^{th}$ boundary-layer iteration. The allowed boundary-layer error is defined by $\epsilon_{bl}$, and when boundary-layer relaxation is applied, $\epsilon_{bl}$ should be multiplied by $\omega_{bl}$. 
5.5 Three-dimensional plate interaction law

In chapter 4 the various requirements for the construction of a suitable interaction law have been discussed. In particular it was described in section 4.10 that the edge velocity and the boundary-layer unknowns should be defined in the node points of the panels in order to obtain an interaction law matrix with the correct structure.

For the construction of a three-dimensional plate interaction law discretisation A of figure 4.5 is therefore to be applied. The velocity components are determined from the potential-flow theory integrals for the velocity components and, following discretisation A, calculated in node point \((x_i, y_j)\). In other words, the interaction law can be constructed from the three-dimensional viscous potential-flow model, as derived in section 3.4.3

\[
\begin{align*}
    u_e(x,y) &= \frac{1}{2\pi} \int_{x_{\text{begin}}}^{x_{\text{end}}} \int_{y_{\text{begin}}}^{y_{\text{end}}} \left( \frac{\partial}{\partial \xi}(q_e \delta_x^*) + \frac{\partial}{\partial \eta}(q_e \delta_y^*) \right) \frac{(x - \xi) d\xi d\eta}{((x - \xi)^2 + (y - \eta)^2)^{\frac{3}{2}}} + u_{\alpha_i} (x,y), \quad (5.8) \\
v_e(x,y) &= \frac{1}{2\pi} \int_{x_{\text{begin}}}^{x_{\text{end}}} \int_{y_{\text{begin}}}^{y_{\text{end}}} \left( \frac{\partial}{\partial \xi}(q_e \delta_x^*) + \frac{\partial}{\partial \eta}(q_e \delta_y^*) \right) \frac{(y - \eta) d\xi d\eta}{((x - \xi)^2 + (y - \eta)^2)^{\frac{3}{2}}} + v_{\alpha_i} (x,y), \quad (5.9)
\end{align*}
\]

where \(w_e(x,y) = \partial/\partial x(q_e \delta_x^*) + \partial/\partial y(q_e \delta_y^*)\). After the discretisation equations (5.8) and (5.9) become

\[
\begin{align*}
u_{eij} &= \sum_{k=1}^{M_p+1} \sum_{l=1}^{M_\nu+1} q_{ekl} \left[ A_{ijkl}^u \delta_x^* + B_{ijkl}^u \delta_y^* \right] + u_{\alpha_i}, \quad (5.10) \\
v_{eij} &= \sum_{k=1}^{M_p+1} \sum_{l=1}^{M_\nu+1} q_{ekl} \left[ A_{ijkl}^v \delta_x^* + B_{ijkl}^v \delta_y^* \right] + v_{\alpha_i}, \quad (5.11)
\end{align*}
\]

A third interaction law formula for \(u_e = W_e\) is not required as both \(u_e = U_e\) and \(v_e = V_e\) are tangent to the surface of the plate.

If the grid is constructed by squares, \(\Delta y = \Delta x\), the following matrix entries of \(A^u\) are obtained:

\[
\begin{align*}
    A_{ijij}^u &= \frac{1}{\pi \Delta x} \ln \left| \frac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}} \right|, \quad (5.12) \\
    A_{iji-1j}^u &= A_{ij-i+1j}^u = -\frac{1}{2\pi \Delta x} \left( \ln \left| \frac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}} \right| + \ln \left| \frac{9 + 3\sqrt{5} - 6\sqrt{2} - 2\sqrt{10}}{9 + 3\sqrt{5} - 6\sqrt{2} + 2\sqrt{10}} \right| \right), \quad (5.13)
\end{align*}
\]

which contain the main influence at node point \((x_i, y_j)\). The other matrix entries of \(A^u\) and those of \(B^u\) are much smaller. For the present square grid the matrix entries of \(B^v\) for the \(y\)-velocity component resemble the entries of \(A^u\) as \(B_{ijij+kj}^v = A_{ijij+kj}^u\) and similarly do the entries of \(A^v\) resemble those of \(B^u\).

The matrix entries for \(B^u\) and \(A^v\) being small, it is decided that for the construction of interaction law formulae for \(u_e\) and \(v_e\) matrices \(B^u\) and \(A^v\) are to be ignored. The formula
for \( u_e \) hence only contains a \( q_e \delta_{x}^{*} \) part and the formula for \( v_e \) only takes the contribution of the \( q_e \delta_{y}^{*} \) part into account. The remaining matrices \( A^u \) and \( B^v \) have the following characteristics:

\[
A_{ijk}^u = A_{kij}^u, \quad B_{ijl}^v = B_{kjl}^v \geq 0, \quad (i \neq k, j \neq l),
\]

which means that both \( A^u \) and \( B^v \) are positive definite \( M \)-matrices (see theorems D.10 and D.12). It follows that interaction law matrices based on \( A^u \) and \( B^v \) by setting certain off-diagonals to zero, have the same characteristics and are therefore suitably constructed for the quasi-simultaneous interaction scheme.

Looking in more detail at the matrix entries of \( A^u \), it is seen that for the interaction law formula for \( u_{eij} \), the entries \( A_{ijkl}^u \) for \( l \neq j \) are small and can be omitted. Similarly for \( v_{eij} \), the entries \( B_{ijkl}^v \) for \( k \neq i \) can be neglected. This results in the following interaction law formulae for \( u_e \) and \( v_e \), respectively:

\[
\Gamma_{ij}^u = \sum_k A_{ikj}^u q_{eij} \delta_{x}^{*} + u_{0eij} = \sum_k I_{ijk}^u q_{eij} \delta_{x}^{*} + u_{0eij}, \quad (5.14)
\]

\[
\Gamma_{ij}^v = \sum_l B_{ijl}^v q_{eij} \delta_{y}^{*} + v_{0eij} = \sum_l I_{ijl}^v q_{eij} \delta_{y}^{*} + v_{0eij}, \quad (5.15)
\]

The number of off-diagonals of matrices \( A^u \) and \( B^v \) can be varied for the construction of a suitable interaction law as discussed in chapter 4.

### 5.5.1 Comparison two- and three-dimensional interaction laws

The above result for three-dimensional plate flow can be shown to correspond with the result found earlier in two dimensions. When comparing the diagonal matrix entries of \( A^u \) for the two- and three-dimensional problem, (3.54) and (5.10), it is seen that

\[
A_{ii}^u = \frac{4}{\kappa \Delta x} \approx 1.27/\Delta x,
\]

\[
A_{ijij}^u = \frac{1}{\kappa \Delta x} \ln \left| \frac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}} \right| \approx 1.12/\Delta x.
\]

Even closer correspondence can be shown by taking the limit \( \Delta y \to \infty \) in the Cauchy principal part of the singular integral (3.58)

\[
\lim_{\Delta y \to \infty} \left( -\frac{\Delta y}{2\pi} \ln \left| \frac{\sqrt{\Delta x^2 + \Delta y^2} + \Delta x}{\sqrt{\Delta x^2 + \Delta y^2} - \Delta x} \right| \frac{\partial w_{e}}{\partial x} \right)_{ij}
\]
\[
\begin{align*}
&= \lim_{\Delta y \to \infty} \left( -\frac{\Delta y}{2\pi} \ln \left| 1 + \frac{4\Delta x \sqrt{\Delta x^2 + \Delta y^2}}{2\Delta x^2 + \Delta y^2 - 2\Delta x \sqrt{\Delta x^2 + \Delta y^2}} \right| \frac{\partial w_x}{\partial x_{ij}} \right) \\
&= \lim_{\Delta y \to \infty} \left( -\frac{\Delta y}{2\pi} \frac{4\Delta x \sqrt{\Delta x^2 + \Delta y^2}}{2\Delta x^2 + \Delta y^2 - 2\Delta x \sqrt{\Delta x^2 + \Delta y^2}} \frac{\partial w_x}{\partial x_{ij}} \right) \\
&= \lim_{\Delta y \to \infty} \left( -\frac{1}{2\pi} \frac{4\Delta x \sqrt{\Delta x^2 + 1}}{2\Delta x^2 + 1 - 2\Delta x \sqrt{\Delta x^2 + 1}} \frac{\partial w_x}{\partial x_{ij}} \right) \\
&= -\frac{2\Delta x}{\pi} \frac{\partial w_x}{\partial x_i},
\end{align*}
\]

which equals the Cauchy principal part in the two-dimensional case (3.52).

It follows from the above that the diagonal entries of the interaction law matrices \( I^1 \) (4.47) and \( I^u \) (5.14) constructed from \( A^u \) are comparable: \( I_{ii}^1 \approx I_{ii}^u \). A similar resemblance exists for the off-diagonals of \( I^1 \) and \( I^u \).

### 5.6 Dirichlet aerofoil interaction law

Before continuing with the construction of an interaction law for the three-dimensional wing problem, an interaction law is constructed for the simpler two-dimensional aerofoil problem.

An often seen choice is the use of a two-dimensional plate interaction law, as constructed in chapter 4, to represent the aerofoil’s edge velocity. The alternative is to derive an interaction law from the actual external flow formulation. The latter option more accurately represents the inviscid flow region, taking into account the Kutta condition, and is to be applied for the present aerofoil flow case.

For the calculation of the actual two-dimensional aerofoil velocity two well-known approaches exist, being the Neumann and the Dirichlet method. The Neumann method, uses the normal velocity boundary condition and determines the velocity integrals in the panel centres. The boundary layer unknowns being defined in the panel endpoints, the Neumann method can be seen to correspond with discretisation B, discussed in section 4.10, leading to an unsuitable interaction law matrix.

The Dirichlet method on the other hand uses a constant potential boundary condition and from the doublet strengths the velocity is determined in the panel endpoints, corresponding with discretisation A. Unlike the Neumann method, the Dirichlet method produces a suitable interaction law for the quasi-simultaneous interaction technique and is to be applied.

In section 3.4.4 the two-dimensional streamwise edge velocity equation for aerofoil flow, using the Dirichlet approach has been derived

\[
U_{s_{eq}} = \sum_{k=1}^{M_b + M_w + 1} A_{ik}^U U_{s_{eq}} \delta_{s_{eq}} + U_{0_{eq}},
\]

(5.16)
for \( i \in [2, M_h] \), and as expected matrix \( A_{i}^{U} \) has the following properties away from the leading and trailing edge:

\[
A_{ii}^{U} > 0, \\
A_{ik}^{U} \approx A_{ki}^{U} \leq 0, \quad (i \neq k).
\] (5.17) (5.18)

Due to the Kutta condition, and its induced strong coupling between the upper and lower surface near the trailing edge (indicated by large matrix entries), \( A_{i}^{U} \) is not weakly diagonally dominant. It follows that assumption 4.1, which assumes that the positive stable matrix \( E + V = A_{i}^{U} + V \) possesses non-positive off-diagonal entries, is not satisfied and the derived theorems in chapter 4 therefore, can not be applied.

The interaction law is based on \( A_{i}^{U} \) by setting certain off-diagonals to zero, leading to

\[
I_{i}^{U} = \sum_{k} A_{ik}^{U} U_{s_{k, i}} \delta_{x_{k}}^{*} + U_{0_{w_{k, i}}} = \sum_{k} I_{ik}^{U} U_{s_{k, i}} \delta_{x_{k}}^{*} + U_{0_{w_{k, i}}}. 
\] (5.19)

Interaction law \( I_{i}^{U} \) maintains the characteristics described by (5.17) and (5.18) and for \( I_{i}^{U} \neq A_{i}^{U} \) constructed according to construction 4.4

\[
\sum_{k=1}^{M_h+M_w+1} I_{ik}^{U} \geq 0, \quad \text{with inequality for at least one } i,
\] (5.20)

which implies that interaction matrix \( I_{i}^{U} \) is an M-matrix (theorem D.10). On an equidistant grid matrix \( I_{i}^{U} \) is symmetric and with (5.17) and (5.18) therefore positive definite (theorem D.12). Interaction law \( I_{i}^{U} \) is therefore suitably constructed for the quasi-simultaneous interaction scheme. A detailed description is also found in [14].

### 5.7 Dirichlet wing interaction law

Like for the two-dimensional aerofoil problem, an interaction law, using the Dirichlet method, is derived from the edge velocity equations derived in section 3.4.4.

In equations (3.72), (3.73) and (3.74) the discretised equations for the velocity components \( U_e, V_e \) and \( W_e \) in the global Cartesian coordinate system have been given

\[
U_{e_{ij}} = \sum_{k=1}^{M_h+2M_w+1} \sum_{l=1}^{M_h+1} q_{e_{kj}} [A_{ij,kl}^{U} \delta_{x_{kj}}^{*} + B_{ij,kl}^{U} \delta_{y_{kj}}^{*}] + U_{0_{e_{ij}}}, 
\] (5.21)

\[
V_{e_{ij}} = \sum_{k=1}^{M_h+2M_w+1} \sum_{l=1}^{M_h+1} q_{e_{kj}} [A_{ij,kl}^{V} \delta_{x_{kj}}^{*} + B_{ij,kl}^{V} \delta_{y_{kj}}^{*}] + V_{0_{e_{ij}}}, 
\] (5.22)

\[
W_{e_{ij}} = \sum_{k=1}^{M_h+2M_w+1} \sum_{l=1}^{M_h+1} q_{e_{kj}} [A_{ij,kl}^{W} \delta_{x_{kj}}^{*} + B_{ij,kl}^{W} \delta_{y_{kj}}^{*}] + W_{0_{e_{ij}}}, 
\] (5.23)

for \( i \in [2, M_h] \) and \( j \in [2, M_s] \). From these equations three approximations are to be derived for the interaction law formulae for the velocity components.
Similar to the three-dimensional dented plate case, the influence on the velocity of matrices \( B^U \approx A^V \), \( A^W \) and \( B^W \) is small and they are therefore to be omitted. The remaining matrices \( A^U \) and \( B^V \) have the following properties away from the leading and trailing edge:

\[
A^U_{ijij}, B^V_{ijij} > 0, \quad \quad (5.24)
\]

\[
A^U_{ijkl} \approx A^U_{klij}, \quad B^V_{ijkl} \approx B^V_{klij}, \quad \leq 0, \quad (i \neq k, j \neq l). \quad (5.25)
\]

Due to the Kutta condition, and its induced strong coupling between the upper and lower surface near the trailing edge, matrices \( A^U \) and \( B^V \) are not weakly diagonally dominant. It follows that assumption 4.1 is not satisfied and the derived theorems in chapter 4 can not be applied.

Interaction law matrices are constructed from \( A^U \) and \( B^V \) by setting certain off-diagonals to zero. As was the case for the three-dimensional plate influence matrices, the entries \( A^U_{ijkl} \) for \( l \neq j \) and the entries for \( B^U_{ijkl} \) for \( k \neq i \) are small and are omitted, leading to the following Dirichlet interaction law matrices:

\[
I^U_{ij} = \sum_k A^U_{ijkl} q_k \delta x_{k,j} + U_{0ij}, \quad \quad (5.26)
\]

\[
I^V_{ij} = \sum_l B^V_{ijil} q_l \delta y_{l,i} + V_{0ij}, \quad \quad (5.27)\]

\[
I^W_{ij} = W_{0ij}. \quad (5.28)
\]

The interaction law formulae have similar properties as described by (5.24) and (5.25) and for \( I^U \neq A^U \) and \( I^V \neq B^V \) constructed according to construction 4.4

\[
\sum_{k=1}^{M_u+2M_w+1} \sum_{l=1}^{M_u+2M_w+1} I^U_{ijkl} \geq 0, \quad \text{with inequality for at least one } ij,
\]

implying that \( I^U \) and \( I^V \) are \( M \)-matrices (theorem D.10). On an equidistant grid the \( M \)-matrices \( I^U \) and \( I^V \) are symmetric and thus positive definite (theorem D.12). Interaction laws \( I^U \) and \( I^V \) are therefore suitable for the quasi-simultaneous interaction scheme.

### 5.7.1 Lumping Dirichlet wing interaction law

For the three-dimensional Dirichlet interaction law it is possible to improve the diagonal dominance of the interaction law matrix via lumping.

As has been discussed, the matrix entries \( A^U_{ijkl} \) for \( l \neq j \) and the entries \( B^U_{ijkl} \) for \( k \neq i \) are small and have previously been omitted for the construction of the interaction law formulae (5.26) and (5.27). However, via lumping they can have a positive influence on the interaction law matrix. The entries of the lumped interaction law matrix \( I^U_L \) can be constructed as follows:

\[
I^U_{ijkl} = A^U_{ijkl} + (A^U_{ij(k)l-1} + A^U_{ijkl+1})/2, \quad (5.29)
\]

entries \( A^U_{ijkl-1} \) and \( A^U_{ijkl+1} \) having the same sign as \( A^U_{ijkl} \). In a similar way a lumped interaction law matrix for \( V_e \) can be constructed.
It is seen that with (5.29) a stronger diagonal is obtained for the interaction law matrices, as $I_{ij}^U > I_{ij}^U$, which is most suitable for the Newton iteration process in order to be able to handle more separation. Lumping has been applied for the construction of the Dirichlet wing interaction law, however, it is noted that with the use of relaxation a similar effect can be obtained.

5.8 Interaction at the trailing edge

At the trailing edge of an aerofoil or wing thus far no interaction law has been constructed. The most straightforward approach to determine an approximation for the upper and lower edge velocities there is by differentiation of the doublet strengths, as was done in (3.69) and (3.75), using the known doublet strengths of the upper and lower trailing edge panels, respectively, and the wake. However, it is known from anti-symmetric thin-aerofoil theory, corresponding with the Kutta condition, that the analytical expression for the velocity degenerates at the trailing edge [101, 86], leading to quite complicated interaction law formula for the quasi-simultaneous interaction method. Two other approaches are therefore suggested here, to be able to deal with the trailing edge.

One possibility is to determine the values at the trailing edge via extrapolation from upstream values. For the present method this is done by constructing the $X$-velocity in the trailing edge points $i = 1$ and $i = M_b + 1$ for every section $j$ as follows:

$$U_{ei} = U_{e_{i-1}} + \frac{U_{ei-1} - U_{ei-2}}{X_{t-1} - X_{t-2}} (X_t - X_{t-1}),$$

(5.30)

where index $t$ indicates the trailing edge point. Similarly, the boundary-layer variables and the velocity components in the $Y$- and $Z$-direction can be determined.

The other possibility is to apply a simpler way of coupling in the trailing edge points, which can be done with the use of the interaction law matrix entries for the upstream wing body points. This should be a more accurate approach than extrapolation.

For the calculations to be presented in chapter 6 the first approach has been followed as the latter is still to be implemented.

5.9 Wake interaction law

In the wake region of a two-dimensional aerofoil or a three-dimensional wing the Dirichlet panel method can not be applied to determine the interaction law formulae for the edge velocities. The Dirichlet approach requires a body with a thickness, whereas the wake geometry is modelled as just a line without thickness.

The edge velocity equations in the wake are therefore to be calculated via another approach. The potential is to be determined in the panel centres and via differencing the edge velocities in the node points can be obtained. As the location of the boundary-layer unknowns and the velocities are taken in the panel endpoints, the constructed wake influence matrix will have a structure suitable for the quasi-simultaneous method as is next to be discussed.
For two-dimensional flow over a flat plate, which corresponds with a two-dimensional wake flow for a non-inclined aerofoil, the perturbation potential \( \phi \), due to the presence of the boundary layer, in point \( x_i \) is given by

\[
\phi_i = \frac{1}{\pi} \sum_{k=1}^{M_w} \int_{x_k}^{x_{k+1}} \frac{d\delta^*_x}{d\xi} \ln |x_i - \xi| \, d\xi,
\]

and similarly the perturbation potential at \( \phi_{i+\frac{1}{2}} \) is to be found. The interaction law for the velocity at point \( x_i \), by taking the gradient of \( \phi \) in point \( x_i \), becomes

\[
I_w^i = \frac{\phi_{i+\frac{1}{2}} - \phi_{i-\frac{1}{2}}}{\Delta x} + u_{0_i} = \sum_{k=1}^{M_w} I_{ik}^w \delta^*_x + u_{0_i},
\]

where index \( w \) implies wake and

\[
I_{ii}^w = \frac{3 \ln 3}{2 \pi \Delta x}, \quad I_{i-1}^w = I_{i+1}^w = \frac{5 \ln 5 - 9 \ln 3}{2 \pi \Delta x}, \quad I_{i-2}^w = I_{i+2}^w = \frac{7 \ln 7 - 15 \ln 5 + 9 \ln 3}{2 \pi \Delta x}.
\]

It is seen that on an equidistant grid the interaction law matrix \( I^w \) has the same characteristics as matrix \( I^A \) in (4.50). Matrix \( I^w \) is a symmetric positive definite M-matrix (theorems D.10 and D.12), suitable for the quasi-simultaneous interaction method. However, a small difference is that the diagonal entries of \( I^w \) are slightly smaller than those of \( I^A \).

The procedure described by (5.32) is to be followed to determine a suitable interaction law for the edge velocities in the wing and aerofoil wake. It is noted that the wake interaction law for the wing is to be further simplified, similarly as was done for the Dirichlet wing interaction law.

Via rotation the geometry of the wake centre line, aligned with the freestream or otherwise located, is included in the wake interaction law.

### 5.10 Inclusion compressibility into interaction law

Thus far, only incompressible flow problems have been discussed for which interaction laws have been derived from potential-flow theory. However, for the quasi-three-dimensional swept tapered wing case the flow is assumed to be transonic \((0.8 < M < 1.2, \text{ say, with } M \text{ being the freestream Mach number})\).

For subsonic flow \((M < 0.8)\) the potential-flow representations are still valid if a compressibility correction term such as the Prandtl-Glauert rule is used \([50]\)

\[
U^\text{compr}_{r_e} = \frac{U^\text{incmpr}_{r_e}}{\sqrt{1 - M^2}}.
\]

An incompressible interaction law can thus be transformed into a subsonic interaction law

\[
I_{i_r}^{\text{compr}} = U^\text{compr}_{0_r} + \sum_k \frac{I_{ik}^{\text{incmpr}}}{\sqrt{1 - M^2}} \delta^*_x = U^\text{compr}_{0_r} + \sum_k I_{ik}^{\text{compr}} \delta^*_x,
\]
where $I_{\text{viscvisc}}$ is an incompressible interaction law matrix as derived before.

The above interaction law matrix, with $I_{\text{viscvisc}}$ being based on thin-aerofoil theory,
has been shown to work successfully for the calculation of transonic flows in the paper by
Veldman et al. [101].

It would be more computationally efficient to use an interaction law that takes account
of the local flow conditions, including the local Mach number of the external flow. For
transonic flow it is not easy to find a more accurate interaction law based on a linear
representation for the external flow from the transonic small perturbation (TSP) equation
[50], not least because the TSP equation is essentially nonlinear. However, the function
of the interaction law is not to define the solution, but to avoid breakdown at separation.

### 5.11 Numerical evaluation aerofoil interaction law

The two-dimensional Dirichlet interaction law $I^{U_s}$ given by (5.19) is to be tested in terms
of basic convergence and convergence rate for various constructions. This is done for the
two-sided Dirichlet interaction laws $I^{U_s}_{0,2}$ with the number of upper and lower off-diagonals
$q$ equal to 0, 1, 20 and 146, for which $I^{U_s} = E = A^{U_s}$. The one-sided interaction laws

<table>
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<th>$I^{U_s}_{20}$</th>
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Table 5.1: Viscous-inviscid and boundary-layer iterations with diagonal viscous-inviscid
relaxation.
\( I_q^{U_{q-1}} \) are used with the number of lower off-diagonals equivalent to \( q = 1 \) and 20. For comparison also the diagonal thin-aerofoil interaction law \( I_0^2 \) for a non-equitandist grid is tested.

The calculations have been performed for the (symmetric) NACA0012 aerofoil section. The freestream Reynolds number is set to \( Re = 3.9 \times 10^6 \) and the number of points used for the discretisation is 121 on the aerofoil and 25 in the wake. Transition to turbulent flow is tripped at \( X/C = 0.0006 \) and 0.65 on the upper and lower surfaces, respectively, except for the incidence \( \alpha = 0 \). For \( \alpha = 0 \) the transition locations are determined with the natural transition formula given in section 2.8.2. Each calculation is started from scratch initialising each individual point during the first iteration with the solution of the previous upstream point. The results are presented in table 5.1, where \( \alpha \) indicates the incidence and \( \omega \) is the applied diagonal viscous-inviscid relaxation.

As discussed in section 5.6 the external flow matrix \( E = A_{U_{q}} \) does have some positive off-diagonal entries and does not satisfy assumption 4.1, which is all due to the Kutta condition. Although the requirements for the theorems derived in chapter 4 therefore are not fulfilled, the numerical results presented here follow the theorems closely.

In terms of basic convergence it is seen that the scheme using interaction law \( I_{146}^{U_{16}} = E \) is the least robust. This is consistent with theorem 4.7, which implies that by increasing the number of lower and/or upper off-diagonals of \( I \), the robustness of the iterative scheme \( I + V \) decreases. Consequently the diagonal interaction laws, in which the diagonal of \( I \) can be increased to make the matrix strongly diagonally dominant, should be the most robust which can also been seen in the table.

In terms of convergence rate it is clear, when comparing the one-sided interaction laws \( I_1^{U_{1}} \), \( I_2^{U_{1}} \) and \( I_0^{U_{2}} \), that for an increasing number of lower off-diagonals the convergence improves. This also follows from theorem 4.5. A similar behaviour is seen for the two-sided interaction laws when comparing \( I_1^{U_{2}} \) and \( I_0^{U_{2}} \).

The diagonal Dirichlet interaction law \( I_0^{U_{2}} \) and the diagonal thin-aerofoil interaction law \( I_0^2 \) both lead to an equally robust scheme. However, in terms of convergence rate it is seen that \( I_0^{U_{2}} \) requires fewer iterations. Physically this is explained by the fact that \( I_0^{U_{2}} \) gives a more accurate description of the inviscid flow, taking into account the Kutta condition which is not present in the thin-aerofoil formulation. Mathematically, as it can be shown that \( I_0^{U_{2}} < I_0^2 \), theorem 4.5 states that interaction law \( I_0^{U_{2}} \) leads to faster convergence.

From the table it is clear, however, that the difference between the two diagonal interaction laws \( I_0^{U_{2}} \) and \( I_0^2 \) is very small. The thin-aerofoil interaction law \( I_0^2 \) is determined by a simple formula. The calculation of Dirichlet interaction law \( I_0^{U_{2}} \) on the other hand requires a more expensive calculation to take into account the Kutta condition.

### 5.11.1 Conclusion

From the numerical evaluation of the Dirichlet aerofoil interaction law for the NACA0012 aerofoil calculations two important conclusions can be drawn.

Firstly, although the requirements for the theorems described in chapter 4 are not fulfilled, due to the presence of the Kutta condition in the external flow matrix, the results obtained with the Dirichlet aerofoil interaction law are very good and in correspondence
with the theory. It is noted that these calculations are started from scratch for every incidence.

The other conclusion follows from the use of the thin-aerofoil interaction law for aerofoil calculations. It was seen that the thin-aerofoil interaction law is equally robust and almost equally fast as the diagonal Dirichlet aerofoil interaction law. Furthermore, the construction of the thin-aerofoil interaction law is very simple, whereas the Dirichlet aerofoil interaction law requires an expensive calculation to include the Kutta condition. It can therefore be concluded that a simple thin-aerofoil interaction law is most suitable for the modelling of aerofoil flow.