# Weil pairing and the Drinfeld modular curve

van der Heiden, Gerrit



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# Chapter 6

# Drinfeld Data on the Compactified Modular Scheme

## 6.1 Introduction

Let $X$ be a projective, non-singular, absolutely irreducible curve over some finite field $\mathbb{F}_q$ and let $\infty \in X$ be some chosen closed point. The ring $A$ is defined to be the ring of all functions on $X$ which are regular outside $\infty$. Let $f \in A \backslash \mathbb{F}_q$ be any non-constant element. Let $A_f := A[f^{-1}]$ and let

$$M^2(f) \longrightarrow \mathrm{Spec}(A_f)$$

be the Drinfeld modular scheme such that $M^2(f)$ represents the functor

$$\mathcal{F} : A_f - \textsc{Schemes} \longrightarrow \textsc{Sets}$$

which associates to each $A_f$-scheme $S$ the set of isomorphy classes of Drinfeld modules $(L, \varphi)$ of rank 2 over $S$ with level $f$-structure $\lambda$. The scheme $M^2(f)$ is affine and we write

$$M^2(f) = \mathrm{Spec}(B).$$

We put $(\varphi, \lambda)$ for the universal Drinfeld module $\varphi$ over $M^2(f)$ together with the universal level $f$-structure $\lambda$. In this notation we forget about the line bundle because the universal line bundle is the trivial one. For a more extensive account of these definitions we refer to the previous chapter. Throughout this chapter we write $R$ for the ring associated to the scheme $M^1(f)$, i.e.,

$$\mathrm{Spec}(R) = M^1(f).$$

In this chapter we describe how to extend the pair $(\varphi, \lambda)$ to Drinfeld data $(X, \overline{\varphi}, \overline{\lambda})$ over the compactification $\overline{M}^2(f)$. In Section 6.2 we describe how we can extend the universal pair $(\varphi, \lambda)$ over $M^2(f)$ and the Tate-Drinfeld module defined over $\oplus R[\![x]\!]_{\mathfrak{p}}$ to a triple $(\mathrm{P}L, \overline{\varphi}, \overline{\lambda})$ over $\overline{M}^2(f)$. Here $\mathrm{P}L$ is the projectivisation of some line bundle $L \longrightarrow \overline{M}^2(f)$. The map

$$\overline{\varphi} : A \longrightarrow \mathrm{End}_{\mathbb{F}_q}(L)$$

is a ring homomorphism with the property that $\overline{\varphi}_a$ restricts to $\varphi_a : L_{|M^2(f)} \longrightarrow L_{|M^2(f)}$ for all $a \in A$. The map

$$\overline{\lambda} : (A/fA)^2 \cup \{\infty\} \longrightarrow \Gamma(\overline{M}^2(f), \mathrm{P}L)$$

is a map of sets whose restriction to $(A/fA)^2$ and $M^2(f)$ gives the level $f$-structure $\lambda$ on $M^2(f)$.
It turns out that some of the sections in $\mathrm{im}(\overline{\lambda})$ still intersect above the scheme of cusps. To get rid of the intersection points, we describe in Section 6.3 the so-called Néron-model $M$ of the Tate-Drinfeld module. This is a model of $\mathbb{P}^1_{\oplus R[\![x]\!]_\mathfrak{p}}$ such that the sections of the image of $\lambda^{\mathrm{td}}$ intersect nowhere. For this we briefly recall the necessary theory exposed by Mumford in [42]. In Section 6.4 we define an open subscheme $M^* \subset M$ which inherits a group scheme structure and an $A$-action from its fibre $\mathbb{P}^1_{\oplus R(\!(x)\!)_\mathfrak{p}}$.
In Section 6.5 we replace $\mathrm{P}L \longrightarrow \overline{M}^2(f)$ by a scheme

$$X \longrightarrow \overline{M}^2(f)$$

which outside the scheme of cusps is isomorphic to $\mathrm{P}L_{|M^2(f)}$ and over the scheme of cusps comes from the Néron model $M$.

### 6.1.1 Notations and results from the previous chapter

In the previous chapter we have described the compactification $\overline{M}^2(f)$ of $M^2(f)$ via a 'pseudo $j$-invariant'

$$j_a : M^2(f) \longrightarrow \mathbb{A}^1_{A_f}.$$

Write $\mathbb{A}^1_{A_f} = \mathrm{Spec}(A_f[j])$ and let $C$ be the integral closure of $A_f[\frac{1}{j}]$ inside the quotient field of $M^2(f)$. The formal neighbourhood of the scheme of cusps $\mathit{Cusps} = \overline{M}^2(f) - M^2(f)$ is given by the topological ring

$$\widehat{C} = \varprojlim C/(\frac{1}{j})^n \cong \bigoplus_{\mathfrak{p}} \varprojlim C/\mathfrak{p}^n.$$

The direct sum runs over all minimal primes $\mathfrak{p} \subset C$ containing $\frac{1}{j}$.

Let as in the previous chapter $\sigma_1, \ldots, \sigma_n \in \mathrm{Sl}_2(A/fA)$ with $\sigma_1 = 1$ and such that the $\sigma_i N$ are all cosets of

$$N = \begin{pmatrix} \mathbb{F}_q^* & A/fA \\ 0 & (A/fA)^* \end{pmatrix} \subset \mathrm{Gl}_2(A/fA).$$

Let $(\psi, \mu)$ be the universal Drinfeld module $\psi$ of rank 1 with level $f$-structure $\mu$ defined over $R$. In the previous chapter we have defined the Tate-Drinfeld module as the ring $\oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$ equipped with a pair $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$. This pair is a Drinfeld module $\varphi^{\mathrm{td}}$ of rank 2 with level $f$-structure $\lambda^{\mathrm{td}}$ over $\oplus R(\!(x)\!)_{(\mathfrak{m},\sigma_i)}$. As in the previous chapter we write $(\varphi^{\mathrm{td}}_\mathfrak{m}, \lambda^{\mathrm{td}}_\mathfrak{m})$ for the restriction of $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$ to $R[\![x]\!]_{(\mathfrak{m},\sigma_1)}$. By definition, the restriction of $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$ to $R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$ equals $(\varphi^{\mathrm{td}}_\mathfrak{m}, \lambda^{\mathrm{td}}_\mathfrak{m} \circ \sigma_i)$. The pair $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$ has the following properties:

(1) For $R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$, the reduction

$$(\varphi^{\mathrm{td}}_\mathfrak{m} \bmod (x), \lambda^{\mathrm{td}}_\mathfrak{m}(0,1) \bmod (x))$$

restricted to this copy equals $(\psi, \mu)$. Consequently, for the restriction of $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$ to this copy we have

$$(\varphi^{\mathrm{td}} \bmod (x), \lambda^{\mathrm{td}} \circ \sigma_i^{-1}(0,1) \bmod (x)) = (\psi, \mu).$$

(2) For every $a \in A$, $\varphi_a^{\mathrm{td}}$ has its coefficients in $\oplus R[\![x]\!]_{\mathfrak{p}}$.

We have shown that there is a 1-1 correspondence between the minimal prime ideals $\mathfrak{p} \subset \widehat{C}$ containing $\frac{1}{j}$ and all pairs $(\mathfrak{m}, \sigma_i) \in \mathrm{Cl}(A) \times \mathrm{Gl}_2(A/fA)/N$. Moreover,

$$\widehat{C} \cong \oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)};$$

cf. Proposition 5.9.1. In the sequel we will sometimes write $R[\![x]\!]_{\mathfrak{p}}$ instead of $R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$. Similarly, we have

$$Cusps \cong \oplus Cusps_{\mathfrak{p}},$$

with $Cusps_{\mathfrak{p}} \cong M^1(f)$.

### 6.1.2 The choice of a representative of $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$

We slightly change the choice of the representative of the isomorphy class of $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$. Note that $\lambda_{\mathfrak{m}}^{\mathrm{td}}(0,1) \in R[\![x]\!]^*_{(\mathfrak{m},\sigma_1)}$. So we may choose for $(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda_{\mathfrak{m}}^{\mathrm{td}})$ the representative of this class with $\lambda_{\mathfrak{m}}^{\mathrm{td}}(0,1) = 1$. This means that for the restriction of $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$ to $R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$ we have $\lambda^{\mathrm{td}} \circ \sigma_i^{-1}(0,1) = 1$.

## 6.2 Extending the universal Drinfeld module

We want to combine the ring homomorphisms

$$\begin{array}{lcl} \varphi & : & A \longrightarrow B\{\tau\}, \\ \varphi^{\mathrm{td}} & : & A \longrightarrow \widehat{C}\{\tau\}, \end{array}$$

together with the $A$-module isomorphisms

$$\begin{array}{lcl} \lambda & : & (A/fA)^2 \xrightarrow{\sim} \varphi[f](B), \\ \lambda^{\mathrm{td}} & : & (A/fA)^2 \xrightarrow{\sim} \varphi^{\mathrm{td}}[f](\oplus R(\!(x)\!)_{\mathfrak{p}}) \end{array}$$

to some Drinfeld data over the compactification $\overline{M}^2(f)$.

**Proposition 6.2.1.** *There exists a line bundle $L \longrightarrow \overline{M}^2(f)$, a ring homomorphism*

$$\overline{\varphi} : A \longrightarrow \mathrm{End}_{\mathbb{F}_q}(L)$$

*and an $A$-morphism*

$$\overline{\lambda} : (A/fA)^2 \longrightarrow L(M^2(f))$$

*such that the restriction of the triple $(L, \overline{\varphi}, \overline{\lambda})$ to $M^2(f)$ equals $(\varphi, \lambda)$ and this triple induces the Tate-Drinfeld data $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$ on $\oplus R[\![x]\!]_{\mathfrak{p}}$.*

*Proof.* Write $Cusps = \coprod Cusps_{\mathfrak{p}}$. Let $h$ denote the ring homomorphism

$$h : B \longrightarrow B[\frac{1}{j}] \longrightarrow C[j] \longrightarrow \widehat{C}[j] \cong \oplus R(\!(x)\!)_{\mathfrak{p}}$$

such that

$$h(\varphi, \lambda) \cong (\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}}).$$

This means that there exists an element $\xi = (\xi_{\mathfrak{p}}) \in (\oplus R((x))_{\mathfrak{p}})^*$ such that

$$(\xi h(\varphi)\xi^{-1}, \xi h(\lambda)) \cong (\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}}).$$

Let $\mathfrak{p}$ correspond to $(\mathfrak{m}, \sigma_i)$, then restricted to the copy $R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$ we have

$$\xi_{\mathfrak{p}} h(\lambda \circ \sigma_i^{-1}(0,1)) = \lambda^{\mathrm{td}}_{\mathfrak{m}} \circ \sigma_i \circ \sigma_i^{-1}(0,1) = 1.$$

So there exists an element $k \in \mathbb{N}$ such that $x^k h(\lambda \circ \sigma_i^{-1}(0,1)) \in R[\![x]\!]^*_{\mathfrak{p}}$. As $\lambda \circ \sigma_i^{-1}(0,1) \in B$, we may conclude that $Cusps_{\mathfrak{p}}$ has an open (affine) neighbourhood $U_{\mathfrak{p}}$ such that $\lambda \circ \sigma_i^{-1}(0,1)$ is invertible on $U_{\mathfrak{p}} \backslash Cusps_{\mathfrak{p}}$. We may assume that $\mathfrak{p}$ is the only minimal containing $\frac{1}{j}$ in $U_{\mathfrak{p}}$.
We construct $L$ as follows. Glue $\mathcal{O}_{\mathrm{Spec}(B)}$ to $\mathcal{O}_{U_{\mathfrak{p}}}$ along the intersection $V = \mathrm{Spec}(B) \cap U_{\mathfrak{p}}$ by

$$\mathcal{O}_{\mathrm{Spec}(B)}(V) \xrightarrow{\lambda \circ \sigma_i^{-1}(0,1)} \mathcal{O}_{U_{\mathfrak{p}}}.$$

Call the result $L_{\mathfrak{p}}$.
For every two prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ the intersection of $\mathrm{Spec}(B) \cup U_{\mathfrak{p}}$ and $\mathrm{Spec}(B) \cup U_{\mathfrak{p}'}$ is simply $\mathrm{Spec}(B)$, so we can glue $L_{\mathfrak{p}}$ and $L_{\mathfrak{p}'}$ along $\mathrm{Spec}(B)$ via the identity map. Repeating this gluing procedure gives the line bundle $L$. □

Note that the sections in the image of $\lambda$ do not extend to sections in $L(\overline{M}^2(f))$. To repair this, we replace the line bundle $L$ by its projectivisation, which we denote by $\mathrm{P}L$. Note that

$$\mathrm{P}L = \mathbb{P}(L \oplus \mathcal{O}_{\overline{M}^2(f)}).$$

It is a $\mathbb{P}^1$-bundle over $\overline{M}^2(f)$ with $\mathrm{P}L_{|M^2(f)} = \mathbb{P}^1_{M^2(f)}$ and $\mathrm{P}L_{|\mathrm{Spec}(C)} = \mathbb{P}^1_{\mathrm{Spec}(C)}$. We define a map of sets

$$\overline{\lambda} : (A/fA)^2 \cup \{\infty\} \longrightarrow \Gamma(\overline{M}^2(f), \mathrm{P}L)$$

as follows:

(1) For every $\alpha \in (A/fA)^2$, we have a section $\overline{\lambda}(\alpha) \in L(M^2(f))$. This section extends to a section

$$\overline{\lambda}(\alpha) \in \Gamma(\overline{M}^2(f), \mathrm{P}L).$$

(2) The image of $\infty$ under $\overline{\lambda}$ is defined to be the $\infty$-section of $\mathrm{P}L$.

The sections in $\mathrm{im}(\overline{\lambda})$ only intersect above the points in *Cusps*. If a section intersects another above a cusp, then it intersects the $\infty$-section as well.

So we have obtained the following Drinfeld data:

$$\begin{array}{rcl} \overline{\varphi} & : & A \longrightarrow \mathrm{End}_{\mathbb{F}_q}(L), \\ \overline{\lambda} & : & (A/fA)^2 \cup \{\infty\} \longrightarrow \Gamma(\overline{M}^2(f), \mathrm{P}L). \end{array}$$

If we delete the $\infty$-section from $\mathrm{P}L$ and restrict this data to $M^2(f)$, then we retrieve the universal Drinfeld module of rank 2 with level $f$-structure.
At the cusps there are sections in the image of $\overline{\lambda}$ which still intersect. To get rid of these intersections, we construct a *Néron model* of $\mathrm{P}L$ at the cusps, which has the property that the sections induced by $\overline{\lambda}$ no longer intersect.

## 6.3 Minimal models and Néron models

We start by constructing minimal models and Néron models for one copy $R[\![x]\!]$ of $\oplus R[\![x]\!]_{\mathfrak{p}}$. In this section we shall simply write $(\varphi^{\text{td}}, \lambda^{\text{td}})$ for the restriction of $(\varphi^{\text{td}}, \lambda^{\text{td}})$ to this copy $R[\![x]\!]$.

### 6.3.1 Definition of a model

We recall here some of the theory we need, which can also be found in Section 3 of [56] and [19]. Let $V$ be a discrete valuation ring, $\mathfrak{v}$ its maximal ideal and let $K_V$ be its quotient field and let $\kappa(\mathfrak{v}) := V/\mathfrak{v}$ be the residue field at $\mathfrak{v}$.

**Definition 6.3.1.** A *model* over $V$ of the projective line $\mathbb{P}^1_{K_V}$ is a projective, flat $V$-scheme $M$ such that

(1) The generic fibre $M \times \operatorname{Spec}(K_V)$ is isomorphic to $\mathbb{P}^1_{K_V}$.

(2) The special fibre $M_{\mathfrak{v}} := M \times \operatorname{Spec}(\kappa(\mathfrak{v}))$ has at most double points as singularities.

The *reduction map* $\rho_M$ attached to the model $M$ is the map

$$\rho_M : \mathbb{P}^1_{K_V}(K_V) \xrightarrow{\sim} M(V) \longrightarrow M_{\mathfrak{v}}(\kappa(\mathfrak{v})).$$

We cite Lemma 3.1 in [56]:

**Lemma 6.3.2.** *Let $E \subset \mathbb{P}^1_{K_V}(K_V)$ be a finite set consisting of at least 3 elements. Then there exists a unique model $M$ over $V$ of $\mathbb{P}^1_{K_V}$, having the following properties:*

*(1) The reduction map $\rho_M$ restricted to $E$ is injective and the image $\rho_M(E)$ does not contain any singularities of $M_{\mathfrak{v}}$.*

*(2) The special fibre $M_{\mathfrak{v}}$ and $E$ form a stable tree, i.e., $M_{\mathfrak{v}}$ is a tree of $\mathbb{P}^1_{\kappa(\mathfrak{v})}$'s and on each $\mathbb{P}^1_{\kappa(\mathfrak{v})}$ there lie at least three special points, where a special point is either a double point or the image of an element of $E$.*

**Definition 6.3.3.** The model $M$ of $\mathbb{P}^1_V$ satisfying the properties of the previous lemma is called the *minimal model* of $\mathbb{P}^1_V$ with respect to $E$.

Let $K_R$ be the quotient field of $R$. Let $V = K_R[\![x]\!]$ and

$$E = \operatorname{im}(\overline{\lambda}) = \varphi^{\text{td}}[f](R(\!(x)\!)) \cup \{\infty\}.$$

In this section we will construct the minimal model described in Lemma 6.3.2. This minimal model already comes close to our goal, but it is not entirely what we want. Instead, we are looking for a *Néron model* of $\mathbb{P}^1_{R(\!(x)\!)}$ over $R[\![x]\!]$.
Let $\mathfrak{m} \subset R$ be a maximal ideal and let $\rho_{\mathfrak{m}}$ be the reduction map

$$\rho_{\mathfrak{m}} : \mathbb{P}^1_{R[\![x]\!]}(R[\![x]\!]) \longrightarrow \mathbb{P}^1_{\kappa(\mathfrak{m})[\![x]\!]}(\kappa(\mathfrak{m})[\![x]\!]).$$

**Definition 6.3.4.** A *Néron model* over $R[\![x]\!]$ of $\mathbb{P}^1_{R[\![x]\!]}$ with respect to $E$ is a projective, flat $R[\![x]\!]$-scheme $M$, having the following properties:

(1) $M \times \mathrm{Spec}(R(\!(x)\!)) \cong \mathbb{P}^1_{R(\!(x)\!)}$.

(2) The scheme $M \times \mathrm{Spec}(K_R[\![x]\!])$ is a minimal model of $\mathbb{P}^1_{K_R[\![x]\!]}$ with respect to $E$.

(3) Let $\mathfrak{m} \subset R$ be a maximal ideal, then

$$M \times \mathrm{Spec}(\kappa(\mathfrak{m})[\![x]\!])$$

is a minimal model of $\mathbb{P}^1_{\kappa(\mathfrak{m})[\![x]\!]}$ with respect to $\rho_{\mathfrak{m}}(E)$.

The following lemma states some properties of the elements of $E$ which we will need in the proof of the existence of the Néron model.

**Lemma 6.3.5.** *The elements of $E$ have the following properties:*

*(1) For every $\alpha \in \varphi^{\mathrm{td}}[f](R(\!(x)\!))$ with $\alpha \neq 0$ there is an $n \in \mathbb{Z}$ such that $x^n \cdot \alpha \in R[\![x]\!]^*$.*

*(2) Using (1), we can consider $E$ as a subset of $\mathbb{P}^1_{R[\![x]\!]}(R[\![x]\!])$. Namely, the point $\infty$ is given by $(1:0)$, $0$ is given by $(0:1)$ and $\alpha \in E\backslash\{\infty, 0\}$ by $(x^n\alpha : x^n)$ with $n$ as in (1). Let $\mathfrak{m} \subset R$ be a maximal ideal. The restriction of $\rho_{\mathfrak{m}}$ to $E$ is injective. For every $e \in E\backslash\{\infty\}$, $v_x(\rho_{\mathfrak{m}}(z)) = v_x(z)$.*

*Proof.* (1) This follows immediately from the definition of the Tate-Drinfeld module.
(2) That $v_x(\rho_{\mathfrak{m}}(e)) = v_x(e)$ follows because either $e = 0$ or $e \in R(\!(x)\!)^*$.
For the injectivity of the restriction of $\rho_{\mathfrak{m}}$ to $E$, note that the elements of $E\backslash\{\infty\}$ form a group. Consequently, if $e_1, e_2 \in E\backslash\{\infty\}$ are two distinct elements, then $e_1 - e_2 \in R(\!(x)\!)^*$ and thus $\rho_{\mathfrak{m}}(e_1) \neq \rho_{\mathfrak{m}}(e_2)$. Finally, it is clear that $\rho_{\mathfrak{m}}(e_i) \neq (1:0)$, which is the image of the point $\infty$. □

Let $e_1, \ldots, e_n$ be an $\mathbb{F}_q$-basis for $\varphi[f](R(\!(x)\!))\backslash\varphi[f](R[\![x]\!])$ such that

$$v_x(e_1) \geq v_x(e_2) \geq \ldots \geq v_x(e_n),$$

then $\varphi^{\mathrm{td}}[f](R(\!(x)\!)) = \varphi^{\mathrm{td}}[f](R[\![x]\!]) + \mathbb{F}_q e_1 + \cdots + \mathbb{F}_q e_n$.

For the description of the Néron model it is convenient to distinguish only between the distinct valuations. Therefore, we introduce the following notation for the vectorspace $\sum \mathbb{F}_q e_i$. Let $k_1, \ldots, k_l$ be distinct elements of $\mathbb{N}$ such that $-k_i$ are all the $x$-valuations of the elements of $\sum \mathbb{F}_q e_i\backslash\{0\}$, i.e.,

$$v_x(e) \in \{-k_1, \ldots, -k_l\} \quad \text{for all non-zero } e \text{ in } \sum \mathbb{F}_q e_i.$$

Moreover, we put $k_0 := 0$ for notational convenience in the sequel. For $j = 1, \ldots, l$, let $E_j$ be the subset of the basis $\{e_1, \ldots, e_n\}$ consisting of exactly those basis elements with valuation $-k_j$, i.e., $E_j = \{e \in \{e_1, \ldots, e_n\} \mid v_x(e) = -k_j\}$. Clearly, the $E_j$ give a partition of the basis $\{e_1, \ldots, e_n\}$, and for each $j$ we have $\#E_j > 0$. Let $\delta_e := x^{k_j} e \in R[\![x]\!]^*$ for every $e \in E_j$. Define

$$F_j := \sum_{e \in E_j} \mathbb{F}_q \delta_e.$$

Clearly, $\sum \mathbb{F}_q \cdot e_i = \sum_{j=1}^{l} F_j x^{-k_j}$.

**Lemma 6.3.6.** *The restriction of the reduction map $R[\![x]\!] \longrightarrow R$ to $F_j$ is injective. Let $\mathfrak{m} \subset R$ be a maximal ideal, then the restriction of the reduction map $R[\![x]\!] \longrightarrow \kappa(\mathfrak{m})$ to $F_j$ is injective too.*

*Proof.* Let $\sum_{e\in E_j} r_e\delta_e$ be any element in $F_j$ with $r_e \in \mathbb{F}_q$ such that $\sum_{e\in E_j} r_e\delta_e \equiv 0 \bmod x$, then the sum $s = \sum_{e\in E_j} r_e e \in \sum \mathbb{F}_q e_i$ has valuation strictly larger than $-k_j$. Therefore, $s$ is linearly dependent on the elements $e \in E_k$ with $1 \leq k < j$. As $\cup_j E_j$ is a basis of $\sum \mathbb{F}_q e_i$ it follows that each $r_e = 0$. This proves that the restriction of the first reduction map to $F_j$ is injective.
For the second map, note that $\delta_e \in R[\![x]\!]^*$ for all $e \in E_j$. As the restiction of $\rho_{\mathfrak{m}}$ to $E$ is injective, cf. Lemma 6.3.5, we can apply the above argument to the restriction map $\kappa(\mathfrak{m})[\![x]\!] \longrightarrow \kappa(\mathfrak{m})$. □

**Remark 6.3.7.** By Lemma 6.3.6 all elements of $F_j$ stay distinct modulo $x$. This is the little technicality needed in the sequel which makes sure that the minimal model over $K_R[\![x]\!]$ gives rise to a Néron model.

**Proposition 6.3.8.** *There exists a Néron model of $\mathbb{P}^1_{R(\!(x)\!)}$ over $R[\![x]\!]$.*

The rest of this section will be concerned with proving this proposition. First we give a construction of the model of Lemma 6.3.2 with $V = K_R[\![x]\!]$ and $E$ as above. For this construction we follow Mumford's paper [42]. It turns out that the equations for the model over $K_R[\![x]\!]$ are already defined over $R[\![x]\!]$. And thus we can consider it as a model over $R[\![x]\!]$. By the special property of the sections in $E$ remarked in Lemma 6.3.7 and Lemma 6.3.6, it will follow that this model is in fact a Néron model.

### 6.3.2 Mumford's paper

We begin by recalling the theory that Mumford describes in his paper [42] and applying it to our case. One can also find something similar in [20]. Consider three points $x_1, x_2, x_3 \in \mathbb{P}^1_{K_R(\!(x)\!)}(K_R(\!(x)\!))$. Let

$$w_i \in K_R(\!(x)\!) \oplus K_R(\!(x)\!)$$

denote homogeneous coordinates of the $x_i$. Then there exists a relation $\sum_i a_i w_i = 0$ with $a_i \in K_R(\!(x)\!)$. This relation is unique up to some scalar in $K_R(\!(x)\!)$. Define the $K_R[\![x]\!]$-module

$$N := \sum_i K_R[\![x]\!] a_i w_i.$$

For any free $K_R[\![x]\!]$-module $N$ of rank 2, we can consider the class of $N$

$$\{\alpha N \mid \alpha \in K_R(\!(x)\!)^*\}.$$

The class of $N$ only depends on the $x_i$. Suppose that $u$ and $v$ generate $N$ as $K_R[\![x]\!]$-module, then we may associate to such a module a scheme

$$P = \mathrm{Proj}(K_R[\![x]\!][X, Y])$$

with $X(u) = Y(v) = 1$ and $X(v) = Y(u) = 0$. This construction gives a bijection between the set of classes of free $K_R[\![x]\!]$-modules of rank 2 and the set of schemes $S_N \cong \mathbb{P}^1_{K_R[\![x]\!]}$ whose generic fibre is $\mathbb{P}^1_{K_R(\!(x)\!)}$. Both these sets are denoted by $\Delta^0$.

A class $\{N_1\}$ is called *compatible* with $\{N_2\}$ if there exist representing modules $N_i$ with basis $\{u_i, v_i\}$ together with elements $\alpha \in K_R((x))^*$ and $\beta \in K_R[\![x]\!]$ such that $u_1 = \alpha u_2$ and $v_1 = \alpha\beta u_2$. This definition is symmetric, and the ideal $(\beta) \subset K_R[\![x]\!]$ is uniquely determined by the classes $\{N_i\}$. The principal ideal $(\beta)$ is called the *distance* between $\{N_1\}$ and $\{N_2\}$. As the dimension of $K_R[\![x]\!]$ is 1, it follows that every pair $\{N_1\}$, $\{N_2\}$ of $K_R[\![x]\!]$-modules is compatible.
Furthermore, we say that the representatives $N_i$ are *representatives in standard position* if

$$N_1 \supset N_2 \supset \beta N_1.$$

A subset $\Gamma \subset \Delta^0$ is called *linked* if

(1) Every pair $\{N_1\}, \{N_2\} \in \Gamma$ is compatible.

(2) For every triple $\{N_1\}, \{N_2\}, \{N_3\}$ with representatives $N_1 \supset N_2$ and $N_1 \supset N_3$ in standard position, the module $N_2 + N_3$ defines a class in $\Gamma$.

Let $\Gamma \subset \Delta^0$ be a linked subset. Two elements $\gamma_1, \gamma_2 \in \Gamma$ are called *adjacent* if there is no $\gamma_3 \in \Gamma$ distinct from $\gamma_1$ and $\gamma_2$ such that the distance of $\gamma_1$ and $\gamma_2$ equals the distance of $\gamma_1$ and $\gamma_3$ multiplied with the distance of $\gamma_2$ and $\gamma_3$.
We can associate a *tree* to $\Gamma$ as follows. The vertices are the module-classes. There is an edge between any two adjacent module-classes. To each edge $\varepsilon$ we associate the distance between its vertices, which is a principal ideal $(\beta_\varepsilon)$. For any two vertices $\{N_1\}$ and $\{N_2\}$ in this tree which are connected by a path consisting of the edges $\varepsilon_1, \dots, \varepsilon_m$, the distance of $\{N_1\}$ and $\{N_2\}$ equals $\prod_{i=1}^m (\beta_{\varepsilon_i})$.

Consider the set of all reduced and irreducible schemes

$$S \longrightarrow \mathrm{Spec}(K_R[\![x]\!])$$

with generic fibre $\mathbb{P}^1_{K_R((x))}$. This set is partially ordered if $S_1 > S_2$ means that there exists a morphism $S_1 \longrightarrow S_2$ whose restriction to the generic fibre is the identity. The least upper bound of two such schemes exists in this set and is called the *join* of those two schemes.

More specifically, let $N_i$ be a $K_R[\![x]\!]$-module for $i = 1, \dots, m$, then the scheme $S_{N_i}$ associated to $N_i$ comes equipped with a canonical isomorphism

$$f_i : S_{N_i} \times \mathrm{Spec}(K_R((x))) \xrightarrow{\sim} \mathbb{P}^1_{K_R((x))}.$$

Consider the morphism

$$\prod_{i=1}^m S_{N_i} \longrightarrow \prod_{i=1}^m \mathbb{P}^1_{K_R((x))}$$

defined by $(f_1, \dots, f_m)$. The join $S$ of $S_{N_1}, \dots, S_{N_m}$ is by definition the closure of the inverse image of the diagonal $D \subset \prod_{i=1}^m \mathbb{P}^1_{K_R((x))}$ inside $\prod_{i=1}^m S_{N_i}$ under $(f_1, \dots, f_m)$.

Using these notions, Mumford proves the following; cf. Proposition 2.3 and Proposition 2.4 in [42].

**Proposition 6.3.9 (D. Mumford).** *Let $\Gamma = \{N_1, \ldots, N_k\} \subset \Delta^0$ be linked and let $S^i$ denote the scheme associated to $N_i$, then the join $S$ of these $S^i$ exists. Let*

$$S_0 := S \times \operatorname{Spec}(K_R[\![x]\!]/(x))$$

*be the closed fibre of $S$, then $S$ has the following properties:*

*(1) $S \longrightarrow Spec(K_R[\![x]\!])$ is normal, proper and flat.*

*(2) $S_0$ is reduced, connected and 1-dimensional.*

*(3) The components of $S_0$ are naturally isomorphic to the closed fibres $S_0^i$ of $S^i$.*

*(4) Two components of $S_0$ meet in at most a double point, are $K_R$-rational and the irreducible components of $S_0$ form a tree of $\mathbb{P}^1_{K_R[\![x]\!]}$.*

In fact, we can say something more on the fourth item. If we let each component of $S_0$ correspond to a vertex and each point of intersection to an edge, then we get the tree of $\Gamma$.

### 6.3.3 Construction of a minimal model

The next step is to apply Mumford's theory to our case. The set $E = \varphi^{\mathrm{td}}[f](R(\!(x)\!)) \cup \{\infty\}$. Let

$$E^{(3)} = \{(e_1, e_2, e_3) \mid e_i \in E \text{ and the } e_i \text{ are mutually distinct}\}.$$

For any $\overline{e} \in E^{(3)}$ let $[N_{\overline{e}}]$ be the module class associated to the triple $\overline{e}$. The subset $\Gamma \subset \Delta^0$ is defined to be

$$\Gamma = \{[N_{\overline{e}}] \mid \overline{e} \in E^{(3)}\}.$$

As before, we write $(1:0)$ for the point $\infty$. Recall that we introduced $F_j = \sum_{e \in E_j} x^{k_j} \delta_e$ with

$$E = (\varphi^{\mathrm{td}}[f](K_R[\![x]\!]) + \sum_{j=1}^{l} F_j x^{-k_j}) \cup \{\infty\}.$$

Let $m \in \{0, 1, \ldots, m\}$, and let $\alpha_i \in F_{l-i}$ for $i = 0, \ldots, m$. Define $e = \sum_{i=0}^{m} \alpha_i x^{-k_{l-i}}$. The elements $e$ and $e + x^{-k_{l-m-1}}$ can be considered as elements in $\mathbb{P}^1(K_R[\![x]\!])$ as follows. If every $\alpha_i = 0$, then $e$ and $e + x^{-k_{l-m-1}}$ give rise to the points

$$(0:1) \quad \text{and} \quad (1 : x^{k_{l-m-1}}),$$

respectively. Otherwise, put $w = \min\{i \mid \alpha_i \neq 0\}$. In this case $e$ and $e + x^{-k_{l-m-1}}$ give rise to

$$(x^{k_{l-w}} e : x^{k_{l-w}}) \quad \text{and} \quad (x^{k_{l-w}}(e + x^{-k_{l-m-1}}) : x^{k_{l-w}}),$$

respectively.

Let $[N(\alpha_0, \ldots, \alpha_m)]$ to be the module-class associated to the triple $(e, e + x^{-k_{l-m-1}}, \infty)$. It is not difficult to see that there exists a representative $N(\alpha_0, \ldots, \alpha_m)$ of this class such that this representative is generated by the following elements.

If $\alpha_i = 0$ for $i = 0, \ldots, m$, then $N(\alpha_0, \ldots, \alpha_m)$ is generated by

$$\begin{aligned} u &= (1, 0), \\ v &= (0, x^{k_{l-m-1}}). \end{aligned}$$

Otherwise, we can choose the representative $N(\alpha_0, \ldots, \alpha_m)$ such that it is generated by

$$\begin{aligned} u &= (x^{k_{l-w}-k_{l-m-1}}, 0), \\ v &= (x^{k_{l-w}}e, x^{k_{l-w}}). \end{aligned}$$

By Lemma 6.3.5 it follows that in both cases the entries of $u$ and $v$ generate the unit ideal in $R[\![x]\!]$. Furthermore, let $N$ be the module generated by

$$\begin{aligned} u &= (1, 0), \\ v &= (0, x^{k_l}). \end{aligned}$$

**Lemma 6.3.10.** *The set $\Gamma$ equals the set of all classes of $[N]$ and $[N(\alpha_0, \ldots, \alpha_m)]$ with $m = 0, \ldots, l-1$. Moreover, $\Gamma$ is linked.*

*Proof.* It follows from a straightforward computation that every class in $\Gamma$ equals one of the $[N(\alpha_0, \ldots, \alpha_m)]$. To show that $\Gamma$ is linked, we first note that $K_R[\![x]\!]$ has dimension 1. Therefore, every pair in $\Gamma$ is compatible.
Let $N_1 = N(\alpha_0, \ldots, \alpha_n, \alpha_{n+1}, \ldots \alpha_m)$ and $N_2 = N(\alpha_0, \ldots, \alpha_n, \alpha'_{n+1}, \ldots \alpha'_{m'})$. Suppose that there is an $i \in \{0, \ldots, n\}$ such that $\alpha_i \neq 0$. We claim that $N_1 + N_2 = N(\alpha_0, \ldots, \alpha_n)$. Put $\tilde{N} = N(\alpha_0, \ldots, \alpha_n)$. The distance between $\tilde{N}$ and $N_1$ equals $\beta_1 = x^{k_{l-n}-k_{l-m-1}}$, and the distance between $\tilde{N}$ and $N_2$ equals $\beta_2 = x^{k_{l-n}-k_{l-m'-1}}$. By our choice of representatives they are in standard position. The claim follows by an easy computation.
In case $\alpha_i = 0$ for all $i = 0, \ldots, n$, then it also follows that $N_1 + N_2 = N(\alpha_0, \ldots, \alpha_n)$. For the computation, however, one should note that the modules need not be in standard position any more (to be precise, this is the case if either $\alpha_{n+1}$ or $\alpha'_{n+1}$ equals 0), but the computation remains essentially the same. This shows that $\Gamma$ is linked. □

This lemma enables us to describe the finite tree associated to $\Gamma$. The vertices of the tree are the module classes $[N]$ and $[N(\alpha_0, \ldots, \alpha_m)]$ for $m = 0, \ldots, l-1$. The vertex $[N]$ is the central vertex. There is an edge between $[N]$ and $[N(\alpha_0)]$ for every $\alpha_0$. Consequently, there are exactly $\#F_l$ edges ending in $[N]$. Similarly, there is an edge between the vertices $[N(\alpha_0, \ldots, \alpha_m)]$ and $[N(\alpha_0, \ldots, \alpha_{m+1})]$ for every $\alpha_i$ and $m = 0, \ldots, l-2$. Consequently, the vertices corresponding to $[N(\alpha_0, \ldots, \alpha_{l-1})]$ are the end vertices.

**Remark 6.3.11.** Note that this description indeed follows from the previous subsection and Lemma 6.3.10 if and only if the module classes $[N]$ and $[N(\alpha_0, \ldots, \alpha_m)]$ are all distinct. To see that this is the case, note first that if $N(\alpha_0, \ldots, \alpha_m)$ and $N(\alpha'_0, \ldots, \alpha'_{m'})$ are in the same module class, then $m = m'$. Suppose that $m = m'$. To see that both modules are distinct if the $m$-tuples $(\alpha_0, \ldots, \alpha_m)$ and $(\alpha'_0, \ldots, \alpha'_{m'})$ are distinct, we refer to Remark 6.3.12.

The tree of the join of the schemes corresponding to the module-classes in $\Gamma$ is dual to the tree of $\Gamma$. We will denote the join by $M_{K_R[\![x]\!]}$. The scheme $M_{K_R[\![x]\!]}$ is our candidate for the minimal model. To show that this scheme actually is the minimal model, we briefly describe the join. We denote by $\mathbb{P}_N$ the scheme corresponding to $[N]$ and with $\mathbb{P}(\alpha_0, \dots, \alpha_m)$ the scheme corresponding to $[N(\alpha_0, \dots, \alpha_m)]$.

Let $u = (1,0)$ and $v = (0, x^{k_l})$ be the generators of $N$ as above. Let $X, Y$ be coordinates of $\mathbb{P}_N$ given by $X(u) = Y(v) = 1$ and $X(v) = Y(u) = 0$. Similarly, let $u_{\alpha_0}$ and $v_{\alpha_0}$ be the generators of $N[\alpha_0]$ as given above, and let $X_{\alpha_0}$ and $Y_{\alpha_0}$ be the corresponding coordinates. An easy computation shows that the join of $\mathbb{P}_N$ and $\mathbb{P}(\alpha_0)$ is given by the closure of

$$XY_{\alpha_0} = \alpha_0 YY_{\alpha_0} + (x^{k_l - k_{l-1}})YX_{\alpha_0} \tag{6.1}$$

inside $\mathrm{Proj}(K_R[\![x]\!][X_{\alpha_0}X, Y_{\alpha_0}X, X_{\alpha_0}Y, Y_{\alpha_0}Y])$.

Therefore, the intersection in the intersection tree is given by

$$(X - \overline{\alpha}_0 Y)Y_{\alpha_0} = 0$$

where $\overline{\alpha}_0 = \alpha_0 \bmod (x)$. In $X, Y$-coordinates, this intersection point is $(\overline{\alpha}_0 : 1)$ and in $X_{\alpha_0}, Y_{\alpha_0}$-coordinates this intersection point is $(1:0)$.

Similarly, we can describe the join of the modules $[N(\alpha_0, \dots, \alpha_m)]$ and $[N(\alpha_0, \dots, \alpha_{m+1})]$ for $m = 0, \dots, l-2$. Let $u, v$ denote the generators as given above of the module $N(\alpha_0, \dots, \alpha_m)$, and let $X, Y$ denote the coordinates with $X(u) = Y(v) = 1$ and $X(v) = Y(u) = 0$. Similarly, let $u', v'$ be the generators of $N(\alpha_0, \dots, \alpha_{m+1})$, and let $X', Y'$ be the corresponding coordinates. The join of $\mathbb{P}(\alpha_0, \dots, \alpha_m)$ and $\mathbb{P}(\alpha_0, \dots, \alpha_{m+1})$ is given by the closure of

$$XY' = x^{k_{l-m-1} - k_{l-m-2}} YX' + \alpha_{m+1} YY' \tag{6.2}$$

inside $\mathrm{Proj}(K_R[\![x]\!][X'X, Y'X, X'Y, Y'Y])$.

Therefore, the intersection in the intersection tree is given by

$$(X - \overline{\alpha}_{m+1} Y)Y' = 0.$$

In $X, Y$-coordinates, this intersection point is $(\overline{\alpha}_{m+1} : 1)$ and in $X_{\alpha_0}, Y_{\alpha_0}$-coordinates this intersection point is $(1:0)$.

**Remark 6.3.12.** By Lemma 6.3.6 the restriction of the reduction map $K_R[\![x]\!] \longrightarrow K_R$ to $F_{l-m-1}$ is injective. Therefore, if $\alpha_{m+1} \neq \alpha'_{m+1}$, then the intersection point in the reduction tree of $M_{K_R[\![x]\!]}$ of $\mathbb{P}(\alpha_0, \dots, \alpha_m, \alpha_{m+1})$ with $\mathbb{P}(\alpha_0, \dots, \alpha_m)$ is distinct from the intersection point of $\mathbb{P}(\alpha_0, \dots, \alpha_m, \alpha'_{m+1})$ with $\mathbb{P}(\alpha_0, \dots, \alpha_m)$.
Therefore, each tuple $(\alpha_0, \dots, \alpha_m)$ gives rise to a distinct projective line in the reduction tree of $\mathbb{P}^1_{K_R[\![x]\!]}$. This also implies that the module classes $[N]$ and $[N(\alpha_0, \dots, \alpha_m)]$ are all distinct; cf. Remark 6.3.11.

Using this description of the tree of $M_{K_R[\![x]\!]}$, it is not difficult to see where the image of the points of $E = \varphi^{\mathrm{td}}[f](K_R(\!(x)\!)) \cup \{\infty\}$ in this tree lie. First of all, the point $\infty$ is mapped to $(1:0)$ on the root of the tree, i.e., on the $\mathbb{P}^1$ corresponding to the module $[N]$.

Furthermore, the element $e = \sum_{i=0}^{l-1} \alpha_i x_{l-i}^k \in E$ lies on the unique endline corresponding to the module class of $N(\alpha_0, \ldots, \alpha_{l-1})$. Also, all elements $e + \varphi^{\mathrm{td}}[f](K_R[\![x]\!])$ are mapped to distinct $K_R$-valued points in this particular $\mathbb{P}^1$.
By the computation of the intersection points, we see that no point of $E$ is mapped to an intersection point of the reduction tree of $M_{K_R[\![x]\!]}$. Similarly, we see that every $\mathbb{P}^1$ of this tree contains at least $q+1$ special points: on each endline we have the points $e + \varphi^{\mathrm{td}}[f](K_R[\![x]\!])$ together with 1 intersection points, on the root we have at least $q$ intersection point and the point $\infty$, and on each other $\mathbb{P}^1$ in the tree we have at least $q+1$ intersection points. These considerations prove the following:

**Lemma 6.3.13.** *The scheme $M_{K_R[\![x]\!]}$ is a minimal model of $\mathbb{P}^1_{K_R(\!(x)\!)}$ with respect to the image of $E$.*

### 6.3.4 The Néron model

Note that the equations (6.1) and (6.2), which define the scheme $M_{K_R[\![x]\!]}$, are also defined over $R[\![x]\!]$. Therefore, the scheme $M_{K_R[\![x]\!]}$ is induced by base-extension by a proper scheme $M \longrightarrow \mathrm{Spec}(R[\![x]\!])$. So we have

$$M \times \mathrm{Spec}(K_R[\![x]\!]) = M_{K_R[\![x]\!]}, \quad M \times \mathrm{Spec}(R(\!(x)\!)) = \mathbb{P}^1_{R(\!(x)\!)}.$$

Let $\mathfrak{m} \subset R$ be a maximal ideal and let $\kappa(\mathfrak{m})$ denote the residue field at $\mathfrak{m}$. Define

$$M_{\mathfrak{m}} := M \times \mathrm{Spec}(\kappa(\mathfrak{m})[\![x]\!]).$$

**Lemma 6.3.14.** *The scheme $M_{\mathfrak{m}}$ is a minimal model of $\mathbb{P}^1_{\kappa(\mathfrak{m})[\![x]\!]}$ with respect to $\rho_{\mathfrak{m}}(E)$.*

*Proof.* The reduction tree of $M_{\mathfrak{m}}$ is a tree of $\mathbb{P}^1_{\kappa(\mathfrak{m})}$'s. We first show that the reduction of $M_{\mathfrak{m}}$ has the same abstract tree as $M_{K_R[\![x]\!]}$. Consider the reductions $\overline{N}(\alpha_0, \ldots, \alpha_m)$ of the modules $N(\alpha_0, \ldots, \alpha_m)$. The scheme $M_{\mathfrak{m}}$ is the join of the schemes corresponding to these modules. According to Lemma 6.3.5, the elements $\alpha_0$ and $\alpha_{m+1}$ are either 0 or inside $R[\![x]\!]^*$. Therefore, the distance between two modules does not change under the reduction.
Consider two distinct elements $\alpha_m, \alpha'_m \in F_{l-m}$, then the schemes $\mathbb{P}(\alpha_0, \ldots, \alpha_{m-1}, \alpha_m)$ and $\mathbb{P}(\alpha_0, \ldots, \alpha_{m-1}, \alpha'_m)$ both intersect $\mathbb{P}(\alpha_0, \ldots, \alpha_{m-1})$ in $(\alpha_m : 1)$ and $(\alpha'_m : 1)$, respectively. Therefore, their reductions intersect in $(\alpha_m \bmod x : 1)$ and $(\alpha'_m \bmod x : 1)$, respectively. By Lemma 6.3.6, these intersection points are distinct. Therefore, distinct lines in the tree of $M_{K_R[\![x]\!]}$ gives rise to distinct lines in the tree of $M_{\mathfrak{m}}$. This shows that the reduction tree of $M_{\mathfrak{m}}$ is the same abstract tree as the reduction tree of $M_{K_R[\![x]\!]}$.
This implies that each endline corresponds to a unique $e \in E \backslash (\varphi^{\mathrm{td}}(R[\![x]\!]) \cup \{\infty\})$, and all points $e + \varphi^{\mathrm{td}}(R[\![x]\!])$ are mapped to $\kappa(\mathfrak{m})$-valued points of this endline. To see that the images of the points $e + \varphi^{\mathrm{td}}(K_R[\![x]\!])$ are mutually distinct, let $y_1, y_2 \in \varphi^{\mathrm{td}}[f](R[\![x]\!])$ be distinct elements. Then the difference $y_1 - y_2$ between the points $e + y_1$ and $e + y_2$ is an element of $\varphi^{\mathrm{td}}[f](R[\![x]\!])$. Therefore $y_1 - y_2 \in R[\![x]\!]^*$. Consequently, the image of $y_1 - y_2$ in $\kappa(\mathfrak{m})$ is not 0.
This shows that $E$ maps injectively into the $\kappa(\mathfrak{m})$-valued points of the special fibre of $M_{\mathfrak{m}}$. Arguing as before, it follows that there are no double points in the image of $E$ inside the set of $\kappa(\mathfrak{m})$-valued points of $M_{\mathfrak{m}} \times \mathrm{Spec}(\kappa(\mathfrak{m})[\![x]\!]/(x))$. This finishes the proof. ☐

This enables us to prove Proposition 6.3.8.

*Proof of Proposition 6.3.8.* We constructed a proper scheme

$$M \longrightarrow \operatorname{Spec}(R[\![x]\!])$$

such that $M \times \operatorname{Spec}(K_R[\![x]\!])$ is the minimal model of $\mathbb{P}^1_{K_R[\![x]\!]}$ with respect to $E$, and by Lemma 6.3.14 the scheme $M_{\mathfrak{m}}$ is the minimal model of $\mathbb{P}^1_{\kappa(\mathfrak{m})[\![x]\!]}$ with respect to $E$. It only remains to show flatness. Let $R_{\mathfrak{m}}$ be the local ring of $R$ at $\mathfrak{m}$ and let $\widehat{R}_{\mathfrak{m}}[\![x]\!]$ be the completion of $R_{\mathfrak{m}}[\![x]\!]$ along its maximal ideal. Then $M \times \operatorname{Spec}(\widehat{R}_{\mathfrak{m}}[\![x]\!])$ is the join of the module classes inside $\widehat{R}_{\mathfrak{m}}[\![x]\!] \oplus \widehat{R}_{\mathfrak{m}}[\![x]\!]$ induced by the module classes $[N]$ and $[N(\alpha_0, \ldots, \alpha_m)]$. Consequently, $M \times \operatorname{Spec}(\widehat{R}_{\mathfrak{m}}[\![x]\!])$ is flat by [42]. As $R_{\mathfrak{m}}[\![x]\!] \longrightarrow \widehat{R}_{\mathfrak{m}}[\![x]\!]$ is a faithfully flat ring homomorphism, it follows by [40, exc. 7.1, p. 53] that

$$M \times \operatorname{Spec}(R_{\mathfrak{m}}[\![x]\!]) \longrightarrow \operatorname{Spec}(R_{\mathfrak{m}}[\![x]\!])$$

is flat and thus $M \longrightarrow \operatorname{Spec}(R[\![x]\!])$ is flat. □

## 6.4 Group structure and $A$-action on $M^*$

Let $M^* \subset M$ be defined as follows. Its reduction tree is the tree of $M \times \operatorname{Spec}(R[\![x]\!]/(x))$, which is a tree of $\mathbb{P}^1_R$'s. Delete from this tree the irreducible components that do not contain an element of $E$ and delete the $\infty$-section. What remains is $\coprod_{e \in E \setminus \{\infty\}} \mathbb{A}^1_R$. Let $M^*$ be the open subscheme in $M$ which is the inverse image of $\coprod_{e \in E \setminus \{\infty\}} \mathbb{A}^1_R$.

**Proposition 6.4.1.** *The generic fibre $\mathbb{P}^1_{K_R(\!(x)\!)}$ induces the following structure on $M^*$:*

*(1) $M^*$ comes equipped with a group scheme structure and an $A$-action.*

*(2) The group law on $M^*$ extends to an action*

$$M^* \times M \longrightarrow M.$$

*(3) The reduction of $M^*$ is as group scheme isomorphic to*

$$\mathbb{G}_{a,R} \times (A/fA)_R,$$

*where $(A/fA)_R$ denotes the constant group scheme over $\operatorname{Spec}(R)$; moreover the $A$-action on $\mathbb{G}_{a,R}$ is given by $\varphi^{\mathrm{td}} \bmod (x) = \psi$.*

*(4) Let $(A/fA)^2_{R[\![x]\!]}$ denote the constant group scheme over $\operatorname{Spec}(R[\![x]\!])$. There is an isomorphism of sheafs of $A$-modules*

$$\overline{\lambda} : (A/fA)^2_{R[\![x]\!]} \longrightarrow \ker(\varphi^{\mathrm{td}}_f, M^*).$$

*Proof.* (1) Write

$$\varphi^{\mathrm{td}}[f](R((x))) = \varphi^{\mathrm{td}}[f](R[\![x]\!]) \oplus W,$$

then $W \cong A/fA$. Note that $W$ is the same set as $\sum \mathbb{F}_q e_i$. Let $e \in W$, then $e$ can be regarded as an element in $\mathbb{P}^1_{R[\![x]\!]}(R[\![x]\!])$. Namely, either $e = (0:1)$ or $e = (z : x^k)$ where $k$ and $z$ are determined by the fact that $z \in R[\![x]\!]^*$. Let $y \in R[\![x]\!]$, then We can add $(y:1)$ and $e$ as follows

$$(y:1) + (z:x^k) = \begin{cases} (y:1) & \text{if } e = (0:1) \\ (yx^k + z : x^k) & \text{otherwise.} \end{cases}$$

Using the construction of the previous section it is not difficult to compute that

$$M^*(R[\![x]\!]) = \{(y:1) + w \mid y \in R[\![x]\!], w \in W\} \subset \mathbb{P}^1_{R((x))}(R((x))).$$

As $W \subset \mathbb{A}^1_{R((x))}(R((x)))$ is an $A$-submodule, as well as

$$\{(a:1) \mid a \in R[\![x]\!]\} \subset \mathbb{A}^1_{R((x))}(R((x))),$$

it follows that $M^*(R[\![x]\!])$ is an $A$-submodule of $\mathbb{A}^1_{R((x))}(R((x)))$. Note that the $A$-action is given by $\varphi^{\mathrm{td}}$.
In general, if $T$ is any $R[\![x]\!]$-algebra, then

$$M^*(T) = \{(t:1) + w \mid t \in T, w \in W\}.$$

Again, $W \subset \mathbb{A}^1_T(T)$ is an $A$-submodule and thus $M^*(T)$ is an $A$-submodule of $\mathbb{A}^1_T(T)$. This induces a group structure and an $A$-action on $M^*$.
(2) From this functorial description of the group law on $M^*$ it is not difficult to see that this group law extends to an action

$$M^* \times M \longrightarrow M$$

by noting that the group action on $\mathbb{A}^1_{R((x))} \subset \mathbb{P}^1_{R((x))}$ extends to an action

$$\mathbb{A}^1_{R((x))} \times \mathbb{P}^1_{R((x))} \longrightarrow \mathbb{P}^1_{R((x))}.$$

(3) Using (1) and the description of the reduction of $M^*$, one sees that $M^*$ is as group scheme isomorphic to

$$\mathbb{G}_{a,R} \times (A/fA)_R.$$

Moreover, this group scheme inherits the $A$-action from $M^*$. The $A$-action on $\mathbb{G}_{a,R}$ is defined by $\varphi^{\mathrm{td}} \bmod (x) = \psi$. The $A$-action on $(A/fA)_R$ comes from the action on $W$ and is consequently the natural one.
(4) This follows from the previous. □

## 6.5 Drinfeld data on the compactification

Section 6.3 gives for every minimal prime $\mathfrak{p} \subset C$ containing $\frac{1}{j}$ a Néron model $M_{\mathfrak{p}}$.

**Definition 6.5.1.** The coproduct

$$M := \coprod M_{\mathfrak{p}}$$

is called the *Néron model of the Tate-Drinfeld module.*

**Remark 6.5.2.** Proposition 6.4.1 also holds for $M$ because it holds for every component of $M$.

The construction of the Néron model of the Tate-Drinfeld module $M$ enables us to replace $L$ by a model $X$ such that the sections in the image of $\overline{\lambda}$ no longer intersect.

**Proposition 6.5.3.** *There is a proper, flat scheme $X \longrightarrow \overline{M}^2(f)$ such that*

*(1)* $X_{|M^2(f)} \cong \mathrm{P}L_{|M^2(f)}$;

*(2)* $X \times_{\overline{M}^2(f)} \mathrm{Spec}(\oplus R[\![x]\!]_{\mathfrak{p}}) \cong M.$

*Proof.* Let as in the proof of Proposition 6.2.1

$$h : B \longrightarrow B[\frac{1}{j}] = C[j] \longrightarrow \widehat{C}[j] \cong \oplus R(\!(x)\!)_{\mathfrak{p}}.$$

Let $(\alpha, \beta) \in (A/fA)^2$. For each $\mathfrak{p}$ there is an element $\xi_{\mathfrak{p}} \in R(\!(x)\!)_{\mathfrak{p}}^*$ with

$$h(\lambda(\alpha, \beta)) = \xi_{\mathfrak{p}} \lambda^{\mathrm{td}}(\alpha, \beta).$$

Here $\lambda^{\mathrm{td}}$ is an abbreviation of $\lambda^{\mathrm{td}}$ restricted to the copy $R[\![x]\!]_{\mathfrak{p}}$. Either $\lambda^{\mathrm{td}}(\alpha, \beta) = 0$ or $\lambda^{\mathrm{td}}(\alpha, \beta) \in R(\!(x)\!)^*$. We may assume that there is an affine open neighbourhood $U_{\mathfrak{p}}$ of $Cusps_{\mathfrak{p}}$ such that for all $(\alpha, \beta)$ for which $\lambda(\alpha, \beta) \neq 0$ the element $\lambda(\alpha, \beta)$ is invertible in $U_{\mathfrak{p}} \backslash Cusps_{\mathfrak{p}}$. Write $U_{\mathfrak{p}} = \mathrm{Spec}(S_{\mathfrak{p}})$. We may assume that $x \in S_{\mathfrak{p}}$ and that $x$ is invertible in $U_{\mathfrak{p}} \backslash Cusps_{\mathfrak{p}}$. The equations (6.1) and (6.2) which we used to define the model $M$, are also defined over $S_{\mathfrak{p}}$. Define $X_{\mathfrak{p}} \longrightarrow U_{\mathfrak{p}}$ to be the scheme given by these equations. Clearly, by definition

$$X_{\mathfrak{p}} \times \mathrm{Spec}(R[\![x]\!]_{\mathfrak{p}}) \cong M_{\mathfrak{p}}.$$

Finally, the restriction of $X_{\mathfrak{p}}$ to $U_{\mathfrak{p}} \backslash Cusps_{\mathfrak{p}}$ is isomorphic to the $\mathbb{P}^1$ over $U_{\mathfrak{p}} \backslash Cusps_{\mathfrak{p}}$. To see this, note that the coefficients of the equations (6.1) and (6.2) only involve $x$ and some of the elements $\lambda^{\mathrm{td}}(\alpha, \beta)$ which are nonzero. Note that both $x$ and the nonzero $\lambda(\alpha, \beta)$ are invertible in $U_{\mathfrak{p}} \backslash Cusps_{\mathfrak{p}}$. Consequently, the restriction of $X_{\mathfrak{p}}$ to $U_{\mathfrak{p}} \backslash Cusps_{\mathfrak{p}}$ is isomorphic to the $\mathbb{P}^1$. Using the same gluing procedure as in the proof of Proposition 6.2.1, we glue the schemes $X_{\mathfrak{p}}$ and $\mathrm{P}L_{|\mathrm{Spec}(B)}$ to a scheme $X \longrightarrow \overline{M}^2(f)$, which has the required properties. ☐

The map $\overline{\lambda}$ extends to a map

$$\overline{\lambda} : (A/fA)^2 \cup \{\infty\} \longrightarrow \Gamma(\overline{M}^2(f), X).$$

By construction the sections in the image of $\overline{\lambda}$ intersect nowhere.

Similar to $M^*$ we define the scheme $X^* \longrightarrow \overline{M}^2(f)$ to be the open part of $X$ obtained by deleting the $\infty$-section from $X$ and by deleting above every cusp of $\overline{M}^2(f)$ the irreducible components which do not carry points of $\overline{\lambda}((A/fA)^2)$. This implies that

$$X^* \times \operatorname{Spec}(\oplus R[\![x]\!]_{\mathfrak{p}}) \cong M^*.$$

Outside the cusps, $X^*$ is simply the affine line over $M^2(f)$ with a group scheme structure, and with an $A$-action induced by the universal rank 2 Drinfeld module. In Proposition 6.4.1 we showed that $M^*$ has a group structure and an $A$-action. The scheme $X^*$ inherits these properties. To describe the structure on $X$ and $X^*$ that we need, we first introduce the following notions; cf. [19].

**Definition 6.5.4.** We have the following definitions:
1. Let $C$ be a connected projective variety over a field $K$, and let $P = (P_1, \ldots, P_n)$ be an $n$-tuple of $k$-rational points of $C$, with $n \geq 3$, then the pair $(C, P)$ is called a *stable $n$-pointed tree* if

(a) every irreducible component of $C$ is isomorphic to $\mathbb{P}^1_k$;

(b) every singular point of $C$ is an ordinary double point and the points of $P$ are regular points;

(c) $C$ is a tree of $\mathbb{P}^1_k$'s;

(d) on every irreducible components there lie at least three special points of $C$, where the special points are the points of $P$ and the singular points of $C$;

2. Let $C \longrightarrow S$ be a projective, flat scheme over $S$ and let $P = (P_1, \ldots, P_n)$ be an $n$-tuple of distinct sections, with $n \geq 3$. The pair $(C, P)$ is called an *stable $n$-pointed tree* if for every $s \in S$ the pair $(C_s, (P_1(s), \ldots, P_n(s)))$ is a stable $n$-pointed tree over the residue field $\kappa(s)$.

**Proposition 6.5.5.** *The scheme $X$ has the following properties:*

*(1) The scheme $X \longrightarrow \overline{M}^2(f)$ together with the set* $\operatorname{im}(\overline{\lambda})$ *is a stable $n$-pointed tree with $n = \#(A/fA)^2 + 1$.*

*(2) The scheme $X^*$ comes naturally equipped with a group scheme structure and an $A$-action. We write*

$$\overline{\varphi} : A \longrightarrow \operatorname{End}_{\mathbb{F}_q}(X^*).$$

*This induces an $A$-module isomorphism*

$$\overline{\lambda} : (A/fA)^2 \xrightarrow{\sim} \ker(\varphi_f, X^*(\overline{M}^2(f))) \subset X^*(\overline{M}^2(f)).$$

*(3) The action $X^* \times X^* \longrightarrow X^*$ extends to an action $X^* \times X \longrightarrow X$.*

In future work we would like to interpret the Drinfeld data $(X, \overline{\varphi}, \overline{\lambda})$ over $\overline{M}^2(f)$ as the scheme which represents the functor

$$\mathcal{F} : A_f - \text{Schemes} \longrightarrow \text{Sets}$$

which sends to every $A_f$-scheme $S$ the set of isomorphy classes of 'generalized Drinfeld modules with level $f$-structure' over $S$. This functor should be similar to the one described in the final part of [56].