Chapter 3

External flow region

3.1 Introduction

For the modelling of the flow outside the boundary layer an inviscid flow solver is needed. As it is of primary importance in the work described in this thesis to develop a fast and efficient computational method, suitable for a design-optimisation environment, a surface singularity method for irrotational subsonic flow is preferred to a physical more complete Euler or full-potential-flow solver.

In the next sections the potential-flow problem is discussed for which solutions can be obtained by distributing singularity elements on the boundaries. First, the purely inviscid potential-flow problem with no boundary layer present is examined. Later on, the boundary-layer displacement effect is taken into account, by modelling an extra source singularity distribution over the profile’s surface and the wake. This is termed the viscous formulation of the potential-flow problem.

The unknown strengths of the singularities are to be determined numerically. This is done straightforwardly with a panel method, which requires the discretisation of the geometry into a number of panel segments.

Various problems are discussed, of which the model problem of the flow over flat plates, which corresponds with non-lifting thin-aerofoil/wing theory, is described in greater detail. The constant potential Dirichlet method is applied for the more practical problems of modelling wing and aerofoil flow. For each of these problems the edge velocity equations are determined, required for the coupled potential-flow/boundary-layer calculations.

3.2 Inviscid steady potential-flow problem

Consider an arbitrary body within an enclosed three-dimensional volume $V$, with the contours of the body surface and wake, $S_b$ and $S_w$, respectively, and the outer control surface $S_\infty$, surrounding the body and the wake surface, as shown in figure 3.1.

The unit normal $\mathbf{n}$ is the vector pointing towards the inside of volume $V$ and towards the outside of $S_b$ and $S_w$. The body, which has a body-fixed Cartesian coordinate system $(X,Y,Z)$, is subject to the inflow velocity $\mathbf{Q}_\infty = (U_\infty, V_\infty, W_\infty)^T$.

The inviscid flow inside domain $V$ over the solid body when assumed incompressible
and irrotational, can be treated as potential flow [71]. This is governed by the Laplace equation

$$ \nabla^2 \Phi = 0, $$

for the velocity potential $\Phi(X,Y,Z)$.

On the solid body surface $S_b$ in the flow regime there is zero normal velocity which can be expressed as the Neumann boundary condition

$$ \nabla \Phi \cdot \mathbf{n} = 0. $$

On the outer control surface $S_{\infty}$ the disturbance due to the presence of the body must vanish in the limit and the potential must be equal to the undisturbed freestream potential $\Phi_{\infty}$

$$ \lim_{S_{\infty} \to \infty} (\nabla \Phi - \nabla \Phi_{\infty}) = 0. $$

Once $\Phi$ has been obtained, the pressure follows from Bernoulli’s equation.

### 3.2.1 Formulation of the integral equation

Using Green’s identity the above constructed boundary-value problem can be transformed into an integral equation for the potential $\Phi [50]$

$$ 4\pi E(p) \Phi(p) = \int_{S_b+S_w+S_{\infty}} \left( \Phi \nabla^2 - \frac{1}{r} \nabla \Phi \right) \cdot \mathbf{n} \, dS, $$

with $r(p;q)$ the distance between the point of integration $q(\xi,\eta,\zeta)$ and a fixed field point $p(X,Y,Z)$ at which the potential is calculated

$$ r(p;q) = \sqrt{(X-\xi)^2 + (Y-\eta)^2 + (Z-\zeta)^2}. $$
The function \( E(p) \) in (3.4) is given by

\[
E(p) = \begin{cases} 
0, & \text{for } p \text{ outside the flow field } V, \\
\frac{1}{2}, & \text{for } p \text{ on the boundary of } V, \\
1, & \text{for } p \text{ inside the flow field } V.
\end{cases}
\] (3.6)

If the flow field inside the body is considered, an equation similar to (3.4) can be derived for the internal potential \( \Phi_{in} \) [50]

\[
4\pi (1 - E(p)) \Phi_{in}(p) = \iint_{S_b+S_w} \left( \frac{1}{r} \nabla \Phi_{in} - \nabla \frac{1}{r} \right) \cdot \mathbf{n} \, dS. \tag{3.7}
\]

Equations (3.4) and (3.7) can be combined to a form relating \( \Phi \) and \( \Phi_{in} \)

\[
4\pi (E(p) \Phi(p) + (1 - E(p)) \Phi_{in}(p)) = 
\iint_{S_b+S_w} \left( \Phi - \Phi_{in} \right) \nabla \frac{1}{r} \cdot \mathbf{n} \, dS + \iint_{S_{\infty}} \left( \Phi \nabla \frac{1}{r} - \frac{1}{r} \nabla \Phi \right) \cdot \mathbf{n} \, dS.
\] (3.8)

The wake surface \( S_w \) is assumed to have zero thickness, such that the normal velocity jump across the wake dividing streamline \( S_w \) is zero, while a jump in the potential is allowed. With this condition the problem reduces to determining the values of \( \Phi - \Phi_{in} \) and \( \nabla (\Phi - \Phi_{in}) \cdot \mathbf{n} \) on the boundaries. The difference in external and internal potential can be provided by a continuous doublet distribution of strength \( \mu \) over the surface of the body and the wake. The difference between the normal derivatives of the external and internal potential can be induced by a continuous source distribution of strength \( \sigma \) on the body

\[
\mu = \Phi - \Phi_{in}, \quad \sigma = \nabla (\Phi - \Phi_{in}) \cdot \mathbf{n}. \tag{3.9}
\]

Using the above definitions for \( \sigma \) and \( \mu \), the potential for points \( p \) inside the flow regime \( V \) is given by

\[
\Phi(p) = -\frac{1}{4\pi} \iint_{S_b} \sigma \, dS + \frac{1}{4\pi} \int_{S_b+S_w} \mu \cdot \nabla \frac{1}{r} \, dS + \Phi_{\infty}(p), \tag{3.10}
\]

with the freestream potential, defined as \( \Phi_{\infty}(p) = U_{\infty}X + V_{\infty}Y + W_{\infty}Z \), to satisfy the condition at infinity. In two dimensions equation (3.10) is written as [50]

\[
\Phi(p) = \frac{1}{2\pi} \int_{S_b} \sigma \ln r \, dS - \frac{1}{2\pi} \int_{S_b+S_w} \mu \cdot \nabla (\ln r) \, dS + \Phi_{\infty}(p). \tag{3.11}
\]

Solving the Laplace equation (3.1) has now been reduced to finding an appropriate singularity distribution over the boundaries so that the boundary conditions (3.2) and (3.3) are satisfied. It is noted, however, that equations (3.10) and (3.11) do not immediately specify a unique combination of sources and doublets for a particular problem. Additional considerations are usually required which depend on the physics of the problem (i.e. Kutta condition to define the flow near sharp trailing edges) to relate \( \mu \) in the wake to \( \mu \) on the body.
3.2.2 Model problem

From the previous section it is clear that a solution for the flow over arbitrary bodies can be obtained by a singularity distribution on the modelled surface. Before applying the described method to practical problems, the method is first investigated for the model problem of two- and three-dimensional non-lifting small disturbance flow over dented plates, which corresponds with non-lifting thin-aerofoil/wing theory.

Let the surface of the dented plate be given by \( Z = Z_p(X,Y) \), with \( Z_p \) assumed to be small compared to the length and the width of the plate. In order to find the normal to the surface, the function \( F(X,Y,Z) \) is defined

\[
F(X,Y,Z) \equiv Z - Z_p(X,Y) = 0. \tag{3.12}
\]

The unit normal vector \( \mathbf{n} \), pointing outward on the upper surface of the dented plate is determined by

\[
\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{1}{|\nabla F|} \left( -\frac{\partial Z_p}{\partial X}, -\frac{\partial Z_p}{\partial Y}, 1 \right)^T. \tag{3.13}
\]

The velocity potential due to a freestream \( Q_{\infty} = (U_{\infty}, V_{\infty}, 0)^T \), corresponds with \( \Phi_{\infty} = U_{\infty}X + V_{\infty}Y \), and the potential can be constructed to be

\[ \Phi = \phi + \Phi_{\infty}, \tag{3.14} \]

which has to fulfil the boundary condition (3.2) of zero normal velocity at the surface of the plate

\[
\nabla \Phi \cdot \mathbf{n} = -\frac{1}{|\nabla F|} \left( \frac{\partial \phi}{\partial X} + U_{\infty} \frac{\partial Z_p}{\partial X} + \left( \frac{\partial \phi}{\partial Y} + V_{\infty} \right) \frac{\partial Z_p}{\partial Y} - \frac{\partial \phi}{\partial Z} \right) = 0. \tag{3.15}
\]

Consequently, the Laplace problem \( \nabla^2 \phi = 0 \) is to be solved for the perturbation potential \( \phi \) with boundary condition (3.15) for \( \partial \phi / \partial Z \) on \( Z = Z_p \). Assuming the geometry of the dent to be shallow, the perturbation velocity can be taken to be small compared to the freestream

\[
\left| \frac{\partial \phi}{\partial X} \right|, \left| \frac{\partial \phi}{\partial Y} \right|, \left| \frac{\partial \phi}{\partial Z} \right| \ll Q_{\infty}, \tag{3.16}
\]

where \( Q_{\infty} = |Q_{\infty}| \) and boundary condition (3.15) reduces to

\[
\frac{\partial \phi}{\partial Z}(X,Y,Z = Z_p) = U_{\infty} \frac{\partial Z_p}{\partial X} + V_{\infty} \frac{\partial Z_p}{\partial Y}. \tag{3.17}
\]

With the use of a Taylor expansion, boundary condition (3.17) at the plate’s surface is transferred to the \((X,Y)\) plane \((Z = 0)\)

\[
\frac{\partial \phi}{\partial Z}(X,Y,0) = U_{\infty} \frac{\partial Z_p}{\partial X} + V_{\infty} \frac{\partial Z_p}{\partial Y}. \tag{3.18}
\]
The above defined potential-flow problem for the perturbation potential \( \phi \) and boundary condition (3.18) can be solved by a source distribution on the \((X, Y)\) plane. Assuming a source distribution of strength \( \sigma \) on the \((X, Y)\) plane over the region \( X \in [0, \infty] \) and \( Y \in [0, \infty] \), the velocity potential and velocity field are given by

\[
\phi(X, Y, Z) = \frac{-1}{4\pi} \int_0^\infty \int_0^\infty \frac{\sigma(\xi, \eta) \, d\xi \, d\eta}{((X - \xi)^2 + (Y - \eta)^2 + Z^2)^{\frac{3}{2}}},
\]

(3.19)

\[
\frac{\partial \phi}{\partial X} = \frac{1}{4\pi} \int_0^\infty \int_0^\infty \frac{\sigma(\xi, \eta)(X - \xi) \, d\xi \, d\eta}{((X - \xi)^2 + (Y - \eta)^2 + Z^2)^{\frac{3}{2}}},
\]

(3.20)

\[
\frac{\partial \phi}{\partial Y} = \frac{1}{4\pi} \int_0^\infty \int_0^\infty \frac{\sigma(\xi, \eta)(Y - \eta) \, d\xi \, d\eta}{((X - \xi)^2 + (Y - \eta)^2 + Z^2)^{\frac{3}{2}}},
\]

(3.21)

\[
\frac{\partial \phi}{\partial Z} = \frac{1}{4\pi} \int_0^\infty \int_0^\infty \frac{\sigma(\xi, \eta)Z \, d\xi \, d\eta}{((X - \xi)^2 + (Y - \eta)^2 + Z^2)^{\frac{3}{2}}},
\]

(3.22)

The normal velocity \( \partial \phi/\partial Z(X, Y, 0) \) at the surface is related to the source strength, see figure 3.2. This can be shown either by a limit process for \( Z \to 0^\pm \) in equation (3.22), \( 0^+ \) and \( 0^- \) representing the upper and lower surface, respectively, or by observing the flux over the area \( \Delta X \) by \( \Delta Y \). The flux due to a source of strength \( \sigma(X, Y) \) over the area \( \Delta X \) by \( \Delta Y \) is given by

\[
\sigma(X, Y) \, \Delta X \Delta Y.
\]

(3.23)

At the same time, the flux due to the flow rate can be found to be

\[
\frac{\partial \phi}{\partial Z}(X, Y, 0^+) \, \Delta X \Delta Y - \frac{\partial \phi}{\partial Z}(X, Y, 0^-) \, \Delta X \Delta Y = 2 \frac{\partial \phi}{\partial Z}(X, Y, 0^+) \, \Delta X \Delta Y,
\]

(3.24)

the fluxes from the side becoming negligible when \( \Delta Z \to 0 \). Modelling the flow over the upper surface of the plate, only \( 0^+ \) is of interest and \( -\partial \phi/\partial Z(X, Y, 0^-) = \partial \phi/\partial Z(X, Y, 0^+) \)

![Figure 3.2: Source distribution for a dented plate.](image-url)
\[ \sigma(X, Y) = 2 \frac{\partial \phi}{\partial Z}(X, Y, 0). \quad (3.25) \]

Comparing (3.25) with boundary condition (3.18), it is seen that modelling the perturbation potential over a dented plate corresponds with a source distribution over the \((X, Y)\) plane when

\[ \sigma(X, Y) = 2 \left( U_\infty \frac{\partial Z_p}{\partial X} + V_\infty \frac{\partial Z_p}{\partial Y} \right). \quad (3.26) \]

The velocity field at the surface of the dented plate, \(Z_p\) being small, is now given by

\[ U(X, Y, Z = Z_p) \approx U(X, Y, 0) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty \frac{W(\xi, \eta, 0)(X - \xi) d\xi d\eta}{((X - \xi)^2 + (Y - \eta)^2)^{3/2}} + U_\infty, \quad (3.27) \]

\[ V(X, Y, Z = Z_p) \approx V(X, Y, 0) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty \frac{W(\xi, \eta, 0)(Y - \eta) d\xi d\eta}{((X - \xi)^2 + (Y - \eta)^2)^{3/2}} + V_\infty, \quad (3.28) \]

where \(W(X, Y, 0) = \partial \phi / \partial Z(X, Y, 0)\) as defined in (3.18). In two dimensions a similar result is found [50]

\[ U(X, Z = Z_p) \approx U(X, 0) = \frac{1}{\pi} \int_0^\infty \frac{W(\xi, 0) d\xi}{X - \xi} + U_\infty, \quad (3.29) \]

where

\[ W(X, 0) = \frac{\partial \phi}{\partial Z}(X, 0) = U_\infty \frac{\partial Z_p}{\partial X}. \quad (3.30) \]

### 3.2.3 Wing/aerofoil problem

In the previous section 3.2.2 the modelling of the inviscid flow over dented plates was discussed with the boundary condition applied on the projection of the dent in the \((X, Y)\) plane \((Z = 0)\). In order to more realistically model the geometry of a wing or aerofoil the boundary conditions should be applied at the actual surface. The singularity elements are to be distributed over the real surface to model the whole flow field, and the problem is reduced to finding the source and doublet strengths so that the boundary conditions (3.2) and (3.3) are satisfied.

Boundary condition (3.3) is automatically fulfilled as the singularity solutions have a velocity field that decays for \(r \to \infty\). For boundary conditions (3.2) two formulations exist. If the boundary-value problem is solved explicitly using boundary condition (3.2) we speak of a Neumann problem, and equations (3.10) and (3.11) are differentiated in order to solve for the total velocity field \(Q_{\text{tot}}\)

\[ Q_{\text{tot}} = \nabla \Phi. \quad (3.31) \]
Instead of applying the direct boundary condition (3.2), it is also possible to define \( \Phi \) as a constant. This is termed the Dirichlet formulation of the problem.

For the work described in this thesis the Dirichlet method is applied on the wing’s surface for the modelling of the really inviscid wing flow. The solution in the wake region follows, once the Dirichlet problem on the body is solved. The modelling of both regions is discussed next in more detail. However, first the Kutta condition is defined to determine the problem uniquely by relating the doublet strengths in the wake to the doublet strengths at the trailing edge.

**Kutta condition**

In the case of modelling wing or aerofoil flow the problem is not uniquely specified by only the boundary condition of no normal velocity through the profile’s surface.

A condition is required at the trailing edge to relate the doublets in the wake to the body doublets in order to uniquely determine the solution. The Kutta condition is used for this and in its most general form requires that the velocity at the trailing edge is bounded

\[
|\nabla \Phi|_{te} < \infty. \tag{3.32}
\]

For the present formulation an implicit Kutta condition is used. The potential jump across the wake is set equal to the difference between the potential values at the upper and lower surface at the trailing edge. This is the same as relating the doublet strength in the wake \( \mu_w \) to the unknown doublets \( \mu_{te_u} \) and \( \mu_{te_l} \) at the upper and lower trailing edge of the wing’s body as

\[
\mu_w = \mu_{te_u} - \mu_{te_l}. \tag{3.33}
\]

**Dirichlet method**

Specifying on the boundary the potential inside the body surface \( \Phi_{in} \), instead of using (3.2), is called the Dirichlet boundary condition and can be employed to determine the solution of the boundary-value problem (3.1). For field points \( p \) inside the body, the internal potential is given by (3.10)

\[
\Phi_{in} = -\frac{1}{4\pi} \int_{S_b} \int_{r} \frac{\sigma}{r} \, dS + \frac{1}{4\pi} \int_{S_b+S_w} \mu \frac{\partial}{\partial n} \frac{1}{r} \, dS + \Phi_\infty. \tag{3.34}
\]

With the Dirichlet boundary condition, \( \Phi_{in} = \phi_{in} + \Phi_\infty = \Phi_\infty \), where the internal perturbation potential \( \phi_{in} \) is set to zero, equation (3.34) simply becomes

\[
\frac{1}{4\pi} \int_{S_b+S_w} \mu \frac{\partial}{\partial n} \frac{1}{r} \, dS = \frac{1}{4\pi} \int_{S_b} \frac{\sigma}{r} \, dS, \tag{3.35}
\]

which for the two-dimensional flow case corresponds with

\[
\frac{1}{2\pi} \int_{S_b+S_w} \mu \frac{\partial}{\partial n} (\ln r) \, dS = \frac{1}{2\pi} \int_{S_b} \sigma \ln r \, dS. \tag{3.36}
\]
From the freestream the source strength \( \sigma \) can be determined, using the definition of source strength \( (3.9) \)

\[
\sigma = \frac{\partial \Phi}{\partial n} - \frac{\partial \Phi_{in}}{\partial n} = 0 - \frac{\partial \Phi_{\infty}}{\partial n} = -Q_{\infty} \cdot n,
\]

(3.37)

with \( \Phi_{in} = \Phi_{\infty} \) and the normal velocity \( \partial \Phi/\partial n \) at the surface being zero.

When applying the implicit Kutta condition \( (3.33) \), linking the doublet strengths in the wake to the upper and lower trailing edge doublets, the above constant-potential Dirichlet problem is uniquely defined and the unknown doublet distribution can be determined.

Assuming the wake to have zero thickness, the velocity distribution everywhere within the flow regime \( V \) is automatically determined by the source and doublet distribution provided by the Dirichlet method on the profile’s surface. Equations \( (3.10) \) and \( (3.11) \), differentiated in the normal direction with respect to the field point \( p \), form the integral equations for the velocity field in the wake and for the three-dimensional flow problem given by equation \( (3.10) \) this results in

\[
Q_{tot}(p) = \nabla \Phi(p) = \nabla \left( -\frac{1}{4\pi} \int_{S_b} \frac{\sigma}{r} dS + \frac{1}{4\pi} \int_{S_b+S_w} \frac{1}{\mu} \frac{\partial}{\partial n} \frac{1}{r} dS + \Phi_{\infty}(p) \right).
\]

(3.38)

### 3.2.4 Full-potential problem

As part of this thesis, contract work has been carried out for the Defence Evaluation and Research Agency (DERA) in Farnborough, United Kingdom, and use has been made of their existing Viscous Full-Potential (VFP) code for transonic flow calculations.

The inviscid flow part of the VFP program is based on the method developed by Forsey and Carr [36] for calculating potential flow over wing-fuselage combinations. It is valid for non-zero, subsonic Mach numbers and the scheme can be applied to configurations with swept, cranked and tapered wings. As the full-potential (FP) part of the VFP code is treated as a blackbox it will not be discussed in this thesis. More information can be found in the papers [33, 34, 4].

### 3.3 Viscous potential-flow problem

In the previous sections the modelling of the inviscid external flow has been discussed, ignoring the presence of the boundary layer. However, as the boundary layer displaces the outer potential flow away from the surface over a distance, termed the displacement thickness, this boundary-layer displacement effect should be taken into account into the external flow modelling.

The potential flow should be modelled over the resulting effective body shape, constructed from the actual physical body plus the displacement thickness. This can be done via the distribution of singularities over the resulting effective body shape which forms a streamsurface [60]. A simpler approach is to change the zero normal velocity boundary condition at the original surface in such a way that the normal velocity at the real surface
in the potential flow corresponds with the normal velocity at the edge of the boundary layer.

Before continuing, a local Cartesian coordinate system \((x, y, z)\) is defined where the \(x\)- and \(y\)-axis are tangent to the surface and the \(z\)-axis is normal to the surface. The local total velocity vector is given by \(\mathbf{q}_{\text{tot}} = (u, v, w)^T\). The velocity at the edge of the boundary layer is indicated with the subscript \(e\), thus \(u_e\) and \(v_e\) are the \(x\)- and \(y\)-velocity components at the edge of the boundary layer in the local Cartesian system.

Using the continuity equation and assuming that \(w\) grows linearly with \(- (\partial u_e / \partial x + \partial v_e / \partial y)\) \(z\) within the boundary layer, the following approximation for the normal velocity in the boundary layer can be obtained:

\[
  w(x, y, z) \approx - \left( \frac{\partial u_e}{\partial x} + \frac{\partial v_e}{\partial y} \right) z + w_1(x, y), \quad z \to \infty, \tag{3.39}
\]

where \(z \to \infty\) indicates the edge of the boundary layer. The disturbance velocity \(w_1 \neq 0\) is the next term of the expansion of \(w\) for \(z \to \infty\) and represents the so-called transpiration velocity, describing the transpiration between the boundary layer and the external flow. The definition for the transpiration velocity \(w_1\) is determined by

\[
  w_1(x, y) = \lim_{z \to \infty} \left[ w(x, y, z) + \frac{\partial u_e}{\partial x} z + \frac{\partial v_e}{\partial y} z \right]
  = \lim_{z \to \infty} \left[ \int_0^z \frac{\partial w}{\partial z} \, dz + \frac{\partial u_e}{\partial x} z + \frac{\partial v_e}{\partial y} z \right]
  = \lim_{z \to \infty} \left[ \int_0^z \left( - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \, dz + \frac{\partial u_e}{\partial x} z + \frac{\partial v_e}{\partial y} z \right]
  = \lim_{z \to \infty} \left[ \int_0^z \left( \frac{\partial u_e}{\partial x} - \frac{\partial u}{\partial x} \right) \, dz + \int_0^z \left( \frac{\partial v_e}{\partial y} - \frac{\partial v}{\partial y} \right) \, dz \right]
  = \frac{\partial}{\partial x} \int_0^\infty (u_e - u) \, dz + \frac{\partial}{\partial y} \int_0^\infty (v_e - v) \, dz
  = \frac{\partial}{\partial x} (q_e \delta_x^*) + \frac{\partial}{\partial y} (q_e \delta_y^*), \tag{3.40}
\]

where \(q_e = \sqrt{u_e^2 + v_e^2 + w_e^2}\) is the velocity magnitude at the edge of the boundary layer. The integral thickness definitions in the local Cartesian coordinate system for the displacement thicknesses \(\delta_x^*\) and \(\delta_y^*\) in the \(x\)- and \(y\)-directions can be found in appendix A.

Physically, the above transpiration velocity represents the rate of change of mass defect, being \(m_x = q_e \delta_x^*\) and \(m_y = q_e \delta_y^*\) in the local Cartesian coordinate system.

In two dimensions equation (3.40) reduces to

\[
  w_1(x) = \frac{\partial}{\partial x} (q_e \delta_x^*). \tag{3.41}
\]
To include the viscous influence in the potential-flow problem described in the previous sections, the ‘zero’ normal velocity boundary condition at the body surface (3.2) is modified, giving

\[
\nabla \Phi \cdot \mathbf{n} = w_1(x, y) \neq 0.
\]

(3.42)

For the discussed potential-flow problems the change in boundary condition, due to the presence of the boundary layer, results in modifying the source strengths. For the three-dimensional dented plate problem, where the global coordinate system corresponds with the local coordinate system, the source strength (3.26) becomes

\[
\sigma_{3D} = 2 \left( \frac{\partial}{\partial x} (U_\infty Z_p) + \frac{\partial}{\partial y} (V_\infty Z_p) \right) + 2 \left( \frac{\partial}{\partial x} (q_e \delta_x^*) + \frac{\partial}{\partial y} (q_e \delta_y^*) \right) = \sigma_{inv} + \sigma_{vis},
\]

(3.43)

or in two dimensions

\[
\sigma_{2D} = 2 \frac{\partial}{\partial x} (U_\infty Z_p) + 2 \frac{\partial}{\partial x} (q_e \delta_x^*) = \sigma_{inv} + \sigma_{vis},
\]

(3.44)

where \( \sigma_{inv} \) represents the inviscid source strength based on the freestream and \( \sigma_{vis} \) represents the viscous source strength due to the presence of the boundary layer.

For the wing and aerofoil problem the viscous sources should be included on the body and in the wake, and equation (3.37) for the source distribution on the body surface changes to

\[
\sigma = -Q_\infty \cdot \mathbf{n} + w_1 = \sigma_{inv} + \sigma_{vis}.
\]

(3.45)

In the wake, the displacement effect is a jump in normal velocity \( \Delta w_1 \) across the wake centre line, dividing the flows coming from the upper and lower surfaces of the profile

\[
\Delta w_1 = \frac{\partial}{\partial x} (q_e \delta_{xu}^* + q_e \delta_{xl}^*) + \frac{\partial}{\partial y} (q_e \delta_{yu}^* + q_e \delta_{yl}^*),
\]

(3.46)

where subscripts \( u \) and \( l \) indicate upper and lower surface values, respectively.

Dependent on the modelling of the boundary layer for the various flow cases, the wake’s upper and lower layers will be treated separately or together as just one layer. In the first case the above boundary condition (3.46) on the wake dividing streamsurface leads to the following viscous source strength in the wake:

\[
\sigma = \frac{\partial}{\partial x} (q_e \delta_{xu}^* + q_e \delta_{xl}^*) + \frac{\partial}{\partial y} (q_e \delta_{yu}^* + q_e \delta_{yl}^*) = \sigma_{vis}.
\]

(3.47)

For the case where the wake is modelled as just one layer with the wake thicknesses defined with \( \delta_{xw}^* = \delta_{xu}^* + \delta_{xl}^* \) and \( \delta_{yw}^* = \delta_{yu}^* + \delta_{yl}^* \), and the velocity \( q_e = q_{eu} = q_{el} \), the viscous source strength becomes

\[
\sigma_{vis} = \frac{\partial}{\partial x} (q_e \delta_{xw}^*) + \frac{\partial}{\partial y} (q_e \delta_{yw}^*).
\]

(3.48)
3.4 Discretisation viscous potential-flow problem

The general formulation of the two-dimensional potential-flow problem in (3.11) with the inclusion of viscous effects can be written

$$
\Phi(p) = \frac{1}{2\pi} \int_{S_b} \sigma_{inv} \ln r \, dS - \frac{1}{2\pi} \int_{S_b+S_w} (\mu \mathbf{n} \cdot \nabla(\ln r) - \sigma_{vis} \ln r) \, dS + \Phi_\infty(p), \quad (3.49)
$$

with the inviscid and the viscous source strengths on the body defined in (3.45). The wake is modelled as just one layer, implying that $\sigma_{vis}$ in the wake is given by the two-dimensional version of (3.48).

3.4 Discretisation viscous potential-flow problem

For the discretisation of the potential-based equations use is made of a panel method. The body surface, and (if present) the wake surface, are divided into a number of panel elements as seen in figure 3.3. For the three-dimensional case planar panels are used, whereas in two dimensions the panels simply reduce to straight lines.

The discretisation for the viscous potential problem for indented plate flow and wing/aerofoil flow is examined, and the velocity equations at the edge of the boundary layer required for the coupled viscous/inviscid problem are determined.

For clarity, first the used local and global Cartesian coordinate systems are re-introduced.

3.4.1 Global and local coordinate systems

In section 3.2 the general formulation of the potential-flow problem has been described in a global body-fixed Cartesian coordinate system $(X, Y, Z)$ with velocity vector $\mathbf{Q}_{tot} = (U, V, W)^T$. When the geometry is discretised, it is useful to define for each panel segment a local panel Cartesian coordinate system $(x, y, z)$ with velocity vector $\mathbf{q}_{tot} = (u, v, w)^T$.

For the model problem of flow over a dented plate, the local and global Cartesian coordinate systems are chosen to correspond with each other, with the origin defined at a corner point of the plate. However, for the panel method applied for a wing or aerofoil section the panels no longer lie in the same plane. Each panel has a different local coordinate system.

![Figure 3.3: Global $(X, Y, Z)$ and local $(x, y, z)$ Cartesian coordinate systems.](image)

**Figure 3.3**: Global $(X, Y, Z)$ and local $(x, y, z)$ Cartesian coordinate systems.
coordinate system, with the origin chosen in the midpoint of each panel, see figure 3.3. The \( z \)-axis is taken to be normal to the panel and the \( x \)- and \( y \)-axis are tangent to the panel.

### 3.4.2 Discretisation two-dimensional model problem

A two-dimensional plate is defined between \( x_1 \) and \( x_{M_p+1} \) along the \( x \)-axis and integral (3.29), which represents the \( x \)-velocity component at the edge of the boundary layer, becomes

\[
 u_e(x) = \frac{1}{\pi} \int_{x_1}^{x_{M_p+1}} \frac{w_e(\xi) d\xi}{x - \xi} + u_\infty(x),
\]

where \( 2w_e = \sigma_{inw} + \sigma_{vis} \) as given in equation (3.44). For the discretisation of the above integral, segment \([x_1, x_{M_p+1}]\) is divided into \( M_p \) straight-line panels each of similar length \( \Delta x = x_{k+1} - x_k \) for \( k \in [1, M_p] \). The collocation points are defined at the panel endpoints \( x = x_i \) for \( i \in [2, M_p] \) and

\[
 u_{e_i} = \frac{1}{\pi} \int_{x_1}^{x_{M_p+1}} \frac{w_e(\xi) d\xi}{x_i - \xi} + u_\infty(i),
\]

It is seen that for \( \xi = x_i \) the above integral is singular and in the region surrounding the singularity \([x_{i-1}, x_{i+1}]\) a linear expansion \( Lw_e \) of \( w_e \) is to be used. The Cauchy principal part \((C.p.p.)\) of the singular integral can be calculated and (3.51) is

\[
 u_{e_i} = \frac{1}{\pi} \sum_{k=1}^{M_p} \delta_{k+1, i} \int_{x_k}^{x_{k+1}} \frac{w_e(\xi) d\xi}{x_i - \xi} + \frac{1}{\pi} \int_{x_{i-1}}^{x_{i+1}} \frac{Lw_e(\xi) d\xi}{x_i - \xi} + u_\infty(i)
\]

\[
 = \frac{1}{\pi} \sum_{k=1}^{M_p} w_{e,k+\frac{1}{2}} \ln \left| \frac{x_i - x_k}{x_i - x_{k+1}} \right| - \frac{2\Delta x}{\pi} \left\{ \frac{\partial w_e}{\partial x} \right\}_{i} + u_\infty(i). \tag{3.52}
\]

where

\[
 Lw_e = w_e + (\xi - x_i) \frac{\partial w_e}{\partial \xi} \bigg|_{i}
\]

\[
 = \frac{\partial}{\partial \xi} \left( u_\infty \frac{Z_p + \delta e x}{Z_x} \right)_{i} + (\xi - x_i) \frac{\partial^2}{\partial \xi^2} \left( u_\infty \frac{Z_p + \delta e x}{Z_x} \right)_{i}. \tag{3.53}
\]

Using the above definition of \( Lw_e \) and centrally discretising the gradients of \( u_\infty Z_p \) and \( \delta e \), equation (3.52) becomes

\[
 u_{e_i} = \sum_{k=1}^{M_p} \frac{q_{e,k+1} \delta e_{x_k+1} - q_{e,k} \delta e_{x_k}}{\pi \Delta x} \ln \left| \frac{i - k}{i - k - 1} \right| - \frac{2}{\pi \Delta x} \left( \frac{q_{e,i+1} \delta e_{x,i+1} - 2q_{e,i} \delta e_{x,i} + q_{e,i-1} \delta e_{x,i-1}}{\pi \Delta x} \right) + u_\infty(i). \tag{3.54}
\]
which can be rewritten to the form

\[ u_{e_i} = \sum_{k=1}^{M_p+1} A^{u}_{i,k} \delta^*_{x_k} + u_{0,e_i}, \quad i = 2, \ldots, M_p, \tag{3.54} \]

where \( u_{0,e} \) is the undisturbed external inviscid flow at the edge of the boundary layer due to the freestream and the geometry of the dent. The summation part on the right-hand side of equation (3.54) describes the disturbance effect due to the presence of the boundary layer, with matrix \( A^u \) constructed as follows:

\[
\begin{align*}
A^{u}_{ii} & = \frac{4}{\pi \Delta x}, & i = 2, \ldots, M_p, \\
A^{u}_{ii-1} & = -\frac{1}{\pi \Delta x}(2 - \ln 2), & i \neq 2, \\
A^{u}_{ii+1} & = -\frac{1}{\pi \Delta x}(2 - \ln 2), & i \neq M_p, \\
A^{u}_{ik} & = \frac{1}{\pi \Delta x} \ln \left| 1 - \frac{1}{(i-k)^2} \right|, & k \neq 1, i-1, i, i+1, M_p, \\
A^{u}_{21} & = -\frac{2}{\pi \Delta x}, \\
A^{u}_{i1} & = -\frac{1}{\pi \Delta x} \ln \frac{i}{i-1}, & i \neq 2, \\
A^{u}_{i,M_p+1} & = -\frac{1}{\pi \Delta x} \ln \frac{M_p + 1 - i}{M_p - i}, & i \neq M_p, \\
A^{u}_{M_p,M_p+1} & = -\frac{2}{\pi \Delta x}.
\end{align*}
\]

The constructed influence matrix \( A^u \) has the following characteristics:

\[
\begin{align*}
A^{u}_{ii} & > 0, \\
A^{u}_{i,k} = A^{u}_{k,i} & < 0, & (i \neq k), \\
\sum_{k=1}^{M_p+1} A^{u}_{ik} & \geq 0, & \text{with at least for one } i \text{ the inequality sign.}
\end{align*}
\tag{3.55}
\]

From (3.55) it follows, that matrix \( A^u \) is a symmetric weakly diagonally dominant M-matrix (definition D.3 and theorem D.10) and therefore positive definite (theorem D.12).

### 3.4.3 Discretisation three-dimensional model problem

In the Cartesian coordinate system a rectangular plate is defined between \( x_1 \) and \( x_{M_p+1} \) and between \( y_1 \) and \( y_{M_l+1} \) in the \((x,y)\) plane. The segments along the \( x \)-axis are defined by the intervals \([x_k, x_{k+1}]\) of length \( \Delta x \) for \( k \in [1, M_p] \) and along the \( y \)-axis by \([y_l, y_{l+1}]\) of length \( \Delta y \) for \( l \in [1, M_l] \). The grid is constructed by rectangulars, resulting in panels of \( \Delta x \) by \( \Delta y \). The double integrals are evaluated in the node points of the panels \((x_i, y_j)\),
for \( i = 2, \ldots, M_p \) and \( j = 2, \ldots, M_q \) and the velocity components (3.27) and (3.28) at the edge of the boundary layer become

\[
\begin{align*}
\mathbf{u}_{eij} &= \frac{1}{2\pi} \int_{x_1}^{x_{M_p+1}} \int_{y_1}^{y_{M_q+1}} \frac{w_e(\xi, \eta)(x_i - \xi) \xi \eta}{(\xi - \eta)^2 + (y_j - \eta)^2} \, d\xi \, d\eta + u_{\infty ij}, \\
\mathbf{v}_{eij} &= \frac{1}{2\pi} \int_{x_1}^{x_{M_p+1}} \int_{y_1}^{y_{M_q+1}} \frac{w_e(\xi, \eta)(y_j - \eta) \eta}{(\xi - \eta)^2 + (y_j - \eta)^2} \, d\xi \, d\eta + v_{\infty ij}.
\end{align*}
\]

(3.56) (3.57)

It is seen that the above integrals, when evaluated in \( \xi = x_i \) and \( \eta = y_j \) are singular. Outside a region surrounding the singularity, the integrals are well-behaved. As in the two-dimensional case it is assumed that a linear source distribution exists over the region \( x_{i-1} < x < x_{i+1} \) and \( y_{j-1} < y < y_{j+1} \), including the singularity. The Cauchy principal part of the singular integral can be calculated and equation (3.56) is rewritten

\[
\begin{align*}
\mathbf{u}_{eij} &= \frac{1}{2\pi} \sum_{k=1}^{M_p} \sum_{l=1}^{M_q} \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \frac{w_e(x_i - \xi) \xi \eta}{(\xi - \eta)^2 + (y_j - \eta)^2} \, d\xi \, d\eta + u_{\infty ij} \\
&= \frac{1}{2\pi} \sum_{k=1}^{M_p} \sum_{l=1}^{M_q} \left( \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \frac{Lw_{eij}(x_i - \xi) \xi \eta}{(\xi - \eta)^2 + (y_j - \eta)^2} \, d\xi \, d\eta + u_{\infty ij} \right) \\
&= \frac{1}{2\pi} \sum_{k=1}^{M_p} \sum_{l=1}^{M_q} \left( \left( \frac{\sqrt{(x_i - x_{i+1})^2 + (y_j - y_{l+1})^2}}{\sqrt{(x_i - x_{i+1})^2 + (y_j - y_{l+1})^2}} \cdot \frac{(x_i - x_{i+1})^2}{(x_i - x_{i+1})^2 + (y_j - y_{l+1})^2} \right) \cdot \frac{\partial w_e}{\partial x}_{ij} + u_{\infty ij},
\end{align*}
\]

(3.58)

with \( Lw_e \) the linear expansion of \( w_e \)

\[
Lw_{eij} = w_{eij} + (x_i - x_i) \frac{\partial w_e}{\partial x}_{ij} + (y_j - y_j) \frac{\partial w_e}{\partial y}_{ij},
\]

(3.59)

and \( 2w_e = \sigma_{\text{inv}} + \sigma_{\text{vis}} \) given by (3.43) is

\[
w_e = \frac{\partial}{\partial x} (u_{\infty} Z_p + q_e \delta_x^*) + \frac{\partial}{\partial y} (v_{\infty} Z_p + q_e \delta_y^*),
\]

(3.60)

Using the definition of \( Lw_e \) (3.59) and centrally discretising the gradients of \( q_e \delta_x^* \), \( q_e \delta_y^* \),


3.4 Discretisation viscous potential-flow problem

\( u_{\infty} Z_p \) and \( v_{\infty} Z_p \), the equations for \( u_{ei} \) and \( v_{ei} \) are found to be of the form

\[
\begin{align*}
    u_{ei} &= \sum_{k=1}^{M_p+1} \sum_{l=1}^{M_s+1} q_{ekl} \left[ A_{ijkl} \delta_{xkl} + B_{ijkl} \delta_{ykl} \right] + u_{0ei}, \\
    v_{ei} &= \sum_{k=1}^{M_p+1} \sum_{l=1}^{M_s+1} q_{ekl} \left[ A_{ijkl} \delta_{xkl} + B_{ijkl} \delta_{ykl} \right] + v_{0ei}.
\end{align*}
\]  

(3.61)

(3.62)

The double summation part contains the disturbance due to the boundary layer, and as before, \( u_{0} \) and \( v_{0} \) are the undisturbed inviscid velocity components, containing the freestream and the discretisation of the geometry.

3.4.4 Discretisation wing/aerofoil problem

The Dirichlet approach of the potential-flow problem is to be applied for the wing/aerofoil problem. With the inclusion of the boundary-layer displacement effect in the Dirichlet formulation of the potential-flow problem on the wing’s surface, equation (3.35) becomes

\[
\frac{1}{4\pi} \int \int_{S_{b}+S_{w}} \frac{1}{r} \frac{\partial}{\partial n} \mu dS = \frac{1}{4\pi} \int \int_{S_{b}} \frac{\sigma_{inv}}{r} dS + \frac{1}{4\pi} \int \int_{S_{b}+S_{w}} \frac{\sigma_{vis}}{r} dS, \tag{3.63}
\]

with the inviscid source strength \( \sigma_{inv} \) and the viscous source strength \( \sigma_{vis} \) as defined in (3.45) on the body and (3.47) in the wake.

In order to calculate the above integrals the wing surface and wake are discretised. Let the number of segments along the contour of the body surface be denoted by \( M_b \), and spanwise along the \( Y \)-axis by \( M_s \), resulting in \( N_b = M_b \times M_s \) panels on the body surface. The wake centre line is divided into \( M_w \) segments, leading to \( N_w = M_w \times M_s \) panels in the wake.

A low-order panel method is used, assuming that the singularity strengths are constant and equal to the value at the centroid of each panel. Each panel will have a constant doublet strength \( \mu_n \) and a constant source strength \( \sigma_n \), for \( n \in [1, N_b + N_w] \). The inviscid source strengths are determined from the freestream and the viscous source strengths are known from the viscous quantities. The discretisation of equation (3.63) then leads to a linear system of equations to be solved for the unknowns \( \mu_n, n \in [1, N_b + N_w] \), defined in the panel centres

\[
\sum_{n=1}^{N_b+N_w} \frac{1}{4\pi} \int \int_{S_n} \frac{\mu_n}{r} \frac{\partial}{\partial n} dS = \sum_{n=1}^{N_b} \frac{1}{4\pi} \int \int_{S_n} \frac{\sigma_{inv}}{r} dS + \sum_{n=1}^{N_b+N_w} \frac{1}{4\pi} \int \int_{S_n} \frac{\sigma_{vis}}{r} dS.
\]

A more simplified way of writing the above system for body collocation points \( m \in [1, N_b] \), located in the midpoint of the panel, is

\[
\sum_{n=1}^{N_b+N_w} A_{mn} \mu_n = \sum_{n=1}^{N_b} B_{mn} \sigma_{inv} + \sum_{n=1}^{N_b+N_w} B_{mn} \sigma_{vis}, \tag{3.64}
\]
and a similar system is found for the two-dimensional flow problem. For the construction of (3.64) the source and doublet strengths, assumed constant per panel, have been taken outside the double integral. Influence matrices $A_{mn}$ and $B_{mn}$, describing the influence at collocation point $m$ due to a constant doublet strength distribution and source strength distribution, respectively, placed at panel $n$, are given by

$$A_{mn} = \frac{1}{4\pi} \int_{S_n} \frac{\partial}{\partial n} \frac{1}{r_{mn}} \, dS,$$

$$B_{mn} = \frac{1}{4\pi} \int_{S_n} \frac{1}{r_{mn}} \, dS,$$

$$A_{mn} = \frac{1}{2\pi} \int_{S_n} \frac{\partial}{\partial n} (\ln r_{mn}) \, dS,$$

$$B_{mn} = \frac{1}{2\pi} \int_{S_n} \ln r_{mn} \, dS,$$

(3.65)

(3.66)

with $r_{mn}$ the distance between collocation point $m$ and the point of integration $n$. The matrices $A_{mn}$ and $B_{mn}$ are defined in a local panel-orientated coordinate system, and the evaluation of the above integrals can be found in the paper of Hess and Smith [44].

The above system (3.64) consists of $N_b$ equations, however, there are $N_b + N_w$ unknown doublet strengths to be determined. For the problem to be unique the implicit Kutta condition is used, to prescribe the unknown doublet strength in the wake. Providing there is no significant flow along the trailing edge, the doublet strength along a wake strip is taken to be constant, and its value, $\mu_w$, is related to the upper and lower trailing edge doublets as

$$\mu_w = \mu_{bu} - \mu_{bt},$$

(3.67)

where $\mu_{bu}$ is the doublet strength of the upper body trailing edge panel and $\mu_{bt}$ is the doublet strength of the lower body trailing edge panel. With the above implicit Kutta condition, the influence matrix $A_{mn}$ can be reduced to a $N_b \times N_b$ matrix. The $N_b$ unknown doublet strengths on the body surface are to be calculated from the resulting system of $N_b$ equations.

**Edge velocity equations body**

For the interaction with the viscous layer, a relation between the velocity from the external flow at the edge of the boundary layer, and the boundary-layer displacement thicknesses, is required. For the Dirichlet method this means that equation (3.64) has to be further manipulated.

The total edge velocity vector $Q_{tot_e}$, located at the endpoint of a panel, instead of at the centroid, can be decomposed as given in the paper by Kerwin et al. [51] and shown in figure 3.4.

$$Q_{tot_e} = Q_{s_e} + Q_\infty + \sigma \, n$$

$$= Q_{s_e} + Q_\infty - (Q_\infty \cdot n) n + w_1 n,$$

(3.68)

and fulfills the boundary condition $Q_{tot_e} \cdot n = w_1$ as given in (3.42). The term $Q_{s_e} = \nabla_s \phi$ corresponds to the component of perturbation velocity tangential to the body surface. If
Figure 3.4: Composition velocity vectors.

t_1 and t_2 are defined as two different unit vectors tangential to the surface, the following holds:
\[
\frac{\partial \phi}{\partial t_1} = t_1 \cdot \nabla_s \phi, \quad \frac{\partial \phi}{\partial t_2} = t_2 \cdot \nabla_s \phi.
\]

When using the above, the tangential velocity \( Q_{s_w} \) can be expressed
\[
Q_{s_w} = \frac{\frac{\partial \phi}{\partial t_1} \left[ t_1 - (t_1 \cdot t_2) t_2 \right] + \frac{\partial \phi}{\partial t_2} \left[ t_2 - (t_1 \cdot t_2) t_1 \right]}{\| t_1 \times t_2 \|^2}. \tag{3.69}
\]

The two perturbation velocity components in the t_1 and t_2 directions are determined by the two components of the doublet gradient in the two corresponding tangential directions
\[
\frac{\partial \phi}{\partial t_1} = \frac{\partial \mu}{\partial t_1}, \quad \frac{\partial \phi}{\partial t_2} = \frac{\partial \mu}{\partial t_2},
\]
which can be discretised using central differencing.

After some manipulations, combining (3.64) with the Kutta condition (3.67) and (3.69), the equation for \( Q_{s_w} \) in the panel endpoints indicated by \( (i,j) \), with \( i \in [2, M_b] \) and \( j \in [2, M_s] \), can be written
\[
Q_{s_{ei}} = \sum_{k=1}^{M_b} \sum_{l=1}^{M_s} (CA^{-1}B)_{ijkl} \sigma_{inv,kl} + \sum_{k=1}^{M_b+M_w} \sum_{l=1}^{M_s} (CA^{-1}B)_{ijkl} \sigma_{vis,kl}, \tag{3.70}
\]
where indices \( k \in [1, M_b] \) and \( l \in [1, M_s] \) define the panel number. Furthermore, matrix \((CA^{-1}B)\) is constructed from three matrices \( A^{-1} \), \( B \) and \( C \). Matrix \( A^{-1} \) is the inverse of \( A \) which represents the doublet influence and matrix \( B \) represents the source influence, both given in (3.65) and (3.66). Matrix \( C \) contains the discretisation of expression (3.69).

It follows that equation (3.68) for the total edge velocity \( Q_{tot,i} \), with \( Q_{s_w} \) given by (3.70), can be rewritten as:
\[
Q_{tot_{ei}} = \sigma_{vis_{ij}} n + \sum_{k=1}^{M_b+M_w} \sum_{l=1}^{M_s} D_{ijkl} \sigma_{vis,kl} + Q_{o_{ei}}, \tag{3.71}
\]
with matrix $D$ being $(CA^{-1}B)$. In the above expression, $Q_{0}$ is the undisturbed inviscid velocity, due to the freestream and the inviscid sources.

Equation (3.71) can be further simplified, using (3.45) and (3.47) for a two-layered wake and centrally discretizing the gradients of $q_{e}\delta_{x}^{e}$ and $q_{e}\delta_{y}^{e}$, giving finally for the global edge velocity components $Q_{\text{tot}} = (U_{e}, V_{e}, W_{e})^{T}$ in the body collocation points $i \in [2, M_{b}]$ and $j \in [2, M_{w}]$

\[
U_{e_{ij}} = \sum_{k=1}^{M_{b}+2M_{w}+1} \sum_{l=1}^{M_{w}+1} q_{e_{il}} [A_{ijkt}^{U} \delta_{x}^{e} + B_{ijkt}^{U} \delta_{y}^{e}] + U_{0_{e_{ij}}}, \tag{3.72}
\]

\[
V_{e_{ij}} = \sum_{k=1}^{M_{b}+2M_{w}+1} \sum_{l=1}^{M_{w}+1} q_{e_{il}} [A_{ijkt}^{V} \delta_{x}^{e} + B_{ijkt}^{V} \delta_{y}^{e}] + V_{0_{e_{ij}}}, \tag{3.73}
\]

\[
W_{e_{ij}} = \sum_{k=1}^{M_{b}+2M_{w}+1} \sum_{l=1}^{M_{w}+1} q_{e_{il}} [A_{ijkt}^{W} \delta_{x}^{e} + B_{ijkt}^{W} \delta_{y}^{e}] + W_{0_{e_{ij}}}. \tag{3.74}
\]

It is noted that the first wake node point corresponds with the upper and lower trailing edge points, defined $(M_{w} + 1, l)$ and $(M_{b} + M_{w} + 1, l)$, and is therefore omitted. Indices $(M_{w}, l)$ and $(M_{b} + M_{w} + 2, l)$ thus represent the upper and lower ‘second’ wake point just after the trailing edge for spanwise station $l$.

In two dimensions a similar manipulation can be followed and relations for the velocity components $U$ and $W$ in the Cartesian coordinate system can be found. However, unlike the three-dimensional flow problem where the velocities in the Cartesian coordinate system are used, the two-dimensional flow problem will make use of a streamline coordinate system $(s, n)$. The tangential velocity in the streamline direction $U_{s} = \sqrt{U^{2} + W^{2}}$ becomes

\[
U_{s_{e_{i}}} = Q_{\text{tot}_{e_{i}} \cdot s_{i}} = \frac{\partial \mu}{\partial s} + \frac{\partial \Phi_{\infty}}{\partial s}, \tag{3.75}
\]

having used (3.68) with $Q_{s_{e}} \cdot s = \partial \phi / \partial s = 0$, $\mu \cdot s = 0$ and $Q_{\infty} \cdot s = \partial \Phi_{\infty} / \partial s$. By manipulating (3.64) and using the Kutta condition (3.67) it is then found that

\[
U_{s_{e_{i}}} = \sum_{k=1}^{M_{b}} (CA^{-1}B)_{ik} \sigma_{i_{k}n_{k}} + \sum_{k=1}^{M_{b}+M_{w}} (CA^{-1}B)_{ik} \sigma_{i_{k}n_{k}} + U_{s_{\text{eq}}}, \tag{3.76}
\]

with matrix $C$ containing the discretisation of (3.75). Modelling the boundary layer in the wake as just one layer with (3.48) and using the integral thickness definitions in a streamline coordinate system, as given in appendix A, $U_{s_{e}}$ becomes

\[
U_{s_{e_{i}}} = \sum_{k=1}^{M_{b}+M_{w}+1} A_{e_{ik}}^{U_{e}} U_{s_{e_{k}}} \delta_{s_{e}^{i}} + U_{0_{\text{eq}}}. \tag{3.77}
\]

From equations (3.72) and (3.74) it follows that:

\[
A_{e_{ik}}^{U_{e}} = \cos \theta_{k} (\cos \theta_{i} A_{i_{k}}^{U} + \sin \theta_{i} A_{i_{k}}^{W}), \tag{3.78}
\]

where $\theta_{k}$ is the angle panel $k$ makes with the $X$-axis.
3.5 Aerodynamic coefficients

Edge velocity equations wake

In the wake region, the doublet strengths are known from the Dirichlet method on the body surface and the implicit Kutta condition. The source strengths are given by the freestream and viscous quantities.

For the discretisation of the potential-flow method in the wake, the potentials are to be determined in the midpoints of the panels. By taking the gradient, the edge velocity equations are calculated back in the node points of the panels. The total potential at the collocation points \( m \in [1, N_w] \), located in the midpoints of the wake panels, is given by

\[
\Phi_m = \Phi_\infty - \sum_{n=1}^{N_w} \frac{1}{4\pi} \int \mu_n \frac{\partial}{\partial n} r dS + \sum_{n=1}^{N_s} \frac{1}{4\pi} \int \frac{\sigma_{inv_n}}{r} dS + \sum_{n=1}^{N_w} \frac{1}{4\pi} \int \frac{\sigma_{vis_n}}{r} dS.
\]

A more simplified way of writing the above system is

\[
\Phi_m = \Phi_\infty - \sum_{n=1}^{N_w} A_{mn} \mu_n + \sum_{n=1}^{N_s} B_{mn} \sigma_{inv_n} + \sum_{n=1}^{N_w} B_{mn} \sigma_{vis_n} \quad (3.79)
\]

with \( A_{mn} \) and \( B_{mn} \) as defined before in (3.65) and (3.66) and \( \phi \) the perturbation potential. The doublet strengths \( \mu \) in (3.79) are known from the Dirichlet method on the body and can be replaced with (3.64), which results in an equation for \( \phi \) only dependent on the sources strengths \( \sigma_{inv} \) and \( \sigma_{vis} \).

The velocity distribution \( Q_{\text{tot}} \) can then be determined in the panel endpoints with index \((i,j), i \in [1, M_w - 1] \) and \( j \in [2, M_i] \), by differencing, using (3.69). Note that \( i = 1 \) represents the ‘second’ wake point after the trailing edge. Relations for the velocity components in the wake are thus found of a similar format as equations (3.72), (3.73) and (3.74).

3.5 Aerodynamic coefficients

In a frame of reference with velocity \( Q_\infty \), Bernoulli’s equation for incompressible and inviscid steady flow is

\[
p_\infty + \frac{1}{2} \rho Q_\infty^2 = p + \frac{1}{2} \rho Q_{\text{tot}}^2,
\]

where \( p \) is the pressure and \( \rho = 1 \) the density. From (3.80) the pressure coefficient follows:

\[
C_p = \frac{p - p_\infty}{\frac{1}{2} \rho Q_\infty^2} = 1 - \frac{Q_{\text{tot}}^2}{Q_\infty^2}.
\]

In two dimensions the lift and the drag coefficient are defined by

\[
C_L = \frac{L}{\frac{1}{2} \rho Q_\infty^2 c}, \quad C_D = \frac{D}{\frac{1}{2} \rho Q_\infty^2 c},
\]
for which the lift $L$, is the force acting in the direction perpendicular to the freestream direction and the drag $D$, the force acting in the direction of the freestream.

Assuming the pressure coefficient $C_p$ and the skin-friction coefficient $C_f$ to be constant over each panel, the forces in the $X$- and $Z$-direction per wing section are given by

\[ F_x = \sum_i (C_p l_i \sin \theta_i + \text{sign} \frac{C_f l_i}{2} \cos \theta_i), \]  
\[ F_z = \sum_i (-C_p l_i \cos \theta_i + \text{sign} \frac{C_f l_i}{2} \sin \theta_i), \]  

where $\theta_i$ the angle between panel $i$ and the $X$-axis and $l_i^2 = \Delta X^2 + \Delta Z^2$, the length of panel $i$. The term $\text{sign}$ is equivalent to $+1$ or $-1$ for panel $i$ on the upper or lower surface, respectively. Then for an incidence $\alpha$ the lift and drag are

\[ L = F_x \cos \alpha - F_x \sin \alpha, \]
\[ D = F_x \cos \alpha + F_z \sin \alpha. \]  

Alternatively, the lift coefficient per wing section can be determined with the Kutta-Joukowski’s formula [50]. This formula states that the resultant aerodynamic force in an incompressible, inviscid, steady and irrotational flow is directly proportional to the circulation $\Gamma$ and acts normal to the freestream, hence

\[ C_{L_{KJ}} = \frac{\Gamma}{\frac{1}{2} \rho Q_{\infty}^2 c}. \]  

The total circulation is determined by

\[ \Gamma = \Phi_{w_{u}} - \Phi_{w_{l}} = \mu_{w_{u}} - \mu_{w_{l}}, \]

having used the implicit Kutta condition (3.33).

The integration formula for the drag coefficient (3.86) can give quite incorrect results. A more suitable prediction is given with Young’s formula [109]

\[ C_{D_{V}} = \frac{2 \theta_{s_{\infty}}}{c} \]
\[ = \left( \frac{U_{\infty}}{c} \right)^{\frac{1}{2}}, \]  

which is derived from the Von Kármán equation, with $\theta_{s_{\infty}}$ the streamwise momentum thickness far downstream in the wake.

### 3.6 Verification two-dimensional Dirichlet method

To verify the two-dimensional Dirichlet method for the calculation of inviscid flow, the described panel method has been tested for the calculation of inviscid flow around an
ellipse for which the analytical solution is known. The geometry of the ellipse is defined by

\[
\frac{(X - \frac{1}{2})^2}{(\frac{1}{2})^2} + \frac{Y^2}{\tau^2} = 1,
\]

(3.89)

where \(2\tau\) is the maximum thickness of the ellipse. The maximum width of the ellipse corresponds with 1. The exact solution for the flow past an ellipse can be determined with complex-function theory and the velocity components in the \(X\)- and \(Y\)-direction on the ellipse are

\[
U_{\text{ellipse}} = \left( \frac{0.5 - \tau}{0.5 + \tau} + 1 \right) \frac{(2 - 8(x - 0.5)^2) \cos \alpha + (4y - 4yx) / \tau \sin \alpha}{1 + \frac{0.5 - \tau}{0.5 + \tau} \left( \frac{0.5 - \tau}{0.5 + \tau} - 16(x - 0.5)^2 + 2 \right)},
\]

(3.90)

\[
V_{\text{ellipse}} = \left( \frac{0.5 - \tau}{0.5 + \tau} - 1 \right) \frac{(4x - 8(x - 0.5)^2) \sin \alpha + (4y(x - 0.5)) / \tau \cos \alpha}{1 + \frac{0.5 - \tau}{0.5 + \tau} \left( \frac{0.5 - \tau}{0.5 + \tau} - 16(x - 0.5)^2 + 2 \right)},
\]

(3.91)

---

**Figure 3.5:** Geometry ellipse for \(\tau = 0.06\).

**Figure 3.6:** Velocity distribution for \(N = 15\).

**Figure 3.7:** Velocity distribution for \(N = 35\).

**Figure 3.8:** Velocity distribution for \(N = 61\).
with $\alpha$ being the angle of attack. The total magnitude of the velocity corresponds with the streamwise velocity at the surface of the ellipse and is given by

$$U_{\text{ellipse}} = \sqrt{U_{\text{ellipse}}^2 + V_{\text{ellipse}}^2}$$ (3.92)

For the present calculations $\tau$ is set to 0.06 for which the geometry of the ellipse is shown in figure 3.5. The trailing edge point corresponds with $x = 1$. The computations are performed for three different incidences, being $\alpha = 0^\circ$, $10^\circ$ and $45^\circ$.

In figures 3.6, 3.7 and 3.8 the velocity distribution obtained with the panel method for the three incidences is shown, compared to the exact results obtained with (3.92). For the results in figure 3.6, 15 points are use for the discretisation of the body surface, and it is seen that the results from the Dirichlet method are not very accurate. Increasing the number of points to 35 gives much better correspondence with the exact solution, as is seen in figure 3.7. However, near the trailing edge the velocity distribution determined by the panel method for the upper surface is seen to cross the velocity distribution for the lower surface, resembling a fish-tail. The analytical Kutta condition, implying equal upper and lower velocities at the trailing edge, is not satisfied.

The results obtained using 61 body points, shown in figure 3.8, are nearly similar to the results for 35 body points in figure 3.7. The accuracy has not really improved and the fish-tail behaviour near the trailing edge for larger angles of attack remains. For the boundary-layer coupling this can cause problems and a modification of the trailing edge values is required and will be discussed in the next section.

The three-dimensional Dirichlet method has been programmed by George Patrianakos, a fellow Ph.D. student at the University of Bristol, and the validation for the three-dimensional Dirichlet method will be presented in his thesis.

### 3.7 Adjustment trailing edge velocity distribution

As is clear from the previous section, the inviscid flow solution obtained at the trailing edge with the two-dimensional Dirichlet panel method is not very accurate. The three-dimensional wing Dirichlet panel method, making use of the same Kutta condition as the two-dimensional method, displays the same inaccuracies.

With the implemented implicit Kutta condition (3.67) the actual Kutta condition is not explicitly fulfilled as (3.67) apparently allows for the trailing edge velocities to be different from each other on the upper and lower surfaces, whereas analytically they should be exactly the same. Already a small difference can have a large influence on the overall solution, especially when coupled to a boundary layer. An even larger problem arises when the inaccurate solution at the trailing induces a steep pressure gradient, which can cause the boundary-layer calculations to breakdown.

For the work described in this thesis the external flow solver is treated as a blackbox. To avoid the breakdown of the boundary-layer calculations, and to possibly improve the overall solution, therefore only some ad hoc modifications have been implemented. To come to a more suitable trailing edge treatment the problem is to be examined more closely.
The velocities at the trailing edge coming from the wing and aerofoil Dirichlet panel method are modified before being used for the boundary-layer calculations. Various possibilities have been tested and in the two-dimensional aerofoil code the following simple procedure is used:

\[ U_{e_{tr}} = \min(U_{e_{tr_u}}, U_{e_{tr_l}}). \] (3.93)

The new value at the trailing edge is set to the minimum of the upper and lower trailing edge velocities.

For a wing section at angle of attack the most critical area is near the trailing edge on the upper surface, where separation can occur for severe positive pressure (negative velocity) gradients. Adjustment by using the maximum value of \( U_{e_{tr}} \) at the trailing edge, instead of the minimum value as is done in (3.93), would decrease the velocity gradient near the trailing edge of the upper surface and one would expect this to improve the boundary-layer calculations. However, using the minimum has shown to work much better. An explanation for this could be that the pressure gradient just after the trailing edge in the wake is more problematic than the pressure gradient at the trailing edge on the upper surface. Taking the maximum value of \( U_{e_{tr}} \) at the trailing edge could in this case increase the (positive) pressure gradient just after the trailing edge in the wake, making the calculations there more difficult, whereas taking the minimum would make the calculations easier.

For the three-dimensional wing external velocity distribution a slightly different modification procedure is used. The solution for the three velocity components in the trailing edge point on the upper and lower surface is determined by averaging the calculated upper and lower trailing edge velocities.

In the wake the solution for the Z-velocity component in the first point downstream of the trailing edge is also redetermined. This is done via interpolation with the new trailing edge velocity and the downstream wake values.

Other adjustment variants have been tested as well, however, the above ones have shown to be the most robust and accurate and have been applied.