Computation, efficiency and endogeneity in discrete choice models

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Chapter 4

Market Equilibrium Models

4.1 Introduction

This chapter discusses versions of a discrete choice model of market demand and supply that have recently gained popularity in econometrics. The development of these models has been driven by the main objectives of modeling theoretically and empirically market demand for differentiated products and market conduct of participating firms in a market.

The literature on discrete choice models (see chapter 2 and the references therein) for estimating demand for differentiated products experienced parallel development with the so-called address approach to modeling product differentiation (e.g., Lancaster (1966, 1971)). This approach assumes that both consumers and products can be described by particular points, or addresses, in the product characteristics space. The two approaches, however, serve essentially the same purposes and it is possible to show that they are equivalent (Anderson, de Palma and Thisse (1992)).

The other main stream of research, which studies market conduct of participating firms, is based on the theory of oligopolistic competition. Its empirical analysis requires simultaneous modeling of demand and supply. The earliest attempts of simultaneous estimation of demand and supply used demand specifications linear in price and assumed vertically differentiated products. In this regard Bresnahan (1987) assumed uniformly distributed consumer characteristics over the quality variable of the products. Bresnahan’s methodology was applied by several authors (e.g., Gasmi, Laffont and Vuong (1992), Kadiyali (1996)), to study various problems of market conduct.

One of the first discrete choice models of oligopolistic competition with differentiated products is due to Perloff and Salop (1985), who employed the
standard logit for specifying demand. Caplin and Nalebuff (1991) identifies some of the conditions in the utility specification required for the existence and uniqueness of the price equilibrium in these models in the case of one-product firms.

Among the first empirical versions of these models are Feenstra and Levinsohn (1995) and Berry, Levinsohn and Pakes (1995, hereafter BLP). The former retains Bresnahan’s (1987) assumption of the uniform distribution of consumer characteristics and uses the address approach to product differentiation. The latter employs a mixed logit model (section 2.3 of chapter 2) allowing for product characteristics that are unobserved by the researcher. This way, it models products that are differentiated in multiple dimensions, consumer heterogeneity through random coefficients, and unobserved product characteristics through price competition.

This chapter has two main objectives. The first is to present the model used by BLP and some versions of it, and the second to investigate the problem of price equilibrium existence and uniqueness in particular versions of the model assuming multi-product firms. We present the demand and supply side of the model in the next two sections. Then we address the problem of existence and uniqueness of price equilibrium in different versions of this model. As we show in the next chapter, uniqueness of the price equilibrium, in addition to its theoretical importance, facilitates the efficient estimation of this model.

4.2 Demand

As mentioned above, the demand model is based on a discrete choice model with differentiated products. In all versions presented below the utility of a consumer \( i \) from consuming product \( j \) is expressed as a function \( U \) of the price of the product, \( p_j \), the vector of characteristics of the product, \( x_j \), the vector of the consumer’s characteristics, \( \tau_i \), and a vector of parameters \( \theta \):

\[
U(p_j, x_j, \tau_i; \theta).
\]  

(4.1)

This utility specification is general and it nests several models from the empirical and theoretical literature (e.g., Caplin and Nalebuff (1991), BLP, Nevo (2001)). First we present the BLP version and then we make reference to other studies.

Let \( J \) be the number of all products available in a certain market and let \( 1, ..., J \) denote these products. For \( j \in \{1, ..., J\} \), let \( x_j \) and \( p_j \) denote the \( K \)-vector of characteristics observed by the researcher and the price of product
BLP assume that the utility of an individual \( i \) who purchases product \( j \) is

\[
u_{ij} = \alpha \ln(y_i - p_j) + x'_j \beta_i + \xi_j + \varepsilon_{ij}, \tag{4.2}\]

where \( \alpha \) is a scalar parameter, \( y_i \) stands for the income of consumer \( i \), \( \beta_i \) is a parameter that weighs the characteristics of product \( j \) in \( i \)'s utility, \( \xi_j \) is the unobserved characteristic of product \( j \), that is, a variable responsible for the utility effect of the characteristics of product \( j \) that are not observed by the researcher. This represents a combination of characteristics of product \( j \) that are quantifiable but not observed by the researcher or that are not easy to measure like style, reputation or past experience of the producer. The error term \( \varepsilon_{ij} \) is assumed to be iid type I extreme value random variable independent of the rest of the variables.

BLP assume that \( \beta_i \sim N(\beta, \Sigma) \) for any consumer \( i \) where \( \beta \) is a \( K \)-vector of (nonrandom) parameters and \( \Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_K^2) \) with \( \sigma_1, \ldots, \sigma_K \) scalar parameters. This yields a utility specification related to the mixed logit (see section 2.3). The random coefficient can be written in the form

\[\beta_i = \beta + \Sigma^{\frac{1}{2}} v_i\]

with \( v_i \) a \( K \)-vector of iid standard normal random vectors. Consumers’ logarithm of income is assumed to have a normal distribution with mean \( \mu_y \) and variance \( \sigma_y^2 \). Hence consumer \( i \)'s income can be represented as \( y_i = \exp(\mu_y + \sigma_y v_{yi}) \), where \( v_{yi} \) is a standard normal random variable assumed to be independent of \( \beta_i \). Then the variables \( v_{yi} \) and \( v_i \) can be interpreted as consumer characteristics that are unobserved by the researcher. The parameters of the income distribution, \( \mu_y \) and \( \sigma_y \), can be estimated from the empirical distribution of the population’s income. The inclusion of the random variables representing consumer characteristics makes it possible to estimate the model using aggregated data. Indeed, BLP estimated their model using data on automobile characteristics, prices and sales.

We note that here income is treated as a random variable with a parametric distribution. It is also possible to treat income as a random variable with a nonparametric distribution, as Nevo (2001) does, by drawing values from the empirical distribution. This author extended the demand model of BLP by including in addition to income other demographic characteristics of consumers like age, family size, race and education. He included these variables in \( \beta_i \) obtaining this way demand that is a function of consumers’ demographic characteristics.
Each variable in equation (4.2) is assumed to be known to consumer \( i \), while the researcher observes only the vector of prices, the characteristics of all cars \( x_j, j = 1, \ldots, J \), and the income distribution of consumers. Hence the utilities are not observed by the researcher either. Nevertheless, the market shares of products are observed, which are equal, up to a small sampling error, to the probabilities that the respective products are purchased by the population. For expressing this probability corresponding to product \( j \) we need to define first the utility of the outside alternative, that is, the utility from not buying any of the \( J \) products.

We define the utility of consumer \( i \) who does not purchase any product by

\[
u_{i0} = \alpha \ln(y_i) + \beta_0 + \sigma_0 v_{i0} + \xi_0 + \varepsilon_{i0}, \tag{4.3}\]

where the term \( \beta_0 + \sigma_0 v_{i0} \), with \( v_{i0} \) standard normal, accounts for the mean and variance of consumers’ utility if they choose the outside alternative, \( \xi_0 \) is the unobserved characteristic of the outside alternative and \( \varepsilon_{i0} \) is a type I extreme value random variable. In practice BLP include a constant term in the utility by putting \( x_{j1} = 1 \), for each \( j = 1, \ldots, J \), and due to identification purposes, they normalize the expression on the right hand side to

\[
u_{i0} = \varepsilon_{i0}. \tag{4.4}\]

We discuss below in a subsection whether this is a normalization that occurs naturally.

Having all these, we can now express the purchase probabilities as a function of the variables and parameters from the utilities. For this we use formula (2.12) derived for the mixed logit (section 2.3). Denoting the probability that product \( j \) is chosen by \( s_j \) we obtain

\[
s_j = \int_{\mathbb{R}^K+1} \frac{\exp \left[ \alpha \ln(y_i - p_j) + x'_{j} \left( \beta + \Sigma^{-1} \xi_0 \right) + \xi_j \right]}{1 + \sum_{r=1}^{J} \exp \left[ \alpha \ln(y_i - p_r) + x'_{r} \left( \beta + \Sigma^{-1} \xi_0 \right) + \xi_r \right]} \phi(v) \, dv, \tag{4.5}\]

where \( v = (v_{yi}, v_{j}')' \). This expression does not have a closed form, a fact that makes it difficult to handle in computations.

A well-known case in which we can compute the integral from (4.5) is when we ignore the unobserved consumer characteristics. This is the same as setting all the variance parameters equal to zero. If in (4.5) we put \( \sigma_y = \sigma_1 = \)
... = \sigma_K = 0 \) we obtain a version of the standard logit probability (2.5):

\[ s_j = \frac{\exp \left[ \alpha \ln(y - p_j) + x'_j \beta + \xi_j \right]}{1 + \sum_{r=1}^n \exp \left[ \alpha \ln(y - p_r) + x'_r \beta + \xi_r \right]}, \quad (4.6) \]

where \( y \) denotes the income common to all consumers.

Another version of the market shares is obtained if we assume that the utility depends linearly on the income-price difference:

\[ u_{ij} = \alpha(y_i - p_j) + x'_j \beta_i + \xi_j + \varepsilon_{ij} \quad \text{and} \quad u_{i0} = \alpha y_i + \varepsilon_{i0}. \quad (4.7) \]

We note that the utility specification (4.7) is a special case of the utility used by Caplin and Nalebuff (1991) with respect to the income-price difference (implying existence of price equilibrium in the one-product firms case), while the previous specification, (4.2), is not since the \( \ln(\cdot) \) function is not defined in 0. These utility specifications yield the probability formula:

\[ s_j = \int_{\mathbb{R}^k} \frac{\exp \left[ -\alpha p_j + x'_j \beta_i + \xi_j \right]}{1 + \sum_{r=1}^n \exp \left[ -\alpha p_r + x'_r \beta_i + \xi_r \right]} \phi(u_i) \, dv_i, \quad (4.8) \]

since the income terms cancel out. If we ignore the unobserved consumer characteristics we obtain the probability formula:

\[ s_j = \frac{\exp \left[ -\alpha p_j + x'_j \beta + \xi_j \right]}{1 + \sum_{r=1}^n \exp \left[ -\alpha p_r + x'_r \beta + \xi_r \right]}, \quad (4.9) \]

This formula is sufficiently simple to be used for illustrating various issues.

We conclude this section with a few remarks. An appealing feature of the BLP utility specification (4.2) is that it is a Cobb-Douglas utility function in the income-price difference. This can be seen by taking its exponential and pooling the rest of the variables in one function:

\[ \exp(u_{ij}) = (y_i - p_j)^\alpha G(x_j, \xi_j, v_i) e^{\varepsilon_{ij}}. \]

Unfortunately, the elegance of the Cobb-Douglas feature is shadowed by a number of difficulties implied. First, in some markets, like that for automobiles, the price of the most expensive product may be higher than the lowest incomes. This yields a negative income-price difference at which the log function is not defined. Second, in the market for automobiles consumers do not
pay the whole price of the product in the period of time in which the choices are made. That is, often the time period considered by researchers is one year but it is rare that consumers pay the whole price of a car in the year they purchase it. This problem may be solved by putting a parameter on price. Then the income-price difference expression would become \((y_i - \zeta p_j)\). The true value of \(\zeta\) is expected to be less than one. Hence this alleviates the first problem also but there is no guarantee that it solves it completely.

### 4.2.1 Remarks on normalization

Here we clarify the details regarding the normalization of the utility corresponding to the outside alternative. This problem was mentioned in the previous section and arose specifically from the normalizing assumptions that lead from (4.3) to (4.4). We show that the normalizing assumptions used by BLP are not natural, and we show that the utility for the outside alternative

\[ u_{i0} = \alpha \ln(y_i) + \varepsilon_{i0}, \]

(4.10)
due to identification reasons, is the one that follows naturally from (4.3).

In order to present our argument, first we compute the purchase probability based on (4.2) and (4.3):

\[
s_j = \int_{\mathbb{R}^{K+2}} \frac{\exp\left[\alpha \ln \frac{y_i - p_j}{y_i} + x_j \left(\beta + \Sigma \varphi \psi_i - \sigma_0 \xi_{i0} \xi_{i0} + \xi_{i0}\right)\right]}{1 + \sum_{r=1}^{K} \exp\left[\alpha \ln \frac{y_i - p_r}{y_i} + x_r \left(\beta + \Sigma \varphi \psi_i - \sigma_0 \xi_{i0} \xi_{i0} + \xi_{i0}\right)\right]} \phi(\nu) \, d\nu.
\]

If, as in BLP, the \(x\)'s are specified such that \(x_{j1} = 1\), for each \(j = 1, ..., J\), then \(\beta_0, \xi_0\) and \(\sigma_0\) cannot be identified since \(\beta_1 - \beta_0 - \xi_0\) and \(\sqrt{\sigma_1^2 + \sigma_0^2}\) are identified only, where \(\beta_1\) and \(\sigma_1\) correspond to \(x_{j1}\). Hence \(\beta_0 = \sigma_0 = \xi_0 = 0\) proves to be a natural normalization.

There is no apparent reason, possibly apart from simplicity, however, to omit the income variable \(y_i\) from (4.3). If we do not omit it, the income-price term in the purchase probability is \(\alpha \ln \frac{y_i - p_j}{y_i}\) as opposed to \(\alpha \ln (y_i - p_j)\) from (4.5). This very likely yields a different estimate for \(\alpha\). From the above reasoning we can conclude that the natural normalization corresponds to (4.10).

### 4.3 Supply

BLP assume that the \(J\) products are produced by a number of \(F\) firms in the market and each firm \(f \in \{1, ..., F\}\) sells a subset \(G_f\) of the \(J\) products.
Let $q_j$ denote the number sold of product $j$, $mc_j$ denote the marginal cost of producing product $j$. Then, following BLP, the profit of firm $f$ is

$$\Pi_f = \sum_{j \in G_f} (p_j - mc_j)q_j. \tag{4.11}$$

A more precise definition of the profit would be that by Nevo (2001):

$$\Pi_f = \sum_{j \in G_f} (p_j - mc_j)q_j - cf,$$

where $cf$ is the fixed cost of firm $f$. There are at least two reasons why the fixed cost term is usually ignored. First, it does not depend on the price so it does not affect profit maximization with respect to price. Second, ignoring it is a convenient simplification because in this case pricing above marginal costs yields non-negative profits.

If we denote by $N$ the number of all consumers in the market then $q_j$ is approximately equal to $Ns_j(p)$, where we use the price vector $p$ as argument in the notation of $s_j$ from (4.5). Then the profit of firm $f$ becomes

$$\Pi_f(p) = N \sum_{j \in G_f} (p_j - mc_j)s_j(p),$$

where for convenience we use $p$ as argument of the profit.

For estimation purposes BLP assume a parametric dependence of the marginal cost of producing $j$ on observed product characteristics affecting cost, say $w_j$, and an unobserved cost characteristic $\omega_j$:

$$\ln(mc_j) = w_j'\gamma + \omega_j. \tag{4.12}$$

The unobserved demand and cost characteristics, $\xi_j$ and $\omega_j$, are expected to have a positive correlation. For example, in the case of cars if power steering is a characteristic not observed by the researcher, then cars having this characteristic will have higher demand unobserved characteristic and are expected to have also higher unobserved cost characteristic.

In some cases data on marginal costs are available as in Nevo (2001). In these cases it is possible to avoid the potential restriction caused by the functional form assumed for the marginal cost. Notice that the marginal cost specification from (4.12) assumes constant returns to scale. Allowing for increasing
or decreasing returns to scale is hampered in BLP by the fact that total production data in the US automobile market is not available (both local and foreign producers sell a part of their products outside the US).

The common assumption on pricing, adopted also by BLP, is that firms maximize their profits by setting the prices of their products given everything else that enters utility and costs. This assumes implicitly that firms observe all product characteristics and consumer characteristics. This framework defines a one-stage normal form game in which the players are the firms and the strategies are own prices.

If we assume that a Nash equilibrium exists for this game then it must satisfy the following first order conditions for any \( j \in G_f, f = 1, \ldots, F \):

\[
\frac{\partial \Pi_f}{\partial p_j}(p^*) = 0,
\]

which is equivalent to

\[
s_j(p^*) + \sum_{r \in G_f} (p^*_r - m_c) \frac{\partial s_r}{\partial p_j}(p^*) = 0. \tag{4.13}
\]

In matrix form this is the same as saying that \( p^* \) solves the (nonlinear) system of equations

\[
p - m_c = \Delta(p)^{-1} s(p) \tag{4.14}
\]

where the element of \( \Delta(p) \) in row \( j \) and column \( r \) is denoted by \( \Delta_{jr} \) and

\[
\Delta_{jr} = \begin{cases} 
-\frac{\partial s_r}{\partial p_j}, & \text{if } r \text{ and } j \text{ are produced by the same firm;} \\
0, & \text{otherwise.}
\end{cases}
\]

If the observed prices correspond to a Nash equilibrium then they must satisfy (4.14). This fact can be used for estimating the parameters of the marginal cost.

We note that the matrix \( \Delta(p) \) is block diagonal and can be written as

\[
\begin{bmatrix}
\Delta_1(p) & 0 \\
& \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \Delta_F(p)
\end{bmatrix},
\]
where the diagonal blocks correspond to the firms and the block corresponding to firm \( f \), \( \Delta_f(p) \), has the elements
\[
\Delta_{f,jr} = -\frac{\partial s_r}{\partial p_j}, \text{ where } r \text{ and } j \text{ are produced by firm } f.
\]

Then the system of equations corresponding to \( f \) and analog to (4.14) can be written as
\[
\mathbf{p}_f - \mathbf{mc}_f = \Delta_f(p)^{-1} \mathbf{s}_f(p), \tag{4.15}
\]
where \( \mathbf{p}_f, \mathbf{mc}_f \) are the vectors of prices and marginal costs corresponding to the products of firm \( f \) and \( \mathbf{s}_f(p) \) is the vector of the probabilities that the products of firm \( f \) are purchased.

### 4.3.1 The standard logit case

Due to the non-closed form of the probabilities corresponding to the mixed logit, (4.5) and (4.8), it is not possible to compute \( \Delta(p)^{-1} \) in a closed form. In order to see better the nature of the relationship satisfied by the equilibrium price, we provide the formulae corresponding to (4.14) in the two standard logit cases (4.6) and (4.9).

1. **The log income-price difference case (4.6).** The first-order derivatives of the probabilities with respect to prices are
\[
\frac{\partial s_j}{\partial p_j} = -\frac{\alpha}{y - p_j} s_j (1 - s_j) \quad \text{and} \quad \frac{\partial s_j}{\partial p_r} = \frac{\alpha}{y - p_r} s_j s_r \quad \text{for } r \neq j.
\]
\[
\tag{4.16}
\]
Denote by \( \mathbf{d}_f \) the column vector with typical elements \( d_j = \frac{s_j}{y - p_j} \) for \( j \in G_f \) and the diagonal matrix formed with the elements of this vector by \( \mathbf{D}_f \). Then, if for simplicity we ignore the argument \( p \) from the notation, we have that
\[
\Delta_f = \alpha \left( \mathbf{D}_f - \mathbf{d}_f d'_f \right).
\]

The inverse of this is (see Lemma 12 on page 74)
\[
\Delta_f^{-1} = \frac{1}{\alpha} \left( \mathbf{D}_f^{-1} + \frac{\mathbf{d}_f d'_f \mathbf{D}_f^{-1}}{1 - \mathbf{d}_f d'_f} \right),
\]
where $\iota_f$ is the column vector of ones of size $|G_f|$. Substituting this in (4.14) we obtain

$$p_f - mc_f = \frac{1}{\alpha} \left( \frac{y - s_f'p_f}{1 - s_f'^t_i f} \cdot \iota_f - p_f \right),$$

which is equivalent to

$$p_f = \frac{\alpha}{\alpha + 1} mc_f + \frac{1}{\alpha + 1} \frac{y - s_f'p_f}{1 - s_f'^t_i f} \cdot \iota_f.$$

We note that the expression $s_f'^t_i f$ is the sum of probabilities that the products of firm $f$ are purchased and hence $s_f'^t_i f < 1$.

2. The linear income-price difference case (4.9). The first-order derivatives of the probabilities with respect to prices are

$$\frac{\partial s_j}{\partial p_j} = -\alpha s_j (1 - s_j) \quad \text{and} \quad \frac{\partial s_j}{\partial p_r} = \alpha s_j s_r \quad \text{for } r \neq j.$$

Then $\Delta_f$ is

$$\Delta_f = \alpha \left( S_f - s_f s_f' \right),$$

which has the inverse (by the same lemma as above)

$$\Delta_f^{-1} = \frac{1}{\alpha} \left( S_f^{-1} + \frac{\iota_f'^t f}{1 - s_f'^t_i f} \right),$$

where $S_f = \text{diag}(s_f)$ and hence

$$p_f - mc_f = \frac{1}{\alpha} \frac{\iota_f}{1 - s_f'^t_i f}. \quad (4.18)$$

Formulae (4.17) and (4.18) provide relationships between the equilibrium prices of the products and their characteristics, which are implicit in the purchase probabilities. Analog relationships exist also in the more general case (4.14) arising as a consequence of the pricing assumption. The correlation between prices and the characteristics of the other firms’ products revealed here is used by BLP for constructing the instruments in the estimation of the parameters, as shown in the next chapter.

Notice the effect on markups (i.e., price-cost differences) of the restrictive substitution patterns induced by the independence of irrelevant alternatives.
property (described in section 2.2 of chapter 2) of the standard logit. In (4.18) all products produced by the same firm have the same markup irrespective of their characteristics. This is in contrast with the intuition that the types of products for which competition is stronger in the market have lower markups than rare products. For example, we expect middle size cars to have lower markups than large luxury cars.

Formula (4.18) offers good intuition about market power when we look at different collusion scenarios. Two extreme cases are the perfect price collusion, when all firms jointly maximize their profits with respect to price, and the perfect competition when the profit from each product is maximized separately in price. In the former case the denominator of the second term on the right hand side is equal to the purchase probability of the outside alternative, which is the largest possible value that it can take. Therefore all the markups take their maximal values. This situation corresponds to the monopoly case, so it is not a surprise that we obtain this result. In the perfect competition case, since profits over single products are maximized, the right hand side of (4.18) will take its lowest value for each product. This implies the lowest possible markups for all products, which, again, is intuitive.

The interpretation of formula (4.17) obtained in the log income-price difference case is slightly different from that in the linear income-price difference case. The difference is that, unlike in (4.18), due to the non-linearity of the income-price difference, markups cannot be separated from marginal costs. More precisely, with \( \alpha \) positive, markups depend negatively on marginal costs. This can be seen from writing (4.17) as

\[
p_f - m_c_f = \frac{1}{\alpha + 1} \left( -mc_f + \frac{y - s_{f'}P_f}{1 - s_{f'}f} \cdot t_f \right). \tag{4.19}
\]

We note that the negative dependence of markups on marginal costs is similar to the case of a monopolist with homogeneous products. The interpretation of this formula from a market power point of view is similar to (4.18) but more difficult to see with multi-product firms. In the case of one-product firms, however, it is easy to see that the right hand side of (4.17) is larger the larger the market share of firm \( f \). This implies, similarly to the linear income-price difference case, high markup for the monopolist and low markups for competing firms if there are many firms in the market.

As already mentioned above, equation (4.14) (and similarly (4.17) and (4.18)) is in fact a system of nonlinear equations in \( p_1, ..., p_J \). Any solution of this system is a stationary point of the firms’ profit functions and hence the system represents necessary but not sufficient conditions for Nash equilibria.
So generally there is no guarantee that the solution of the system will be a Nash equilibrium. However, in the next section we will show that in the standard logit case with linear income-price difference the system has a unique solution that corresponds to a Nash equilibrium.

Finally we make two additional remarks on the issues presented in this section. First, in the model presented, firms maximize their profits with respect to the prices of their own products assuming the characteristics of the products fixed. In reality this is generally not true. It turns out, however, that the assumption of fixed characteristics is useful for estimating the parameters of the model and its validity is an empirical matter. We return to this issue in more detail in the next chapter when discussing the identifying assumptions of the model.

The second remark is on price collusion. The assumption that firms maximize their profits by setting the prices of their own products reflects that they compete with each other in price. Tacit price collusion, however, when some firms maximize their profits with respect to price jointly may also occur in reality. If prior information about tacit price collusion is available, this can be incorporated in the model easily by considering joint profit maximization of the colluding firms with respect to the prices of their products. We may test whether our prior information corresponds to reality. As an example we mention Nevo (2001), who tested the collusive behavior in the market of ready-to-eat cereals.

4.4 Price equilibrium

Existence and uniqueness of price equilibrium is important from both a theoretical and an empirical viewpoint. Among the theoretical studies dealing with this problem we mention Caplin and Nalebuff (1991), Anderson, de Palma and Thisse (1992) and Peitz (2000). All these studies assume that firms produce one product. The first study analyzes existence and uniqueness of price equilibrium for several models including random coefficient discrete choice models. These authors allow for fairly general distributions of the random coefficients. They establish existence of price equilibrium for, among others, mixed logit models and existence and uniqueness of price equilibrium for the standard logit, both for the linear income-price difference specification. Unfortunately their proofs are difficult or maybe even impossible to generalize for multi-product firms. Anderson, de Palma and Thisse (1992) provide a review of equilibrium results regarding the standard logit model. Peitz (2000) extends the results of Caplin and Nalebuff (1991) to cases that can be viewed as more
realistic, like utility maximization with a budget constraint or boundedly rational consumers. Anderson and de Palma (1992) show existence and uniqueness of price equilibrium in a model with nested logit demand with multi-product firms and symmetric products. This latter feature refers to the fact that the observed demand characteristics of all products are the same. This turns out to be a crucial assumption in proving existence and uniqueness of price equilibrium in this model because it allows the first order conditions to be reduced to an equation of a single variable. Hence the proof employed here cannot be generalized to a model with asymmetric products.

In empirical models that use the pricing assumption in the estimation procedure (i.e., a relationship of type (4.14)) existence of price equilibrium is important for the consistency of the parameter estimator. Such studies are, among others, Feenstra and Levinsohn (1995), BLP, Nevo (2000) and Nevo (2001). In all these papers, in line with reality, multi-product firms are considered. A way to deal with multi-product firms is shown by Milgrom and Roberts (1990), who study supermodular and log-supermodular games. We show below, however, that the pricing games we deal with do not satisfy these properties in general. Specifically, we prove that in the case of multi-product firms with standard logit demand (4.9) the pricing game is neither supermodular nor log-supermodular.

In empirical studies the difficulty caused by the possible non-existence of price equilibrium can be alleviated by verifying empirically ex post whether the estimates are consistent with a price equilibrium. This fact obviously does not play down the importance of price equilibrium existence. Moreover, uniqueness of price equilibrium is likely to facilitate the efficient estimation of the parameters, as we show in the next chapter.

In this section we show the existence of price equilibrium in the case of multi-product firms with standard logit demand and uniqueness of this equilibrium for the linear income-price difference case. This result is of limited use empirically due to the non-attractive substitution patterns of the standard logit, but more interesting theoretically since, to our knowledge, it is the first result for multi-product firms and asymmetric products. Below in the first subsection we present some useful lemmas. Then in the following subsection we provide proofs for the two standard logit cases. After that we give some negative results: the non-supermodularity mentioned above and an example when there is no price equilibrium. Finally we make some remarks regarding the price equilibrium problem for the multi-product mixed logit model.
4.4.1 Useful lemmas

First we prove a lemma that provides the final result of existence and uniqueness. Assume a game with a finite number of players denoted \( i = 1, \ldots, n \) whose strategies are multi-dimensional real convex compact sets \( S_i \). Let \( u_i : S \rightarrow \mathbb{R} \) denote their continuously differentiable profit functions, where \( S = S_1 \times \ldots \times S_n \). We use the common notation that \( v_{-j} \) is the vector \( v \) without its \( j \)'th component, \( v_{-j} \) is the part of vector \( v \) without the components corresponding to the vector \( v_j \), and the analog notation for intervals. By the notation of multi-dimensional intervals that are open on one side, i.e., \([a, b) \) we mean that all one-dimensional components of the Descartes product are open on the same side, i.e., \([a_1, b_1) \times \ldots \times [a_n, b_n) \).

**Lemma 9** Assume that the game satisfies the conditions:

1. There is an \( s^* \) for which \( \frac{\partial u_i (s^*)}{\partial s_i} = 0 \) for any \( i \in \{1, \ldots, n\} \).

2. For any \( i \in \{1, \ldots, n\} \) there is exactly one \( s_i \) for which \( \frac{\partial u_i (s_i, s_{-i}^*)}{\partial s_i} = 0 \).

3. For any \( i \in \{1, \ldots, n\} \), \( u_i (s_i, s_{-i}^*) \) has an interior global maximum with respect to \( s_i \in S_i \).

Then \( s^* \) is a Nash equilibrium of the game. If, in addition, \( s^* \) from 1 is unique then it is the unique Nash equilibrium of the game.

**Proof.** In order to show that \( s^* \) is a Nash equilibrium we need to prove that \( u_i (s_i^*, s_{-i}^*) \geq u_i (s_i, s_{-i}^*) \) for any \( s_i \in S_i \). By condition 3 there is an \( \pi_i \in S_i \) that is an interior global maximum point of \( u_i (\cdot, s_{-i}^*) \). This satisfies \( \frac{\partial u_i (\pi_i, s_{-i}^*)}{\partial s_i} = 0 \). By condition 1 we know that \( \frac{\partial u_i (s_i^*, s_{-i}^*)}{\partial s_i} = 0 \) is also true. Then by 2 \( \pi_i = s_i^* \) and hence \( s^* \) satisfies \( u_i (s_i^*, s_{-i}^*) \geq u_i (s_i, s_{-i}^*) \). Hence it is a Nash equilibrium. Since any Nash equilibrium of the game necessarily satisfies condition 1, the uniqueness follows.

We note that in the case of our pricing game, conditions 1 and 2 from the above lemma are equivalent to the fact that equation (4.14) has a solution and, given this solution, equation (4.15) has a unique solution. We observe that any solution of (4.14) happens to be a fixed point of the function

\[
g(p) = mc + \Delta(p)^{-1} s(p) .
\]
Similarly, any solution of \((4.15)\), when \(p_{-f}\) is given, is the fixed point of the function

\[
g_f(p_f|p_{-f}) = mcf + \Delta_f(p_f, p_{-f})^{-1} s_f(p_f, p_{-f}).
\]  

(4.21)

In order to demonstrate that conditions 1 and 2 of this lemma are satisfied we use the fixed point uniqueness result by Kellogg (1976).

**Lemma 10** (Kellogg (1976)) Let \(F : D \rightarrow D\) be a continuously differentiable function on a convex compact set \(D \subset \mathbb{R}^n\). If \(\det \left( \frac{\partial F(x)}{\partial x} - I_n \right) \neq 0\) for any \(x \in D\), and \(F\) has no fixed points on the boundary of \(D\) then \(F\) has a unique fixed point.

The proofs of our main results for the standard logit can be summarized in the following way. First we show the existence of \(J\)-dimensional compact intervals that \(g\) transforms into themselves. Then we verify that \(g\) satisfies the conditions of Lemma 10. This way we show that conditions 1 and 2 of this lemma are satisfied. For showing that condition 3 holds we use the next result.

**Lemma 11** Assume that the profit function of firm \(f\), \(\Pi_f\), is defined on the interval \([mc, H]\), and there exists a vector \(\mathbf{p}_{f} \in (\mathbf{mcf}, \mathbf{H}_f)\) such that for any \(j \in G_f\), any \(p_j \in [\mathbf{p}_j, H_j]\) and any \(p_{-j} \in [mc, H]_{-j}\) we have

\[
\frac{\partial \Pi_f}{\partial p_j} (p_{j}, p_{-j}) < 0 \quad \text{and} \quad \frac{\partial \Pi_f}{\partial p_j} (mc_j, p_{-j}) > 0.
\]

(4.22)

Then for any \(p_f \notin (\mathbf{mcf}, \mathbf{p}_f)\) and for any \(p_{-f} \in [mc, H]_{-f}\) there is a \(\tilde{p}_f \in (\mathbf{mcf}, \mathbf{p}_f)\) such that

\[
\Pi_f(\tilde{p}_f, p_{-f}) > \Pi_f(p_f, p_{-f}).
\]

(4.23)

**Proof.** Take an arbitrary \(p_f \notin (\mathbf{mcf}, \mathbf{p}_f)\). For any \(j \in G_f\) define \(\tilde{p}_j\) by its components

\[
\tilde{p}_j = \begin{cases} 
\mathbf{p}_j, & \text{if } p_j > \mathbf{p}_j \\
mc_j + \varepsilon_j, & \text{if } p_j = mc_j \\
p_j, & \text{otherwise},
\end{cases}
\]

where \(\varepsilon_j > 0\) will be specified below. Denote the products of firm \(f\) by \(f_1, f_2, ..., f_L\). Then we can show that

\[
\Pi_f(\tilde{p}_{f1}, ..., \tilde{p}_{fL}, p_{-f}) \geq \Pi_f(p_{f1}, \tilde{p}_{f2}, ..., \tilde{p}_{fL}, p_{-f})
\]

\[
\geq \Pi_f(p_{f1}, p_{f2}, \tilde{p}_{f3}, ..., \tilde{p}_{fL}, p_{-f})
\]

\[
\geq \Pi_f(p_{f1}, ..., p_{fL}, p_{-f})
\]

(4.24)
step by step using (4.22). For example, we can show that the first inequality holds by treating the different cases for \( p_{f1} \) separately. If \( p_{f1} \in (mc_{f1}, \bar{p}_{f1}] \) then \( \tilde{p}_{f1} = p_{f1} \) and there is nothing to prove. If \( p_{f1} > \bar{p}_{f1} \) then by the first inequality from (4.22) we have that \( \Pi_f (\cdot, \tilde{\bar{p}}_{-f1}, \bar{p}_{-f}) \) is strictly decreasing and hence \( \Pi_f (\tilde{p}_{f1}, \tilde{\bar{p}}_{-f1}, \bar{p}_{-f}) = \Pi_f (\bar{p}_{f1}, \tilde{\bar{p}}_{-f1}, \bar{p}_{-f}) > \Pi_f (\bar{p}_{f1}, \bar{p}_{-f1}, \bar{p}_{-f}) \). If \( p_{f1} = mc_{f1} \) then by the second inequality from (4.22) \( \Pi_f (\cdot, \bar{p}_{-f1}, \bar{p}_{-f}) \) is strictly increasing and therefore there is a small \( \varepsilon_{f1} > 0 \) for which \( \tilde{p}_{f1} = mc_{f1} + \varepsilon_{f1} \) satisfies

\[
\Pi_f (\tilde{p}_{f1}, \tilde{p}_{-f1}, \bar{p}_{-f}) > \Pi_f (mc_{f1}, \bar{p}_{-f1}, \bar{p}_{-f}) = \Pi_f (p_{f1}, \bar{p}_{-f1}, \bar{p}_{-f}).
\]

For showing the other steps of inequality (4.24) we proceed similarly. The strict inequality from (4.23) is implied by the fact that if \( p_f \notin (mc_f, \bar{p}_f] \) then at least one component of \( p_f \), say \( j \), satisfies that \( p_j \notin (mc_j, \bar{p}_j] \). □

An implication of this lemma is that a profit function that satisfies these conditions and whose definition domain is reduced from \([mc, H]\) to a compact interval will have interior global maximum. The lemma, translated in words, shows that if such a profit function is decreasing in the prices of the firm beyond a certain bound and is increasing at the marginal cost values, then the global maximum of the profit function is kept between some bounds. These bounds prevent the profit function from having a global maximum on the boundary of its definition domain.

Finally, we state a result that we use for verifying non-singularity of some matrices.

**Lemma 12** Let \( M \) be an \( n \times n \) invertible matrix, \( \lambda \) a scalar and \( u \) and \( v \) column vectors of size \( n \). Then

\[
\det (M - \lambda uv') = (1 - \lambda v'M^{-1}u) \det M,
\]

and hence the matrix \( M - \lambda uv' \) is invertible if and only if \( 1 - \lambda v'M^{-1}u \neq 0 \). If this last non-equality holds, then

\[
(M - \lambda uv')^{-1} = M^{-1} + \frac{\lambda}{1 - \lambda v'M^{-1}u} v'M^{-1}.
\]

For a proof we refer to Dhrymes (1984, p. 40).

For showing that the conditions of Lemma 10 and 11 are satisfied in the two standard logit (linear and log income-price difference) cases, we need to treat these separately. In the log income-price case we need to make some additional assumptions.
4.4.2 The standard logit case

We apply the above lemmas in the two standard logit cases: when the income price difference is in log and when it is linear.

1. The log income-price difference case (4.6)

We show that the pricing game has a Nash equilibrium in this case. First we make an assumption A1 and then we show that the conditions of Lemma 10 are satisfied. We introduce the notation $B_j \equiv \frac{\alpha}{\alpha + 1} mc_j + \frac{1}{\alpha + 1} \frac{y}{s_0(mc)}$ for $j = 1, ..., J$, where $s_0(mc)$ is the probability of the outside alternative computed at prices equal to $mc$.

(A1) $B_j < y$ for any $j \in \{1, ..., J\}$.

We observe that an implication of A1 is that the income is greater than the marginal cost of any product, since for any $j \in \{1, ..., J\}$

$$B_j = \frac{\alpha}{\alpha + 1} mc_j + \frac{1}{\alpha + 1} \frac{y}{s_0(mc)} > \frac{\alpha}{\alpha + 1} mc_j + \frac{1}{\alpha + 1} y,$$

which implies that $y > mc_j$ for any $j \in \{1, ..., J\}$. Throughout we assume that $\alpha > 0$, which is intuitive since it implies that the purchase probabilities of a certain product are decreasing in own prices and increasing in the prices of rival products (see (4.16)).

It is impossible to assess theoretically how realistic A1 is since this depends on the specific values of the parameters, $mc$ and $y$. For example, if $\alpha$ is close to zero, A1 is not likely to be satisfied since $s_0(mc) < 1$. Empirically, however, it is possible to verify this assumption.

**Proposition 13** If A1 holds then for any $p \in [mc, y_{i,j}]$ we have $g_j(p) \in (mc, B)$, where $i,j$ is the $J$-vector of ones.

**Proof.** We need to show that $mc_j < g_j(p) < B_j$ for any $j \in \{1, ..., J\}$. From (4.17) we have that (ignoring the argument $p$ from the notation)

$$g_j(p) = \frac{\alpha}{\alpha + 1} mc_j + \frac{1}{\alpha + 1} \frac{y - s_f'p_f}{1 - s_f'\mu_f}$$

for $j \in G_f$.

Since $p_j < y$ for any $j \in G_f$, we have $s_f'p_f < y s_f'\mu_f$ and therefore

$$g_j(p) > \frac{\alpha}{\alpha + 1} mc_j + \frac{1}{\alpha + 1} y > mc_j.$$
This proves the first part of the inequality. For showing the second part we observe that
\[ g_j(p) < \frac{1}{\alpha + 1} mc_j + \frac{1}{\alpha + 1} \frac{y}{1 - s_f'} \leq \frac{1}{\alpha + 1} mc_j + \frac{1}{\alpha + 1} \frac{y}{s_0(p)} = B_j, \]
since \( s_0(\cdot) \) is increasing in all price components. This completes the proof. ■

Next we show that the other condition of Lemma 10 is also satisfied. Let \( g_f \) denote the vector function corresponding to firm \( f \). Then
\[
\frac{\partial g}{\partial p} = \begin{bmatrix}
\frac{\partial g_1}{\partial p_1} & \cdots & \frac{\partial g_1}{\partial p_f} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_F}{\partial p_1} & \cdots & \frac{\partial g_F}{\partial p_F}
\end{bmatrix}.
\]
In order to compute this matrix we need the formulae of the partial derivatives
\[
\frac{\partial g_j}{\partial p_h} = -\frac{s_h}{1 - s_f'} \quad \text{for } j, h \in G_f, \quad \text{and}
\frac{\partial g_j}{\partial p_k} = \frac{\alpha}{\alpha + 1} \frac{s_k}{y - p_k} \left(1 - s_f'\right) \quad \text{for } j \in G_f, \quad k \notin G_f.
\]
These imply
\[
\frac{\partial g_f}{\partial p_f'} = -\frac{1}{1 - s_f'} s_{f'}.
\]
We can verify now whether the matrix \( \frac{\partial g_f(p)}{\partial p_f'} - I_f \) is non-singular. The condition implied by Lemma 12 is that
\[ 1 - \frac{1}{1 - s_f'} s_{f'} = \frac{1}{1 - s_f'} \neq 0, \]
where \( I_f \equiv I_{|G_f|} \) and \( |G_f| \) is the number of elements of the set \( G_f \). This condition is certainly satisfied. Thus we can state the following.

**Proposition 14** \( \det \left( \frac{\partial g_f(p)}{\partial p_f'} - I_f \right) \neq 0 \) for any firm \( f \) and \( p \in [mc, y_{-f}) \).
For expressing the whole partial derivatives matrix we adopt the following notation:

\[ \lambda_f = \frac{\alpha}{\alpha + 1} \left( \frac{s_f'(y - p_f)}{1 - s_f'p_f} \right), \quad \mu_f = \left( \frac{s_j}{y - p_j} \right)_{j \in G_f} \quad \text{and} \]

\[ \rho_f = \left( \frac{s_j}{1 - s_f'p_f} + \lambda_f s_j \right)_{j \in G_f}. \]

Then

\[
\frac{\partial g}{\partial y^f} - I_J = \begin{bmatrix}
\lambda_1 \nu_1 \\
\vdots \\
\lambda_{F \times F} \\
\mu_F
\end{bmatrix}
\begin{bmatrix}
\mu_1' \\
\vdots \\
\vdots \\
\mu_F'
\end{bmatrix}'
- \begin{bmatrix}
\nu_1 \rho_1' + I_1 \\
\vdots \\
\vdots \\
\nu_F \rho_F' + I_F
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \nu_1 \\
\vdots \\
\lambda_{F \times F}
\end{bmatrix}.
\]

where the second matrix on the right hand side is block-diagonal. We note that this second matrix is invertible since its each diagonal block is invertible. This latter statement can be seen by applying Lemma 12 and using the fact that \( 1 + \rho_f'p_f > 0 \). For invertability of \( \frac{\partial g}{\partial y^f} - I_J \), applying again Lemma 12 we need that

\[
1 - \begin{bmatrix}
\mu_1' \\
\vdots \\
\mu_F'
\end{bmatrix}' \begin{bmatrix}
\nu_1 \rho_1' + I_1 \\
\vdots \\
\vdots \\
\nu_F \rho_F' + I_F
\end{bmatrix}^{-1} \begin{bmatrix}
\lambda_1 \nu_1 \\
\vdots \\
\lambda_{F \times F}
\end{bmatrix} \neq 0.
\]

This is equivalent to

\[
\sum_{f=1}^{F} \mu_f' (\nu_f \rho_f' + I_f)^{-1} \lambda_{f \times f} \neq 1.
\]

If we use \( (\nu_f \rho_f' + I_f)^{-1} = I_f - \frac{1}{1 + \rho_f'p_f} \nu_f \rho_f' \) we obtain

\[
\sum_{f=1}^{F} \mu_f' (\nu_f \rho_f' + I_f)^{-1} \lambda_{f \times f} = \sum_{f=1}^{F} \frac{\lambda_f (\mu_f' \rho_f)}{1 + \rho_f' p_f}.
\]
We notice that \(1 + \rho_f^{t,f} = \frac{1}{1 - s_f^{t,f}} + \lambda_f \left( \mu_f^{t,f} \right)\) and therefore

\[
\sum_{f=1}^{F} \mu_f^{t} (t_f \mu_f^{t} + I_f)^{-1} = \sum_{f=1}^{F} \frac{\lambda_f \left( \mu_f^{t,f} \right)}{1 - s_f^{t,f}} + \lambda_f \left( \mu_f^{t,f} \right).
\]

Unfortunately it is not clear whether the right hand side expression can be equal to 1. Hence we cannot draw the conclusion that \(G_{103}/G_{104}/G_{119}\) for any \(G_{115}\).

Next we show that the conditions of Lemma 11 are met. For this we need an additional assumption. We recall that \(N\) denotes the number of all consumers in the market.

\[(A2)\] For any \(f\) and any \(j \in G_f\) there exists a \(\overline{p}_j < y\) such that

\[
\alpha \left( \frac{\Pi_f (p_j, p_{-j})}{N} - (p_j - mc_j) \right) + y - p_j < 0
\]

for any \(p_j \in [\overline{p}_j, y]\) and any \(p_{-j} \in [mc, \min_{j \neq j} y_{-j}]\).

As we show formally in the next paragraph, assumption A2 means that as the price of product \(j\) approaches the income \(y\) the profit of the firm producing \(j\) decreases. A2 will be satisfied if

\[
\lim_{p_j \to y} \alpha \left( \sup_{p_{-j}} \frac{\Pi_f (p_j, p_{-j})}{N} - (p_j - mc_j) \right) + y - p_j < 0,
\]

which implies that

\[
\lim_{p_j \to y} \sup_{p_{-j}} \frac{\Pi_f (p_j, p_{-j})}{N} < (y - mc_j).
\]

From here we can see that, similarly to A1, assumption A2 is difficult to verify theoretically but possible to verify empirically.

This assumption assures that the conditions of Lemma 11 are satisfied. In order to see this, we express the first-order derivatives of firm \(f\)'s profit function as

\[
\frac{\partial \Pi_f}{\partial p_j} = \frac{s_j}{y - p_j} \left[ \alpha \left( \frac{\Pi_f}{N} - (p_j - mc_j) \right) + y - p_j \right].
\]
Thus assumption A2 indeed guarantees the existence of a $\overline{p}_j < y$ such that
$$\frac{\partial \Pi_f(p_j, p_{-j})}{\partial p_j} < 0$$
for any $p_j \in [\overline{p}_j, y)$ and any $p_{-j} \in [mc, y_{i,j})_{-j}$. For the other condition of Lemma 11 we observe that
$$\frac{\partial \Pi_f(m_{c,j}, p_{-j})}{\partial p_j} = \frac{s_j}{y - m_{c,j}} \left[ \frac{\Pi_f}{N} (m_{c,j}, p_{-j}) + y - mc_j \right],$$
which is obviously positive for any $p_{-j} \in [mc, y_{i,j})_{-j}$. We summarize the above findings in the following statement.

**Proposition 15** If assumption A2 holds then for any firm $f$ and any $j \in G_f$ there exists a $\overline{p}_j \in [mc_j, y)$ such that for any $p_j \in [\overline{p}_j, y)$ and any $p_{-j} \in [mc, y_{i,j})_{-j}$ we have
$$\frac{\partial \Pi_f(p_j, p_{-j})}{\partial p_j} < 0 \quad \text{and} \quad \frac{\partial \Pi_f(m_{c,j}, p_{-j})}{\partial p_j} > 0.$$  \hspace{1cm} (4.26)

Now we state the final result on the existence of price equilibrium.

**Theorem 16** In the standard logit model with log income-price difference (4.6) if A1 and A2 hold and $\alpha > 0$ then there exists a price equilibrium.

**Proof.** By A2, for any $j$ there exists a $\overline{p}_j < y$ with the property specified in (4.25). Let $C_j = \max \{ B_j, \overline{p}_j \}$. Then $mc_j < C_j < y$. By Proposition 15 we know that (4.26) is satisfied for any $p_j \in [C_j, y)$ and any $p_{-j} \in [mc, y_{i,j})_{-j}$. In this case Lemma 11 implies that for firm $f$, any $p_f \notin (mc_f, C_f)$ and any $p_{-f} \in [mc, y_{i,f})_{-f}$ there is a $\overline{p}_f \in (mc_f, C_f)$ such that $\Pi_f(\overline{p}_f, p_{-f}) > \Pi_f(p_f, p_{-f})$. Thus $\Pi_f(\cdot, p_{-f}) : [mc_f, C_f] \to \mathbb{R}$ has either interior global maximum or a global maximum $p^*_f$ on the boundary such that for some $j \in G_f$ $p^*_f = C_j$. If we increase all such $C_j$'s by a small number and let these be the right hand side bounds of the intervals on which now $\Pi_f(\cdot, p_{-f})$ is defined, then $\Pi_f(\cdot, p_{-f})$ can only have interior global maximum. For simplicity we keep denoting the new right hand side bounds of the intervals by $C_j$. So we have established that $\Pi_f(\cdot, p_{-f}) : [mc_f, C_f] \to \mathbb{R}$ has interior global maximum. We define the profit function of each firm $f$ on $[mc, C] \equiv \Pi_f^*_{j=1} [mc_f, C_f]$, on which it is obviously continuously differentiable. Hence condition 3 of Lemma 9 is satisfied by this profit function.

Proposition 13 implies that $g(p) \in [mc, B] \subset [mc, C]$ for any $p \in [mc, C]$. Since $g : [mc, C] \to [mc, C]$, given in (4.20) is a continuous function
defined on a compact convex set, by Brouwer’s fixed point theorem there exists \( p^* \) such that \( g(p^*) = p^* \). This is equivalent to

\[
\frac{\partial \Pi_f}{\partial p_f}(p^*) = 0 \quad \text{for any firm } f.
\]

Thus condition 1 of Lemma 9 is satisfied.

For establishing condition 2 of this lemma we use Lemma 10. For a firm \( f \) we consider the function \( g_f(\cdot|p_{-f}^*): [mc_f, C_f] \to \mathbb{R}^{G_f} \) defined in (4.21), when \( p_{-f}^* \) is given. This function is continuously differentiable on the convex compact set \([mc_f, C_f]\), it cannot have any fixed point on the boundary of \([mc_f, C_f]\) due to Proposition 13, and by Proposition 14

\[
\det \left( \frac{\partial g_f}{\partial p_f}(p_f|p_{-f}^*) - I_f \right) \neq 0 \quad \text{for any } p_f \in [mc_f, C_f].
\]

Thus Lemma 10 implies that \( g_f(\cdot|p_{-f}^*) \) has a unique fixed point. Since \( p_f^* \) is a fixed point of this function, we have now established that it is its unique fixed point. This is equivalent to the fact that there is exactly one \( p_f \) with

\[
\frac{\partial \Pi_f}{\partial p_f}(p_f, p_{-f}^*) = 0,
\]

which is in fact condition 2 of Lemma 9.

Applying now Lemma 9 the proof is completed. ■

In conclusion, assumptions A1 and A2 are sufficient for proving the existence of price equilibrium. The way the whole proof is constructed suggests that they are not necessary. There are several reasons for this, which can be traced back from the proof. We mention just one of them, namely that a fixed point of the function \( g \) may exist even if the conditions of Brouwer’s fixed point theorem are not satisfied. Another important question regarding the assumptions is how realistic they are. This problem is not simple since both assumptions depend on the values of underlying variables like \( \alpha \) and \( mc \).

2. The linear income-price difference case (4.9)

In this case we show that the pricing game has a unique Nash equilibrium. This case is different from that discussed above since we do not need to make any assumptions. The only necessary assumption is the intuitive \( \alpha > 0 \). The proof of the final result, similarly to the previous case, relies on Lemma 9. Below we build up the proof by showing that the conditions of this lemma are satisfied.

We start by showing that the conditions of Lemma 10 are satisfied. The
function \( g \) from (4.18) has the components

\[
g_j(p) = mc_j + \frac{1}{\alpha} \frac{1}{1 - s'_{j \ell_f}} \quad \text{for any } f \text{ and } j \in G_f.
\]

We define

\[
B_j \equiv mc_j + \frac{1}{\alpha} \frac{1}{s_0(mc)} \quad \text{for any } j \in \{1, ..., J\}.
\]

**Proposition 17** For any \( p \in [mc_1, \infty) \times ... \times [mc_J, \infty) \) \( g_j(p) \) satisfies

\[
mc_j < g_j(p) < B_j.
\]

**Proof.** The first part of the inequality is clear. For the second part, since the probabilities of all alternatives sum to one, we write

\[
g_j(p) = mc_j + \frac{1}{\alpha} \frac{1}{s_0} + \sum_{r \notin G_f} s_r < mc_j + \frac{1}{\alpha} \frac{1}{s_0(mc)}.
\]


**Proposition 18** The function \( g \) is continuously differentiable on \( \mathbb{R}^d \) and for any \( p \in \mathbb{R}^d \)

1. \( \det \left( \frac{\partial g_f(p)}{\partial \mathbf{p}_f^t} - I_f \right) \neq 0 \) for any firm \( f \),

2. \( \det \left( \frac{\partial g(p)}{\partial \mathbf{p}^t} - I_J \right) \neq 0. \)

**Proof.** The partial derivatives of the components of \( g \) corresponding to a firm \( f \) have the expressions

\[
\frac{\partial g_j}{\partial p_h} = -\frac{s_h}{1 - s'_{j \ell_f}} \quad \text{for } j, h \in G_f, \quad \text{and}
\]

\[
\frac{\partial g_j}{\partial p_k} = \frac{s_k}{1 - s'_{j \ell_f}} \frac{s'_{j \ell_f}}{1 - s'_{j \ell_f}} \quad \text{for } j \in G_f, \quad k \notin G_f.
\]

These imply that the diagonal blocks of the derivatives matrix are

\[
\frac{\partial g_f}{\partial \mathbf{p}_f^t} = -\frac{1}{1 - s'_{j \ell_f}} f s'_{j \ell_f}, \quad f = 1, ..., F.
\]
Hence proving the first statement of this proposition requires the same verifications as in Proposition 14. Thus this statement has already been proved.

For the second statement we need the formulae of the off-diagonal blocks of the derivatives matrix as well, which are

\[
\frac{\partial g_f}{\partial p_q} = \frac{s_{f,t}^{q} s_{q}}{1 - s_{f,t}^{q}}, \quad f, q = 1, \ldots, F, \quad f \neq q.
\]

For simplifying the occurring expressions we introduce the notation:

\[
\rho_f = \frac{s_{f,t}^{q}}{1 - s_{f,t}^{q}} \quad \text{and} \quad \psi_f = \frac{1}{1 - s_{f,t}^{q}}, \quad f = 1, \ldots, F.
\]

We can write \( \frac{\partial g}{\partial p} - I_J \) in the form

\[
\frac{\partial g}{\partial p} - I_J = \begin{bmatrix} \rho_1 & s_1 \\ \vdots & \vdots \\ \rho_{F-t,F} & s_F \end{bmatrix}' \begin{bmatrix} \psi_1 s_1' + I_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \psi_{F-t,F}s_F' + I_F \end{bmatrix}
\]

where the matrix on the right hand side is block-diagonal. A diagonal block of this matrix \( \psi_{f,t} s_f' + I_f \) is invertible if \( 1 + \psi_f s_f' \neq 0 \) for any \( p \in \mathbb{R}^J \) (Lemma 12). This property is clearly satisfied. Then its inverse is

\[
(\psi_{f,t} s_f' + I_f)^{-1} = I_f - \frac{\psi_f}{1 + \psi_f s_f'} s_f'.
\]

From (4.27) conform Lemma 12 the matrix \( \frac{\partial g}{\partial p} - I_J \) is invertible if

\[
1 - \begin{bmatrix} s_1 \\ \vdots \\ s_F \end{bmatrix}' \begin{bmatrix} \psi_1 s_1' + I_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \psi_{F-t,F}s_F' + I_F \end{bmatrix}^{-1} \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_{F-t,F} \end{bmatrix} \neq 0
\]

for any \( p \in \mathbb{R}^J \). Using the inverse (4.28), after some simple calculus this condition becomes

\[
1 - \sum_{f=1}^{F} \frac{\rho_f}{1 + \rho_f s_f'} s_f' \neq 0 \quad \text{for any} \quad p \in \mathbb{R}^J.
\]
4.4. PRICE EQUILIBRIUM

Since \( \frac{\rho_f}{1 + \rho_f} < 1 \)

\[
\sum_{f=1}^{F} \frac{\rho_f}{1 + \rho_f} s_f t_f < \sum_{f=1}^{F} s_f t_f = 1 - s_0 < 1.
\]

This implies that

\[
1 - \sum_{f=1}^{F} \frac{\rho_f}{1 + \rho_f} s_f t_f > 0 \quad \text{for any } p \in \mathbb{R}^J
\]

and hence \( \frac{\partial g}{\partial \eta_j} - I_J \) is invertible, that is, \( \det \left( \frac{\partial g(p)}{\partial \eta_j} - I_J \right) \neq 0 \) for any \( p \in \mathbb{R}^J \).

In order to prove that condition 3 of Lemma 9 is satisfied first we show that the conditions of Lemma 11 are satisfied.

**Proposition 19** For any firm \( f \) and any \( j \in G_f \) there exists a \( \overline{p}_j > mc_j \) such that

\[
\frac{\partial \Pi_f}{\partial p_j} (p_j, p_{-j}) < 0 \text{ and } \frac{\partial \Pi_f}{\partial p_j} (mc_j, p_{-j}) > 0
\]

for any \( p_j \geq \overline{p}_j \) and \( p_{-j} \geq mc_{-j} \).

**Proof.** The derivative of \( \Pi_f \) with respect to \( p_j \) can be written as

\[
\frac{\partial \Pi_f(p)}{\partial p_j} = s_j \left( 1 - \alpha (p_j - mc_j) + \alpha \frac{\Pi_f(p)}{N} \right), \tag{4.29}
\]

where \( N \) is the number of all consumers in the market.

First we note that the profit \( \Pi_f(p) \) is bounded in \( p \). This can be seen from looking at the limit of \( \Pi_f(p) \) when for some \( j \)'s \( p_j \to \infty \). First consider the case when \( p_j \to \infty \) for a \( j \in G_f \). Then the part of \( \Pi_f(p) \) corresponding to \( j, s_j (p_j - mc_j) \) goes to zero irrespective of the other prices since \( \exp \left( -\alpha \rho_j + x_j^f \beta + \xi_j \right) (p_j - mc_j) \to 0 \). Here \( \alpha > 0 \) is used explicitly. The rest of the profit is either bounded or goes to zero depending on whether the corresponding prices go to infinity or stay bounded. Hence in this case the profit is bounded. In the second case consider \( p_j \to \infty \) for some \( j \notin G_f \) and \( p_j \) is bounded for all \( j \in G_f \). Then since each \( s_j \in [0, 1] \), the profit of firm \( f \) is bounded.
With the above considerations we have that for large \( p_j \) the right hand side of the inequality below is negative:

\[
\frac{\partial \Pi_f (p)}{\partial p_j} < 1 - \alpha (p_j - mc_j) + \alpha \sup \Pi_f (p) \frac{N}{N},
\]

This implies the existence of \( p_j \) with the announced property.

The second inequality from the statement of the proposition follows directly from (4.29).

The final result regarding the existence and uniqueness of price equilibrium is contained in the following statement.

**Theorem 20** In the standard logit model with linear income-price difference (4.9) if \( \alpha > 0 \) then there exists a unique price equilibrium in \( [mc_1, \infty) \times \ldots \times [mc_J, \infty) \).

**Proof.** The proof is based on Lemma 9. For all \( j \in \{1, \ldots, J\} \) we define \( C_j \) such that

\[
C_j = \max \{\bar{p}_j, B_j\}.
\]

We define the profit function of any firm \( f \) as \( \Pi_f : [mc, C] \to \mathbb{R} \). With the same reasoning as that used for Theorem 16, conform Proposition 19 we have that \( \Pi_f (\cdot, p_{-f}) : [mc_f, C_f] \to \mathbb{R} \) has interior global maximum. Then the profit functions are defined on a convex compact set, are continuously differentiable and satisfy condition 3 of Lemma 9.

Conditions 1 and 2 of this lemma are implied by the property showed in Proposition 17 that the function \( g \) has values in the set \( [mc, B] \subset [mc, C] \) and by Proposition 18 Lemma 10 can be applied. Hence there is a unique price equilibrium in the set \( [mc, C] \). Since this statement is true for any \( C \) with the property (4.30), we have that there is exactly one equilibrium in the set \( [mc_1, \infty) \times \ldots \times [mc_J, \infty) \).

This result is interesting since it shows the existence and uniqueness of price equilibrium using only the intuitive assumption that the purchase probabilities are decreasing in own price (\( \alpha > 0 \)). While this result is simple to obtain if the firms in the market are assumed to produce a single product (e.g., Milgrom and Roberts (1990), Caplin and Nalebuff (1991)), we are not aware of any previously published work that has established it in the multi-product firm case.
4.4.3 Some negative results

Here we provide two negative results. First we show that in the standard logit case with linear income-price difference the pricing game is neither supermodular nor log-supermodular. Then we provide an example in which the price equilibrium does not exist.

Supermodularity of games was introduced by Topkis (1979) and developed further by Milgrom and Roberts (1990), who apply them to, among others, showing the existence and uniqueness of equilibrium in pricing games. Specifically, they show that with one-product firms the pricing game corresponding to the standard logit with linear income-price difference has unique equilibrium. Supermodularity, however, as they show, guarantees the existence of equilibrium also in games where the strategy sets are multi-dimensional.

The definition of supermodularity from Milgrom and Roberts (1990) implies that the pricing game in the case of the standard logit is supermodular if the profit functions are twice continuously differentiable and for any \( p \)

\[
\frac{\partial^2 \Pi_f (p)}{\partial p_j \partial p_h} \geq 0 \text{ for any } f, j, h \in G_f, j \neq h, \text{ and}
\]

\[
\frac{\partial^2 \Pi_f (p)}{\partial p_j \partial p_k} \geq 0 \text{ for any } f, j \in G_f, k \notin G_f.
\]

The pricing game is log-supermodular if the logarithms of profit functions satisfy the above criteria. We show that the first inequality does not hold generally in the case of linear income-price difference. By Theorem 20 we know that there is a unique price equilibrium, \( p^\alpha \), and this solves the first-order conditions for profit maximization, that is, \( \frac{\partial \Pi_f (p^\alpha)}{\partial p_j} = 0 \) for any \( f \) and \( j \in G_f \).

Formula (4.29) implies that

\[
1 - \alpha (p^\alpha_j - mc_j) + \alpha \frac{\Pi_f (p^\alpha)}{N} = 0.
\]

(4.31)

The second order derivative of the profit function for \( j, h \in G_f, j \neq h \) is

\[
\frac{\partial^2 \Pi_f (p)}{\partial p_j \partial p_h} = \frac{\partial s_j}{\partial p_h} \left( 1 - \alpha (p_j - mc_j) + \alpha \frac{\Pi_f (p)}{N} \right) + \alpha s_j \frac{\partial \Pi_f (p)}{\partial p_h} N.
\]

Computed at the equilibrium price, this is zero:

\[
\frac{\partial^2 \Pi_f (p^\alpha)}{\partial p_j \partial p_h} = 0,
\]

(4.32)

conform (4.31) and \( \frac{\partial \Pi_f (p^\alpha)}{\partial p_h} = 0 \). We use this fact for showing the following.
Proposition 21  For arbitrary \( j, h \in G_f \) there exists a \( p \) such that

1. \( \frac{\partial^2 \Pi_f (p)}{\partial p_j \partial p_h} < 0 \) and
2. \( \frac{\partial^2 \ln \Pi_f (p)}{\partial p_j \partial p_h} < 0. \)

Proof. Take any \( \varepsilon > 0 \) and define \( p \) such that

\[ p_j = p_j^* + \varepsilon, \quad p_h = p_h^* + \varepsilon \quad \text{and} \quad p_r = p_r^* \quad \text{for all} \quad r \neq j, h. \]

Conform Theorem 20, \( p_j^* \) is a unique global maximum of \( \Pi_f \left( \cdot, p_j^* \right) \). Thus for any \( \varepsilon > 0 \) we have \( \Pi_f (p) < \Pi_f (p^*) \). We observe another way of writing the second order derivatives:

\[ \frac{\partial^2 \Pi_f (p)}{\partial p_j \partial p_h} = \alpha s_j s_h \left( 2 - \alpha (p_j - mc_j) - \alpha (p_h - mc_h) + 2\alpha \frac{\Pi_f (p)}{N} \right). \]

(4.33)

Then (4.32) implies that

\[ 2 - \alpha (p_j^* - mc_j) - \alpha (p_h^* - mc_h) + 2\alpha \frac{\Pi_f (p^*)}{N} = 0. \]

From the definition of \( p \) we have

\[ 2 - \alpha (p_j - mc_j) - \alpha (p_h - mc_h) + 2\alpha \frac{\Pi_f (p)}{N} < 2 - \alpha (p_j^* - mc_j) - \alpha (p_h^* - mc_h) + 2\alpha \frac{\Pi_f (p^*)}{N} = 0. \]

This together with (4.33) implies statement 1 of the proposition.

For showing statement 2 we write

\[ \frac{\partial^2 \ln \Pi_f (p)}{\partial p_j \partial p_h} = \frac{\partial^2 \Pi_f (p)}{\partial p_j \partial p_h} \frac{\Pi_f (p)}{\Pi_f (p)} - \frac{\partial \Pi_f (p)}{\partial p_j} \frac{\partial \Pi_f (p)}{\partial p_h}. \]

(4.34)

Since

\[ 1 - \alpha \left( p_j - mc_j \right) + \alpha \frac{\Pi_f (p)}{N} < 1 - \alpha \left( p_j^* - mc_j \right) + \alpha \frac{\Pi_f (p^*)}{N} = 0, \]
by (4.29) we obtain that $\frac{\partial \Pi_f (p)}{\partial p_j} < 0$ and similarly $\frac{\partial \Pi_f (p)}{\partial p_h} < 0$. Thus
\[
\frac{\partial \Pi_f (p)}{\partial p_j} \frac{\partial \Pi_f (p)}{\partial p_h} > 0.
\]
Therefore the right hand side of (4.34) is negative and the proof is complete.

This proposition implies that the pricing game in the standard logit case with linear income-price difference is neither supermodular nor log-supermodular. Our results are robust in the sense that we show that these properties are violated at prices arbitrarily close to the equilibrium price. Hence the results by Milgrom and Roberts (1990) on equilibrium existence and uniqueness of (log-) supermodular games cannot be applied to the standard logit with linear income-price difference and multi-product firms. In a similar way it is possible to show that the same conclusion holds for the standard logit with log income-price difference. We note that these results do not mean that supermodularity is not useful for the standard logit in the case of multi-product firms. This property may still be useful if an appropriate transformation of the profit function is employed. The above result, however, shows that applying supermodularity to the pricing game in the standard logit case is problematic and it may not lead to any results.

Now we discuss the second topic of this section, an example when there is no price equilibrium. For this we assume that the utility of consumer $i$ who purchases product $j$ is
\[
u_{ij} = \alpha (\ln y - \ln p_j) + x'_j \beta + \xi_j + \varepsilon_{ij},
\]
and for the outside alternative
\[
u_{i0} = \alpha \ln y + \varepsilon_{i0},
\]
where all variables are as in section 4.2 and, obviously, the utility is a special case of (4.1). This implies that the purchase probability corresponding to $j$ is equal to
\[
s_j = \frac{\exp \left[ -\alpha \ln p_j + x'_j \beta + \xi_j \right]}{1 + \sum_{r=1}^J \exp \left[ -\alpha \ln p_r + x'_r \beta + \xi_r \right]}.
\]
If there is a price equilibrium, this should satisfy the first order conditions for profit maximization of any firm $f$ and hence also the relationship (4.15), which
in this case has the form

\[ p_f - mc_f = \frac{1}{\alpha} \left( p_f + \frac{s'_f p_f}{1 - s'_f \ell_f} \ell_f \right). \]

We take now \( \alpha = 1 \) and assume that \( mc > 0 \), which is a realistic assumption. The above equality becomes

\[ mc_f + \frac{s'_f p_f}{1 - s'_f \ell_f} \ell_f = 0. \]

This is impossible, however, since the left hand side is strictly positive. Hence the first order conditions in this case do not have solution and, consequently, there is no price equilibrium.

### 4.4.4 Remarks on the mixed logit case

In section 4.4.2 we have shown that in the standard logit case there exists price equilibrium, and if the income-price difference in the utility is linear then the equilibrium is unique. The mathematical result that the proof is based on is Kellogg’s (1976) theorem (Lemma 10) on the uniqueness of fixed point. It would be of interest to know whether the same equilibrium results are true in the mixed logit case.

An approach to see this would be to apply the same method as in the standard logit case. We are not aware of any other ways to show existence and uniqueness of price equilibrium in the case of this problem. The methods presented by Milgrom and Roberts (1990) and Caplin and Nalebuff (1991) for one-product firms are close to impossible to extend to multi-product firms. The former failed in the standard logit case, as we have shown in the previous subsection, and hence we suspect it would not work in the mixed logit case either. The latter uses generalized concavity properties of probabilities to show that the profit functions are generalized concave. In the multi-product case this method amounts to establishing similar generalized concavity properties for a sum of functions (i.e., sum of profits corresponding to each product of a firm). Such result, to our knowledge, is not available in the mathematical literature.

The main difficulty of applying the method used in the standard logit case is, however, that the matrix \( \Delta \) from the markup equation (4.14) does not have a closed form and hence it is impossible to verify the conditions of Lemma 10 using standard mathematics, as we did in the standard logit case. In the remaining part of this section we present the problem that arises for proving the non-singularity of the first order derivatives matrix of \( g \) from (4.20) minus
the identity matrix, when the purchase probabilities take a general form. The columns of the matrix \( \frac{\partial g}{\partial p_j} \) can be computed like

\[
\frac{\partial g}{\partial p_j} = \Delta^{-1} \left( \frac{\partial s}{\partial p_j} - \frac{\partial \Delta}{\partial p_j} \Delta^{-1} s \right)
\]

so the matrix itself can be written as

\[
\frac{\partial g}{\partial p_j} = \Delta^{-1} \left( \frac{\partial s}{\partial p_j} - T \right),
\]

where \( T \) is the matrix formed by the column vectors

\[
t_j = \frac{\partial \Delta}{\partial p_j} \Delta^{-1} s \quad \text{for} \quad j = 1, \ldots, J.
\]

We obtain that

\[
\frac{\partial g}{\partial p_j} - I_J = \Delta^{-1} \left( \frac{\partial s}{\partial p_j} - T - \Delta \right),
\]

and hence

\[
\det \left( \frac{\partial g}{\partial p_j} - I_J \right) = \det \left( \Delta^{-1} \right) \det \left( \frac{\partial s}{\partial p_j} - T - \Delta \right).
\]

Consequently, we should prove that

\[
\det \left( \Delta \right) \neq 0 \quad \text{and} \quad \det \left( \frac{\partial s}{\partial p_j} - T - \Delta \right) \neq 0.
\]

(4.35)

The first non-equality is easier to show; for example, in the mixed logit case with linear income-price difference it is not difficult to show that \( \Delta \) is positive definite. The second non-equality is more difficult, and we have not yet found the appropriate way of verifying it.

The above relations are for verifying equilibrium uniqueness. For equilibrium existence the situation may be slightly less complicated, at least in the mixed logit case with linear income-price difference. Here \( \Delta_J = -\frac{\partial s_J}{\partial p_J} \) and hence

\[
\det \left( \frac{\partial s_J}{\partial p_J} - T_J - \Delta_J \right) = \det \left( 2 \frac{\partial s_J}{\partial p_J} - T_J \right).
\]
where $T_f$ is the matrix with columns

$$t_{f,j} = \frac{\partial \Delta f}{\partial \Delta f} \Delta f^{-1} s_f$$ for $j \in G_f$.

This determinant has a simpler form than that from (4.35) but still it is difficult to see whether it can be zero or not.

In conclusion, in the mixed logit case, even in the simplest version with linear income-price difference, it is difficult to prove with standard mathematical methods that the conditions of Lemma 10 are satisfied. There are, however, some intuitive reasons that make us believe that a price equilibrium generally exists in the mixed logit case. In the special case of linear income-price difference and one-product firms this was established by Caplin and Nalebuff (1991). Moreover, as we have shown in an example on substitution patterns in chapter 2, the mixed logit has a behavior similar to the standard logit if the variances of the random coefficients are low. This suggests that, for sufficiently low variances, the price equilibrium results established for the standard logit remain true for the mixed logit as well. Obviously, the above intuition may be wrong in the multi-product firm case with random coefficients having high variances.

### 4.5 Summary and conclusions

We conclude this chapter by summarizing what we have presented here. Within this we also provide an overview of the market equilibrium model in terms of exogeneity and endogeneity of the variables. Then we point to some topics for future research.

In this chapter we presented versions of a market equilibrium model that can be used through econometric techniques for estimating demand parameters and market conduct of participating firms. The models are based on the logit model of demand; we discuss two versions based on the standard logit and two based on the mixed logit. The two versions in both cases differ on the way the income-price difference is specified in the utility: in logarithm or linear. The appealing features of these models are that they allow for products that are differentiated in multiple dimensions and product characteristics that are unobserved by the researcher. The mixed logit versions, in addition, are able to model consumer heterogeneity through random coefficients. Hence these latter versions are more important due to this realistic feature. The treatment of product characteristics that are not observed by the researcher, included in both standard and mixed logit versions, is possible if we assume price competition
of participating firms in the market. This implies that prices are endogenous in
the model.

Here we provide an overview of the model presented in this chapter from
an exo- and endogeneity viewpoint. This will be relevant in the next chapter
when discussing optimal instruments for estimating the model. The exogenous
variables of the model are the observed and unobserved product characteristics
\( x_j, w_j, \xi_j, \omega_j, j = 1, \ldots, J \). From these the model determines the endogenous
variables, which are the purchase probabilities and prices. The purchase prob-
abilities are determined by the expressions (4.5), (4.8), (4.6), (4.9), in the four
different versions, as functions of the prices, and the prices are computed as a
Nash equilibrium of the pricing game.

Due to this, the question of existence and uniqueness of price equilibrium
becomes important. As we have shown above, this question has an affirmative
answer in the standard logit case with linear income-price difference. Adopting
some assumptions, we have shown equilibrium existence in the standard logit
case with log income-price difference. An issue that needs further attention
is investigating whether our assumptions are realistic. This is important since
this model is a simplified version of that used by Berry, Levinsohn and Pakes
(1995), and a sound analysis of the simple version would provide intuition on
the existence of price equilibrium in this model.

A major challenge remaining for future work is the question whether in the
mixed logit case a price equilibrium exists and whether this is unique. Very lit-
tle is known about this problem; the only result available in the literature is that
an equilibrium exists in the linear income-price version with one-product firms
(Caplin and Nalebuff (1991)). We believe that our proving strategy, developed
in this chapter for the standard logit, combined with more sophisticated math-
ematics is able to shed some more light on this issue.