Weil pairing and the Drinfeld modular curve

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Chapter 5

Weil Pairing and the Drinfeld Modular Curve

5.1 Introduction

The main theme of this chapter will be the compactification of the Drinfeld modular curve, a description of the scheme of cusps, and a computation of the number of geometric components of the Drinfeld Modular Curve. The description of the compactification is more or less analogous to the description of the classical modular curve as is done by N.M. Katz and B. Mazur in chapter 8 of their book [33]. We will define the universal Tate-Drinfeld module, which is the analogue of the Tate-curve, prove its universal property, and show that the Tate-Drinfeld module describes the formal neighbourhood of the cusps. Once we have described the scheme of cusps, it is not difficult to compute the number of geometric components.

Let us discuss this in slightly more detail. Consider a smooth, projective, geometrically irreducible curve X over $\mathbb{F}_q$, and fix some point ∞ on this curve. Let

$$A := \Gamma(X - \infty, \mathcal{O}_X)$$

be the ring of functions on X which are regular outside ∞. Let $f \in A \backslash \mathbb{F}_q$ be a non-constant element, and let

$$A_f := A[f^{-1}].$$

The moduli schemes $M^r(f)$ play an important role in this chapter. They represent the functor which associates to every A_f-scheme S the set of isomorphy classes of Drinfeld modules with a level f-structure over S.
In this chapter we will address the following problems:

(*i*) Construct a morphism

$$w_f : M^r(f) \longrightarrow M^1(f).$$

This morphism is induced by the Weil pairing for Drinfeld modules. The Weil pairing is defined in the previous chapter.

(*ii*) Define the Tate-Drinfeld module and describe its universal property, using ideas of G. Böckle in chapter 2 of [3] and of M. van der Put and J. Top in [57] and [56].

(*iii*) Describe a compactification $\overline{M}^2(f)$ of $M^2(f)$. The Tate-Drinfeld module will enable us to describe the scheme of cusps

$$Cusps = (\overline{M}^2(f) - M^2(f))^{\text{red}}.$$

(*iv*) Compute the number of geometric components of $M^2(f)$ and describe the cusps of the analogue of the classical curve $X_0(N)$.

Another description of the compactification of $M^2(f)$ can be found in T. Lehmkuhl's 'Habilitation' [37]. The treatment given in this chapter distinguishes itself from the one given in [37] in the following ways. The most important feature here is the use of the Weil pairing. In particular, the morphism w_f is not considered in [37]. This morphism was already known by Drinfeld when he wrote his paper [11], but its interpretation in terms of the Weil pairing is new. Instead of the Tate-data that Lehmkuhl studies, we give an explicit description of the Tate-Drinfeld module. This explicit description provides us with the means which hopefully leads up to an alternative modular interpretation of the compactified modular scheme. In the next chapter we develop the Néron model of the Tate-Drinfeld module, analogous to the Deligne and Rapoport's construction of the Néron model of the Tate-elliptic curve in [10].
Lehmkuhl uses his Tate-data to define the compactification. We use the 'elementary construction', following [33]. This means that we have to do some work in Section 5.9 to show that the Tate-Drinfeld module indeed is the modular interpretation at the cusps.

Sections 5.2 and 5.3 give a brief introduction to the moduli problem and to the moduli schemes. At the end of Section 5.3 we state the assumptions which are used throughout this chapter. In Section 5.2 we recall the definition of Drinfeld modules over schemes and level structures. In Section 5.3 we describe the moduli problem that Drinfeld considers in his original paper [11]. The goal of Section 5.4 is to prove Theorem 5.4.1, i.e., to construct the morphism w_f considered in problem (*i*). In Sections 5.5, 5.6 and 5.7 we discuss problem (*ii*). In Section 5.5 we classify the Drinfeld modules of rank 2 with level f-structure over the quotient field of some complete discrete valuation ring V which have stable reduction of rank 1. The main result of this section is Theorem 5.5.8. In Section 5.6 we define the Tate-Drinfeld module of type $\mathfrak{m}$ with level f-structure. In Section 5.7 we define the universal Tate-Drinfeld module $\mathcal{Z}$; cf. Theorem 5.7.4. For the application to problem (*iii*) we need the universal property of the scheme $\mathcal{Z}$ as stated in Theorem 5.7.8. In Sections 5.8 and 5.9 we study problem (*iii*). In Section 5.8 we give a compactification of $M^2(f)$. This enables us in Section 5.9 to identify the formal neighbourhood of the scheme of cusps $\overline{M}^2(f) - M^2(f)$ with the universal Tate-Drinfeld module. The main results are Proposition 5.9.1 and Theorem 5.9.2. In Section 5.10 we compute, using the scheme of cusps, the number of geometric components for all characteristics; cf. Theorem 5.10.1.

5.2 Drinfeld modules over schemes

Throughout this chapter we will denote the quotient field of any integral domain D by K_D. We recall the definition of Drinfeld modules over schemes and level structures. There are many texts available for a more extensive account of these definitions; cf. [11], [9], [49], [37] and [54].

5.2.1 Line bundles and morphisms

Let B be a commutative $\mathbb{F}_q$-algebra with 1 and let $\mathbb{G}_{a,B}$ denote the additive group over B. The ring of $\mathbb{F}_q$-linear endomorphisms $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,B})$ of $\mathbb{G}_{a,B}$ is isomorphic to the skew polynomial ring $B\{\tau\}$. In this skew polynomial ring multiplication is determined by the rule $\tau b = b^q\tau$ for all $b \in B$.

This can be generalized to schemes. Let S be an $\mathbb{F}_q$-scheme, and let $L \longrightarrow S$ be a line bundle. As usual, L is also a group scheme due to its additive group scheme structure. A *trivialization of* L is a covering $\mathrm{Spec}(B_i)$ of open affines of S together with isomorphisms $L_{|\mathrm{Spec}(B_i)} \cong \mathbb{G}_{a,B_i}$. By $\mathrm{End}_{\mathbb{F}_q}(L)$ we denote the $\mathbb{F}_q$-linear S-group scheme endomorphisms of L. Let $\mathcal{L}$ be the invertible $\mathcal{O}_S$-sheaf corresponding to L, and let

$$\tau^i : \mathcal{L} \longrightarrow \mathcal{L}^{q^i} \quad \text{by} \quad s \mapsto s \otimes \ldots \otimes s.$$

The ring $\mathrm{End}_{\mathbb{F}_q}(L)$ is isomorphic to the ring of all formal expressions $\sum_i \alpha_i\tau^i$ which are locally finite where $\alpha_i : \mathcal{L}^{q^i} \longrightarrow \mathcal{L}$ is an $\mathcal{O}_S$-module homomorphism for every i. Multiplication in the ring of formal expressions is given by $\alpha_i\tau^i\beta_j\tau^j = \alpha_i \otimes \beta_j^{q^i}\tau^{i+j}$. If $\{\mathrm{Spec}(B_i)\}_{i\in I}$ is a trivialization of L, then the restriction of $\mathrm{End}_{\mathbb{F}_q}(L)$ to B_i is simply $B_i\{\tau\}$.
Furthermore, we denote by ∂_0 the point derivation at 0:

$$\partial_0 : \mathrm{End}_{\mathbb{F}_q}(L) \longrightarrow \Gamma(S, \mathcal{O}_S) \quad \text{by} \quad \sum_i \alpha_i\tau^i \mapsto \alpha_0.$$

5.2.2 Drinfeld modules over a scheme

Definition 5.2.1. Let K be an A-field equipped with an A-algebra structure given by $\gamma : A \longrightarrow K$. Let $L = \mathbb{G}_{a,K}$. A *Drinfeld module over* K is a ring homomorphism

$$\varphi : A \longrightarrow \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K})$$

such that

(*i*) $\partial_0 \circ \varphi = \gamma$;

(*ii*) there is an $a \in A$ such that $\varphi_a \neq \gamma(a)$.

A Drinfeld module over a field K has a *rank*, i.e., there is an integer $r > 1$ such that $\deg_\tau \varphi_a = r\deg(a)$ for all $a \in A$.

Definition 5.2.2. Let S be a scheme equipped with a morphism $\gamma_S : S \longrightarrow \mathrm{Spec}(A)$. A *Drinfeld module of rank* r *over* S is a pair $(L, \varphi)_S$ of a line bundle $L \longrightarrow S$ and a ring homomorphism $\varphi : A \longrightarrow \mathrm{End}_{\mathbb{F}_q}(L)$ such that

(*i*) $\partial_0 \circ \varphi = \gamma_S^{\#}$;

(*ii*) For all $a \in A$ the morphism φ_a is finite of degree $q^{r\deg(a)}$.

Remark 5.2.3. The pull-back of a Drinfeld module (L, φ, S) along a morphism

$$\operatorname{Spec}(K) \longrightarrow S$$

for some field S is a Drinfeld module over K in the sense of Definition 5.2.1.

If $S = \operatorname{Spec}(B)$ and L is isomorphic to $\mathbb{G}_{a,B}$, then we simply write φ instead of $(L, \varphi)_S$. The morphism γ_S is called the *characteristic* of $(L, \varphi)_S$. An ideal $\mathfrak{n} \subset A$ is called *away from the characteristic* if $V(\mathfrak{n})$ is disjoint with the image of γ_S.

Definition 5.2.4. A *morphism* ξ *of Drinfeld modules over* S

$$\xi : (L, \varphi)_S \longrightarrow (M, \psi)_S$$

is a map $\xi \in \operatorname{Hom}_{\mathbb{F}_q}(L, M)$ such that $\xi \circ \varphi_a = \psi_a \circ \xi$ for all $a \in A$. A morphism ξ is called an *isomorphism*, if it gives an isomorphism between the line bundles L and M over S. An *isogeny* of Drinfeld modules is a finite morphism.

Remark 5.2.5. An isogeny exists only between Drinfeld modules of the same rank.

Remark 5.2.6. A Drinfeld module $(L, \varphi)_S$ of rank r and a morphism

$$f : T \longrightarrow S$$

give by pull-back rise to a Drinfeld module $(f^*L, f^*\varphi)_T$ over T of rank r.

5.2.3 Level structures

For any non-zero $f \in A$, let $\varphi[f] = \ker(\varphi_f : L \longrightarrow L)$. This is a finite, flat group scheme over S, namely

$$\ker(\varphi_f : L \longrightarrow L) = L \times_L S$$

where the fibre product is taken over $\varphi_f : L \longrightarrow L$ and the unit-section $e : S \longrightarrow L$ of the group scheme L. If $\mathfrak{n} \subset A$ is any non-zero ideal, then

$$\varphi[\mathfrak{n}] = \prod_{f \in \mathfrak{n}} \varphi[f]$$

where the product is the fibre product over L. The scheme $\varphi[\mathfrak{n}]$ is étale over S if and only if $\mathfrak{n}$ is away from the characteristic.

Remark 5.2.7. Let K be an algebraically closed A-field. If $S = \operatorname{Spec}(K)$, then

$$\varphi[\mathfrak{n}] = \operatorname{Spec}(K[X]/(\varphi_f(X), \varphi_g(X)))$$

with $\mathfrak{n} = (f, g)$.

Definition 5.2.8. Suppose that $\mathfrak{n}$ is away from the characteristic, then a level $\mathfrak{n}$-structure λ over S of $(L, \varphi)_S$ is an A-isomorphism

$$\lambda : (A/\mathfrak{n})^r \xrightarrow{\sim} \varphi[\mathfrak{n}](S).$$

Remark 5.2.9. If $\mathfrak{n}$ is not away from the characteristic, then the definition of a level $\mathfrak{n}$-structure is more involved. One can view $\varphi[\mathfrak{n}]$ as an effective Cartier divisor on L. By definition a level $\mathfrak{n}$-structure of $(L,\varphi)_S$ is an A-homomorphism

$$\lambda : (A/\mathfrak{n})^r \longrightarrow L(S)$$

which induces an equality of Cartier-divisors:

$$\sum_{\alpha \in (A/\mathfrak{n})^r} \lambda(\alpha) = \varphi[\mathfrak{n}].$$

We will not expand on this. In this chapter we restrict to the cases for which Definition 5.2.8 is enough.

Let the triple $(L,\varphi,\lambda)_S$ denote a Drinfeld module of rank r over S with level $\mathfrak{n}$-structure λ.

Definition 5.2.10. A *morphism* between two triples $(L,\varphi,\lambda)_S$ and $(M,\psi,\mu)_S$ is a morphism $\xi : (L,\varphi)_S \longrightarrow (M,\psi)_S$ of Drinfeld modules over S such that $\xi(S) \circ \lambda = \mu$ where $\xi(S) : L(S) \longrightarrow M(S)$ is induced by ξ. A morphism is called an *isomorphism* if ξ is an isomorphism of Drinfeld modules.

5.3 The moduli problem

Let $\mathfrak{n} \subset A$ be a non-zero, proper ideal. In his original paper, Drinfeld considers the following moduli problem for Drinfeld modules. Let

$$\mathcal{F}^r(\mathfrak{n}) : A - \text{SCHEMES} \longrightarrow \text{SETS}$$

be the functor which associates to each A-scheme S the set of isomorphy classes of Drinfeld modules over S of rank r with level $\mathfrak{n}$-structure over S. Drinfeld showed the following; cf. Proposition 5.3 and its corollary in [11].

Theorem 5.3.1 (V.G. Drinfeld). *If $\mathfrak{n} \subset A$ is an ideal divisible by at least 2 distinct primes, then there exists a fine moduli space*

$$M^r(\mathfrak{n}) \longrightarrow \mathrm{Spec}(A)$$

representing the moduli problem $\mathcal{F}^r(\mathfrak{n})$. Moreover, this scheme has the following properties:

(i) $M^r(\mathfrak{n})$ is affine and smooth of dimension r;

(ii) $M^r(\mathfrak{n}) \longrightarrow \mathrm{Spec}(A)$ is smooth of relative dimension $r-1$ over $\mathrm{Spec}(A) - V(\mathfrak{n})$.

For arbitrary non-zero ideals $\mathfrak{n} \subset A$, the functor $\mathcal{F}^r(\mathfrak{n})$ has in general only a coarse moduli scheme. This coarse moduli scheme will also be denoted by $M^r(\mathfrak{n})$. We recall here briefly the construction of this scheme; cf. [54] for a nice exposition of this. Let $\mathfrak{n}, \mathfrak{m} \subset A$ be ideals such that $\mathfrak{n}\mathfrak{m}$ is divisible by at least two distinct prime ideals, then by Theorem 5.3.1 there exists an affine scheme $M^r(\mathfrak{n}\mathfrak{m})$ representing the moduli functor $\mathcal{F}^r(\mathfrak{n}\mathfrak{m})$. Let

$(L,\varphi,\lambda)_S$ be a Drinfeld module of rank r with full level $\mathfrak{mn}$-structure λ over an A-scheme S. On these triples the group $\mathrm{Gl}_r(A/\mathfrak{mn})$ acts by

$$\sigma(L,\varphi,\lambda) := (L,\varphi,\lambda\circ\sigma) \quad \text{for all} \quad \sigma\in\mathrm{Gl}_r(A/\mathfrak{n}).$$

Note that $\mathrm{Gl}_r(A/\mathfrak{n})$ is isomorphic to the kernel of the $\bmod\ \mathfrak{m}$ reduction map

$$\mathrm{Gl}_r(\mathfrak{mn}) \longrightarrow \mathrm{Gl}_r(\mathfrak{m}).$$

This induces an action of $\mathrm{Gl}_r(A/\mathfrak{n})$ on $M^r(\mathfrak{mn})$. The coarse moduli scheme of $\mathcal{F}^r(\mathfrak{m})$ is defined as

$$M^r(\mathfrak{m}) := M^r(\mathfrak{mn})/\,\mathrm{Gl}_r(A/\mathfrak{n}).$$

This quotient exists, because $\mathrm{Gl}_r(A/\mathfrak{n})$ is finite. It is, however, not obvious that this scheme is coarse for the given moduli problem. See [54] for a proof of the coarseness of the scheme. The scheme $M^r(\mathfrak{m})$ does not depend on the choice of $\mathfrak{n}$.

5.3.1 Actions on $M^r(\mathfrak{n})$

Let $\hat{A} = \varprojlim A/\mathfrak{n}$, and let $\mathbb{A}_f = \hat{A}\otimes_A K_A$. In this subsection we describe the natural action of $\mathbb{A}_f\cdot\mathrm{Gl}_r(\hat{A})$ on $M^r(\mathfrak{n})$; cf. [11, 5D]. Using this action, we can define the action of $\mathrm{Gl}_r(A/\mathfrak{n})$ and $\mathrm{Cl}(A)$ on $M^r(\mathfrak{n})$. To keep this chapter self-contained, we recall here the treatment given in Section 3.5 of [37], where the reader can find proofs and details.

A *total level structure* of a Drinfeld module $(L,\varphi)_S$ is a homomorphism

$$\kappa : (K_A/A)^r \longrightarrow L(S)$$

such that its restriction to $(\mathfrak{n}^{-1}A/A)^r$ defines a level $\mathfrak{n}$-structure. Let

$$M^r := \varprojlim M^r(\mathfrak{n})$$

where $\mathfrak{n}$ runs through the non-zero ideals of A. This is an affine scheme, and M^r represents the functor which associates to each A-scheme S the set of isomorphy classes of Drinfeld modules with a total level structure over S.

There is a natural action of $\mathrm{Gl}_r(\mathbb{A}_f)$ on M^r, which is defined as follows. Let S be an A-scheme, and let $(L,\varphi,\kappa)_S$ be a Drinfeld module with a total level structure over S. Let $\sigma\in\mathrm{Gl}_r(\mathbb{A}_f)$ such that the entries of σ are elements of $\hat{A}$, then σ gives rise to a map

$$\sigma : (K_A/A)^r \longrightarrow (K_A/A)^r.$$

Let H_σ denote the kernel of σ. The kernel H_σ gives rise to a finite subgroup scheme of L. We can divide out the pair $(L,\varphi)_S$ by this subgroup scheme. This gives us a pair $(L',\varphi')_S$. The following diagram equips the pair $(L',\varphi')_S$ with a total level structure κ':

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_\sigma & \longrightarrow & (K_A/A)^r & \xrightarrow{\sigma} & (K_A/A)^r & \longrightarrow & 0 \\ & & \downarrow & & \kappa\downarrow & & \downarrow\kappa' & & \\ 0 & \longrightarrow & H_\sigma(S) & \longrightarrow & L(S) & \longrightarrow & L'(S). & & \end{array} \qquad (5.1)$$

If σ comes from an element in $A\backslash\{0\}$, then its action is trivial. This implies that we get an action of $\mathrm{Gl}_r(\mathbb{A}_f)/K_A^*$ on $M^r(S)$. As this action is functorial in S, this defines an action of $\mathrm{Gl}_r(\mathbb{A}_f)/K_A^*$ on M^r.

For the moduli scheme $M^r(\mathfrak{n})$ we have $M^r(\mathfrak{n}) = \Gamma(\mathfrak{n})\backslash M^r$ with

$$\Gamma(\mathfrak{n}) := \ker(\mathrm{Gl}_r(\hat{A}) \longrightarrow \mathrm{Gl}_r(A/\mathfrak{n})).$$

The restriction of the universal triple (L, φ, κ) on M^r to $M^r(\mathfrak{n})$ gives the universal pair (φ, λ) on $M^r(\mathfrak{n})$ (Recall that the line bundle of the universal Drinfeld module on $M^r(\mathfrak{n})$ is trivial.) As $\mathbb{A}_f^* \cdot \mathrm{Gl}_r(\hat{A})$ commutes with $\Gamma(\mathfrak{n})$ in $\mathrm{Gl}_r(\mathbb{A}_f)$, it follows that the action of $\mathrm{Gl}_r(\mathbb{A}_f)$ on M^r defines an action of $\mathbb{A}_f^* \cdot \mathrm{Gl}_r(\hat{A})$ on $M^r(\mathfrak{n})$. The normal subgroup $K_A^* \cdot \Gamma(\mathfrak{n}) \subset \mathbb{A}_f^* \cdot \mathrm{Gl}_r(\hat{A})$ acts trivially on $M^r(\mathfrak{n})$. Let

$$G := \mathbb{A}_f^* \cdot \mathrm{Gl}_r(\hat{A})/K_A^* \cdot \Gamma(\mathfrak{n}).$$

As $\mathbb{A}_f^*/K_A^* \cdot \hat{A}^* \cong \mathrm{Cl}(A)$, it is not difficult to see that we have the following exact sequence

$$0 \longrightarrow \mathrm{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^* \longrightarrow G \longrightarrow \mathrm{Cl}(A) \longrightarrow 0.$$

To describe the action of $\mathrm{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^*$, let $\sigma \in \mathrm{Gl}_r(\hat{A})$ and let $\tilde{\sigma}$ be the image of σ under the reduction map $\mathrm{Gl}_r(\hat{A}) \longrightarrow \mathrm{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^*$. Then

$$\sigma : (\varphi, \lambda) \mapsto (\varphi, \lambda \circ \tilde{\sigma}^{-1}).$$

Remark 5.3.2. However, in the sequel we prefer to drop the inverse. If we talk about the action of $\sigma \in \mathrm{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^*$ on (φ, λ), then we mean the action given by

$$\sigma : (\varphi, \lambda) \mapsto (\varphi, \lambda \circ \sigma).$$

Consequently, $\mathrm{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^*$ acts on the right of $M^r(\mathfrak{n})$ and not on the left.

Let $m \in \hat{A} \cap \mathbb{A}_f^*$, then m defines a unique ideal $\mathfrak{m} = (m) \cap A \subset A$. We suppose that $\mathfrak{m}$ is a non-zero, proper ideal which is relatively prime to $\mathfrak{n}$. Let I_r denote the identity element in $\mathrm{Gl}_r(\mathbb{A}_f)$, and let $\sigma = m \cdot \mathrm{I}_r$. We describe the action of σ on M^r. Clearly, $H_\sigma = (\mathfrak{m}^{-1}A/A)^r$, and H_σ maps to $\varphi[\mathfrak{m}](M^r)$ under κ. This means that the isogeny

$$\xi_\mathfrak{m} : (L, \varphi) \longrightarrow (L', \varphi')$$

defined by σ has kernel $\varphi[\mathfrak{m}]$. The total level structure κ' is given by

$$\kappa' = \xi_\mathfrak{m} \circ \kappa \circ m^{-1}.$$

Let φ' denote the restriction of (L', φ') to $M^r(\mathfrak{n})$. Let $\overline{m}$ denote the image of m under the reduction map $\hat{A} \longrightarrow A/\mathfrak{n}$. As $\mathfrak{m} + \mathfrak{n} = A$, we see that $\overline{m} \in (A/\mathfrak{n})^*$. Let φ be the restriction of (L', φ') to $M^r(\mathfrak{n})$, then the action of m on the universal pair (φ, λ) on $M^r(\mathfrak{n})$ is given by

$$m : (\varphi, \lambda) \mapsto (\varphi', \xi_\mathfrak{m} \circ \lambda \circ \overline{m}^{-1}).$$

This describes the action of m on $M^r(\mathfrak{n})$.

Let $\mathfrak{m} \subset A$ be a non-zero ideal relatively prime to $\mathfrak{n}$, i.e., $\mathfrak{m} + \mathfrak{n} = A$. Choose $m \in \hat{A}$ such that $(m) = \mathfrak{m}\hat{A}$ and $m \equiv 1 \bmod \mathfrak{n}$. We define the action of $\mathfrak{m}$ on (φ, λ) to be the action of (m):

$$\mathfrak{m} : (\varphi, \lambda) \mapsto (\varphi', \xi_{\mathfrak{m}} \circ \lambda).$$

This action is well-defined: the chosen element m is unique up to an element $\alpha \in \hat{A}^*$ with $\alpha \equiv 1 \bmod \mathfrak{n}$. For such an element α we have $\alpha \mathrm{I}_r \in \Gamma(\mathfrak{n})$. Consequently, $m \cdot \mathrm{I}_r$ and $\alpha m \cdot \mathrm{I}_r$ give the same element in G.
Using this, we can define the *action of* $\mathrm{Cl}(A)$ *on* $M^r(\mathfrak{n})$. First note that by Lemma 5.6.4 we can represent every class in $\mathrm{Cl}(A)$ by a non-zero ideal $\mathfrak{m}$ with $\mathfrak{m} + \mathfrak{n} = A$. Namely, suppose that $\mathfrak{m}$ and $\mathfrak{m}'$ are both non-zero ideals relatively prime to $\mathfrak{n}$ which represent the same class in $\mathrm{Cl}(A)$, then there is an element $x \in K_A^*$ with $\mathfrak{m} = x\mathfrak{m}'$. Let $m, m' \in \hat{A}$ be elements which define the action of $\mathfrak{m}$ and $\mathfrak{m}'$, respectively. Then there is an element $\alpha \in \hat{A}^*$ with $\alpha \equiv 1 \bmod \mathfrak{n}$ such that $m' = x\alpha m$ with $x\alpha \cdot \mathrm{I}_r \in K_A^* \cdot \Gamma(\mathfrak{n})$. Therefore, (m) and (m') give the same element in G.

These considerations also imply the following lemma:

Lemma 5.3.3. *The exact sequence*

$$0 \longrightarrow \mathrm{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^* \longrightarrow G \longrightarrow \mathrm{Cl}(A) \longrightarrow 0$$

splits. Therefore,

$$G \cong \mathrm{Cl}(A) \times \mathrm{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^*.$$

Remark 5.3.4. Let us once more stretch that we use the convention that the action of $\sigma \in \mathrm{Gl}_r(A/\mathfrak{n})$ on $M^r(\mathfrak{n})$ is given by

$$\sigma : (\varphi, \lambda) \longrightarrow (\varphi, \lambda \circ \sigma).$$

So $G = \mathrm{Cl}(A) \times \mathrm{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^*$ acts on $M^r(\mathfrak{n})$ on the right.

5.3.2 Assumptions in this chapter

In this chapter we will make the following two assumptions.

(1) Throughout the chapter we will assume that $\mathfrak{n} = (f)$ is a non-zero, proper, principal ideal. This simplifies the description of the Tate-Drinfeld module. Dropping this assumption does not seem to give rise to different results.
If $\mathfrak{m} \subset A$ is a non-zero proper ideal containing f, then by the previous, we see that $M^r(\mathfrak{n}) = M^r(f)/G$ where G is given by dividing out the action of the kernel

$$\ker(\mathrm{Gl}_r(A/fA) \longrightarrow \mathrm{Gl}_r(A/\mathfrak{n})).$$

(2) We will not consider the moduli problem over A-schemes, but over A_f-schemes S, i.e., f is invertible in S. This implies that f is away from the characteristic of $(L, \varphi)_S$. This assumption is used because the Weil pairing plays an important role

in our description, and in the previous chapter we did not describe the Weil pairing for f-torsion which is not away from the characteristic.
This assumption implies that a level f-structure is an isomorphism

$$\lambda : (A/fA)^r \xrightarrow{\sim} \varphi[f](S).$$

So in this chapter we will be considering the moduli problem

$$\mathcal{F}^r(f) : A_f - \text{SCHEMES} \longrightarrow \text{SETS}$$

which associates to each A_f-scheme S, the set of isomorphy classes of Drinfeld modules of rank r with full level f-structure over S. We write $M^r(f)$ for the scheme which represents $\mathcal{F}^r(f)$. It follows from the proof of Theorem 5.3.1 that the moduli scheme $M^r(f)$ is a fine moduli scheme if $f \neq 1$.

Throughout this chapter we will write $\mathrm{Spec}(R) = M^1(f)$. The ring R is regular and $M^1(f)$ is connected. In fact, R is the integral closure of A_f in a field extension of K_A. The Galois group of K_R/K_A is the group $G \cong (A/fA)^*/\mathbb{F}_q^* \times \mathrm{Cl}(A)$ that we discussed above; cf. Section 8 in [11].

5.4 The Weil pairing on the modular schemes.

In this section we will show that the Weil pairing for Drinfeld modules over an A_f-field K as defined in the previous chapter gives rise to the following theorem:

Theorem 5.4.1. *The Weil pairing induces an A_f-morphism*

$$w_f : M^r(f) \longrightarrow M^1(f).$$

The Weil pairing is equivariant with respect to the action of $\mathrm{Cl}(A) \times \mathrm{Gl}_r(A/fA)$.

Let (φ, λ) be a Drinfeld module φ of rank r over K with level f-structure. The Weil pairing is an A/fA-isomorphism

$$w_f : \wedge^r \varphi[f](K) \xrightarrow{\sim} \psi[f](K) \otimes_A \Omega_f^{\otimes r-1}.$$

It is unique up to a unique isomorphism of ψ. Once and for all we fix a generator ω of the A/fA-module Ω_f. This gives an A/fA-isomorphism

$$w_f : \wedge^r \varphi[f](K) \xrightarrow{\sim} \psi[f](K).$$

The level f-structure λ induces a canonical isomorphism

$$\wedge^r \lambda : \wedge^r (A/fA)^r \xrightarrow{\sim} \wedge^r \varphi[f](K).$$

Because $\wedge^r (A/fA)^r$ is canonically isomorphic to A/fA, ψ comes equipped with a level f-structure μ over K via the following commutative diagram:

$$\begin{array}{ccc} \wedge^r (A/fA)^r & \xrightarrow{\wedge^r \lambda} & \wedge^r \varphi[f](K) \\ \uparrow & & \downarrow{\scriptstyle w_f} \\ A/fA & \xrightarrow{\mu} & \psi[f](K). \end{array}$$

Note that if ξ is an isomorphism between ψ and ψ', then also the pairs (ψ, μ) and (ψ', μ') are isomorphic via ξ. Here μ and μ' are defined by the previous diagram by ψ and ψ' respectively. So the pair (ψ, μ) is unique up to isomorphy. These considerations show the following:

Lemma 5.4.2. *The Weil pairing gives for all A_f-fields K rise to a map*

$$w_K : M^r(f)(K) \longrightarrow M^1(f)(K).$$

This map depends functorially on K.

Proof. The construction of the map w_K is described above. It associates to each isomorphy class (φ, λ) of rank r over K a unique isomorphy class (ψ, μ) of rank 1 over K. That this map depends functorially on K follows immediately from the construction in terms of A-motives. □

From this lemma, we proceed as follows to prove the existence of the map w_f.

Proof of Theorem 5.4.1. Let (φ, λ) be the universal pair over $M^r(f)$. The moduli scheme $M^r(f)$ is affine and regular over $\mathrm{Spec}(A_f)$. So we may write

$$M^r(f) \cong \coprod_{i=0}^{n} \mathrm{Spec}(S_i)$$

such that each S_i is an integrally closed domain of relative dimension $r - 1$ over A_f. Moreover, from Drinfeld's description we know that

$$M^1(f) = \mathrm{Spec}(R)$$

where R is an integrally closed domain. Let K_{S_i} be the quotient field of S_i. Lemma 5.4.2 gives rise to a unique isomorphy class

$$(\psi, \mu) \in M^1(f)(\prod_i K_{S_i}).$$

This means that there exists a unique A_f-ring homomorphism

$$h : R \longrightarrow \prod_j K_{S_j}.$$

Hence, the theorem follows if we can show that $h(R) \subset \prod_j S_j$.
Let $\mathfrak{p} \in \mathrm{Spec}(S_j)$ be a closed point of height 1. By Lemma 5.4.3 it follows that there is a map $\mathrm{Spec}(S_{j,\mathfrak{p}}) \longrightarrow M^1(f)$, inducing the map

$$\mathrm{Spec}(K_{S_j}) \longrightarrow M^1(f).$$

Consequently,

$$h(R) \cap K_{S_j} \subset S_{j,\mathfrak{p}}.$$

As $S_j = \cap_{\mathfrak{p}} S_{j,\mathfrak{p}}$ where the intersection runs over all primes of height 1 in S_j, one has $h(R) \cap K_{S_j} \subset S_j$.

For the $\mathrm{Cl}(A) \times \mathrm{Gl}_r(A/fA)$-equivariance of w_f, note that the $\mathrm{Gl}_r(A/fA)$-equivariance is obvious. Let $\mathfrak{m} \subset A$ be a non-zero, proper ideal with $f \notin \mathfrak{m}$ representing an element in $\mathrm{Cl}(A)$. For the $\mathrm{Cl}(A)$-equivariance we recall its definition in the previous section. Using the notations of the previous section, we see that the action of $\mathfrak{m}$ on (φ, λ) is given by

$$(\varphi, \lambda) \mapsto (\varphi', \xi_{\mathfrak{m}} \circ \lambda),$$

with $\xi_{\mathfrak{m}}\varphi_a = \varphi'_a\xi_{\mathfrak{m}}$ for all $a \in A$. Let (ψ, μ) be the image of (φ, λ) under the Weil pairing, and let (ψ', μ') be the image of $(\varphi', \xi_{\mathfrak{m}} \circ \lambda)$ under w_f. Let $F = \prod_i \overline{K}_{S_i}$. The isogeny $\xi_{\mathfrak{m}}$ induces an isogeny $\zeta_{\mathfrak{m}} : \psi \longrightarrow \psi'$. The kernel of $\zeta_{\mathfrak{m}}$ is

$$\ker(\zeta_{\mathfrak{m}})(F) = \wedge^r \ker(\xi_{\mathfrak{m}})(F) = \wedge^r \varphi[\mathfrak{m}](F) \cong \psi[\mathfrak{m}](F).$$

Therefore, the action of $\zeta_{\mathfrak{m}}$ on ψ coincides with the action of $\mathfrak{m}$ on ψ. So we have

$$\mathfrak{m} : (\psi, \mu) \mapsto (\psi', \zeta_{\mathfrak{m}} \circ \mu) = (\psi', \mu').$$

□

Lemma 5.4.3. *Let S be a regular local A_f-ring, let K_S be its quotient field, and let $(\varphi, \lambda) \in M^r(f)(S)$. The unique class $(\psi, \mu) \in M^1(f)(K_S)$ associated to (φ, λ) by the Weil pairing comes from a unique class $(\psi', \mu') \in M^1(f)(S)$ via the canonical embedding $S \longrightarrow K_S$.*

Proof. Via the ring homomorphism $S \longrightarrow K_S$ we may view the pair (φ, λ) over K_S, and we can associate via the Weil pairing a pair (ψ, μ) of rank 1 over K_S. We want to prove that there is a representing pair in the isomorphy class of (ψ, μ) which is defined over S. Let V be the set of all height one primes of S. The ring S is a UFD; cf. Theorem 20.3 in [40]. Consequently, every $\mathfrak{p} \in V$ is of the form $\mathfrak{p} = (h_{\mathfrak{p}})$ for some irreducible $h_{\mathfrak{p}} \in S$; cf. Theorem 20.1 in [40]. Let $v_{\mathfrak{p}}$ denote the valuation at $\mathfrak{p}$. The invertible elements of S are given by

$$S^* = \{s \in S \mid v_{\mathfrak{p}}(s) = 0 \text{ for all } \mathfrak{p} \in V\}.$$

Let $S_{\mathfrak{p}}$ denote the local ring of S at $\mathfrak{p}$. This is a discrete valuation ring.
As the f-torsion of ψ is K_S-rational, it follows that ψ has good reduction at every $\mathfrak{p} \in V$. This implies that for all $\mathfrak{p} \in V$ there is an element $x_{\mathfrak{p}} \in K_S$ such that $x_{\mathfrak{p}}\psi x_{\mathfrak{p}}^{-1}$ is a Drinfeld module of rank 1 defined over $S_{\mathfrak{p}}$. In fact, we may assume $x_{\mathfrak{p}} = h_{\mathfrak{p}}^{m_{\mathfrak{p}}}$ for some $m_{\mathfrak{p}} \in \mathbb{Z}$. There are only finitely many $m_{\mathfrak{p}} \neq 0$, as we show below. So we can define $x = \prod_{\mathfrak{p} \in V} h_{\mathfrak{p}}^{m_{\mathfrak{p}}}$. The pair $(x\psi x^{-1}, x\mu)$ is defined over S. So this is the pair that we are looking for.
To see that there are only finitely many $m_{\mathfrak{p}} \neq 0$, let $a \in A \backslash \mathbb{F}_q$ and consider the leading coefficient c of ψ_a. Under the isomorphism $x_{\mathfrak{p}}$, the leading coefficient becomes $x_{\mathfrak{p}}^{1-q^{\deg(a)}}c$. As $v_{\mathfrak{p}}(c) = 0$ for all but finitely many $\mathfrak{p}$'s, it follows that $m_{\mathfrak{p}} = 0$ for all but finitely many $\mathfrak{p}$'s. □

Proposition 5.4.4. *The morphism w_f induces the maps w_K for any A_f-field K, where w_K is as in Lemma 5.4.2.*

Proof. By the functoriality in K of the maps w_K, it suffices to prove the statement for algebraically closed fields $\overline{K}$. So let $\overline{K}$ be algebraically closed and let

$$\zeta : \mathrm{Spec}(\overline{K}) \longrightarrow M^r(f)$$

be a geometric point of $M^r(f)$. If ζ is a generic point of one of the connected components of $M^r(f)$, then w_f induces $w_{\overline{K}}$ by construction. If ζ is a closed point we have to do something more. Clearly, ζ factors over $\mathrm{Spec}(k_\zeta)$ where k_ζ is the residue field at (the image of) ζ. To see that w_{k_ζ} and consequently $w_{\overline{K}}$ is induced by w_f, we have to dive into the language of A-motives a little; cf. the previous chapter.
Let $V = \mathcal{O}_\zeta$, then V is a regular local A_f-ring and let (φ, λ) be defined over V of rank r. Let K_V be the quotient field of V. Then the construction of A-motives associates to φ the Drinfeld module ψ of rank 1 for which

$$M(\psi) \cong \wedge^r M(\varphi).$$

By Lemma 5.4.3, the pair (ψ, μ) is also defined over V. Because φ is defined over V, it makes sense to consider $M^0(\varphi) = V\{\tau\}$ with the obvious $V\{\tau\} \otimes_{\mathbb{F}_q} A$-action such that

$$M^0(\varphi) \otimes_V K_V = M(\varphi).$$

Similarly, we can define $M^0(\psi)$ because ψ is defined over V. Clearly,

$$\wedge^r_{K_V \otimes A} M(\varphi) \cong K_V \otimes_V \wedge^r_{V \otimes A} M^0(\varphi).$$

Consequently, the $K_V\{\tau\} \otimes A$-isomorphism

$$M(\psi) \cong \wedge^r M(\varphi)$$

comes from a $V\{\tau\} \otimes A$-isomorphism

$$M^0(\psi) \cong \wedge_{V \otimes A} M^0(\varphi).$$

This construction can be reduced modulo the maximal of V. This gives us the construction of w_{k_ζ}. Therefore, w_f induces $w_{\overline{K}}$. □

5.5 Drinfeld modules of rank 2 with stable reduction of rank 1

For this section we fix the following notation. Let V be a complete discrete valuation ring which is also an A_f-algebra. Let K_V be the quotient field of V, and let $\pi \in V$ be a generator of the maximal ideal of V. Let $v(x)$ denote the π-valuation of x for every $x \in V$.

Definition 5.5.1. Let φ be a Drinfeld module of rank r over K_V, then φ has *stable reduction at v of rank r'* if φ is isomorphic over K_V to a Drinfeld module φ' over K_V such that for all $a \in A$ each coefficient $\beta_i(a)$ of the sum $\varphi'_a = \sum \beta_i(a)\tau^i$ is an element of V and the reduction φ' mod πV is a Drinfeld module of rank r' over $V/\pi V$.
The Drinfeld module φ has *potentially stable reduction at v of rank r'* if there is a finite field extension L of K_V and a valuation w of L extending v such that φ has stable reduction at w of rank r'.

Let φ be a Drinfeld module of rank 2 over K_V with full level f-structure λ over K_V such that φ has potentially stable reduction of rank 1. The goal of this section is Theorem 5.5.8 which describes the pairs (φ, λ) in terms of Drinfeld modules of rank 1 and lattices.

Lemma 5.5.2. *Let φ be a Drinfeld module with K_V-rational f-torsion. If φ has potentially stable reduction at (π), then φ has stable reduction at (π).*

Proof. As φ has potentially stable reduction, there is an element $k \in \mathbb{Q}$ such that for all $x \in K_V^{\mathrm{sep}}$ with $v(x) = k$ we have the following: $\tilde{\varphi} := x\varphi x^{-1}$ is a Drinfeld module, $\tilde{\varphi}_a$ has all coefficients in V for all $a \in A$, and $\tilde{\varphi}$ mod πV is a Drinfeld module over $V/\pi V$. Cf. Section 4.10 in [22]. As $f \in V^*$, it is not difficult to see that k is the smallest slope of the Newton polygon of $\frac{1}{X}\varphi_f(X)$. Therefore,

$$k = \max\{v(\alpha) \mid \alpha \in \varphi[f](\overline{K}_V)\backslash\{0\}\}.$$

As the f-torsion of φ is K_V-rational, we have $k \in \mathbb{Z}$. Therefore, we may choose $x \in K_V$. □

To abbreviate notation, we introduce the following two properties P and P'. Let χ be a Drinfeld module of rank r over K_V.

$P(\chi)$: χ has stable reduction of rank 1.

$P'(\chi)$: χ has stable reduction of rank 1, χ_a has all coefficients in V for all $a \in A$ and $\chi[f](V) \cong A/fA$.

Suppose that κ is a level f-structure of χ over K_V. If $r = 1$, then $P'(\chi)$ implies that (χ, κ) is defined over V.

By Lemma 5.5.2 we have $P(\varphi)$ for the pair (φ, λ). Therefore, we may assume that φ_a has all its coefficients in V for all $a \in A$. Moreover, we have that the smallest slope of the Newton polygon of φ_f equals $v(\varphi_f) = 0$. Consequently, $\varphi[f](V) \cong A/fA$. So in the isomorphy class of (φ, λ), there is a representing element (φ, λ) with $P'(\varphi)$ and this element is unique up to V^*. In fact,

$$\{(\varphi, \lambda)_{K_V} \text{with } P'(\varphi)\}/V^* \overset{\text{bij}}{\cong} \{(\varphi, \lambda)_{K_V} \text{with } P(\varphi)\}/K_V^*. \tag{5.2}$$

Note that the Weil pairing equips V with an R-structure. The isomorphy class of (φ, λ) is induced by an A_f-morphism $\mathrm{Spec}(K_V) \longrightarrow M^r(f)$. Composing this morphism with w_f gives rise to an A_f-linear ring homomorphism $R \longrightarrow K_V$. As R is integral over A_f, it follows that this ring homomorphism gives an A_f-linear ring homomorphism $h : R \longrightarrow V$.

5.5.1 Drinfeld's bijection without level structure

To classify the isomorphy classes (φ, λ) with stable reduction of rank 1, we recall Drinfeld's classification of Drinfeld modules of rank 2 with potentially stable reduction of rank 1; cf. Proposition 7.2 in [11].

An *A-lattice of rank 1 in K_V^{sep}* is a projective A-module of rank 1 which lies discretely in K_V^{sep} and which is invariant under the action of $G_{K_V} := \mathrm{Gal}(K_V^{\mathrm{sep}}/K_V)$. Two A-lattices Λ_1 and Λ_2 are called *isomorphic* if there is an element $x \in (K_V^{\mathrm{sep}})^*$ such that $x\Lambda_1 = \Lambda_2$. Then Drinfeld's result states the following:

Theorem 5.5.3 (V.G. Drinfeld). *There is a bijection between the set of isomorphy classes over K_V of Drinfeld modules of rank 2 over V with potentially stable reduction of rank 1 and the set of isomorphy classes over K_V of pairs (ψ, Λ), where ψ is a Drinfeld module of rank 1 over V and Λ is an A-lattice of rank 1 inside K_V^{sep}.*

Sketch of the proof. Applying Proposition 5.2 in [11] for the rings $V_n = V/(v)^n$ with $n \in \mathbb{Z}_{\geq 1}$ gives us the existence of unique elements $s_n \in V_n\{\tau\}$ such that $s_n \varphi_a s_n^{-1}$ is in standard rank 1 form over V_n. Moreover, each s_n has the form

$$s_n = 1 + \sum_{i=1}^{k_n} v_i \tau^i \quad \text{with} \quad v_i \in (v).$$

Let $s = \varprojlim s_n$, then s is an element in $V\{\{\tau\}\}$, the set of skew formal power series in τ over V. In the proof of Proposition 7.2 in [11], Drinfeld shows that the homomorphism s is in fact analytic. One has by construction

$$s = 1 + \sum_{i \geq 1} v_i \tau^i \quad \text{with} \quad v_i \in (v).$$

This implies both that $\Lambda := \ker(s)$ is contained in K_V^{sep}. Moreover, each element in $\Lambda \backslash \{0\}$ has strictly negative valuation.
Let $\psi' = s \varphi s^{-1}$, then $\psi' \mod (v) = \psi$. We get the following diagram, which is commutative for all $a \in A$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & K_V^{\mathrm{sep}} & \xrightarrow{e_\Lambda} & K_V^{\mathrm{sep}} \\ & & \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \varphi_a} \\ 0 & \longrightarrow & \Lambda & \longrightarrow & K_V^{\mathrm{sep}} & \xrightarrow{e_\Lambda} & K_V^{\mathrm{sep}}, \end{array} \tag{5.3}$$

where

$$e_\Lambda(z) = z \prod_{\alpha \in \Lambda \backslash \{0\}} \left(1 - \frac{z}{\alpha}\right) = s(z).$$

Let $a \in A \backslash \mathbb{F}_q$, then $\psi_a^{-1}\Lambda$ is mapped surjectively to $\varphi[a]$, hence $\psi_a^{-1}\Lambda/\Lambda \cong (A/aA)^2$ as A-module. On the other hand, the kernel of the surjective map

$$\psi_a^{-1}\Lambda/\Lambda \longrightarrow \Lambda/\psi_a\Lambda$$

is isomorphic to $\psi[a](K_V^{\mathrm{sep}})$. So we may conclude that $\Lambda/\psi_a\Lambda \cong A/aA$. This implies that Λ is a projective A-module of rank 1. We already saw that Λ consists of elements with strictly negative valuation, hence Λ lies discretely in K_V^{sep}. Finally, for any element $\sigma \in G_{K_V}$, we have $\sigma \circ s(z) = s \circ \sigma(z)$, hence Λ is G_{K_V}-invariant. We conclude that Λ is an A-lattice of rank 1 in K_V^{sep}. □

5.5.2 The bijection with level f-structure

Let $(\psi^{\mathrm{un}}, \mu^{\mathrm{un}})$ be the universal pair of rank 1 defined over R. We will assume that $\mu^{\mathrm{un}}(1) = 1$, which is possible because f is invertible in R. Drinfeld's construction in

Theorem 5.5.3 lifts the rank 1 Drinfeld module φ mod πV to a unique Drinfeld module ψ of rank 1 defined over V. Also, the f-torsion of ψ is V-rational. We would like to equip ψ with a natural level f-structure μ which comes from λ. For this we use the ring homomorphism

$$h : R \longrightarrow V$$

which arises from the Weil pairing.
Suppose that ψ is equipped with a level f-structure $\tilde{\mu}$, then the isomorphy class of $(\psi, \tilde{\mu})$ comes from an A_f-linear ring homomorphism

$$\tilde{h} : R \longrightarrow V,$$

i.e., there is a unique element $v \in V^*$ such that

$$\tilde{h}((\psi^{\mathrm{un}}, \mu^{\mathrm{un}})) = (v\psi v^{-1}, v\tilde{\mu}).$$

The pair $(\tilde{h}, v)$ uniquely determines $(\psi, \tilde{\mu})$.

As R is integral over A_f, there exists an A_f-automorphism

$$g_\mu : R \longrightarrow R$$

with $g_{\tilde{\mu}} \in G = \mathrm{Gal}(R/A_f)$ such that $\tilde{h} = h \circ g_{\tilde{\mu}}$. We described this Galois group in Section 5.3:

$$G \cong \mathrm{Cl}(A) \times (A/fA)^*/\mathbb{F}_q^*.$$

So the element $g_{\tilde{\mu}}$ is given by a pair $(\mathfrak{m}, \sigma) \in \mathrm{Cl}(A) \times (A/fA)^*/\mathbb{F}_q^*$.
If we have another level f-structure μ', then $\mu' = \alpha\tilde{\mu}$ for some $\alpha \in (A/fA)^*$ and we see that $g_{\mu'}$ corresponds to the pair $(\mathfrak{m}, \alpha\sigma)$.

We equip ψ with the level f-structure μ such that g_μ is given by the pair $(\mathfrak{m}, 1)$. More specifically, let $h_\mu := h \circ g_\mu$. Let $v \in V^*$ be an element with $v\psi v^{-1} = \tilde{h}(\psi^{\mathrm{un}})$, then this element v is unique up to $\mathbb{F}_q^*$. Let (ψ, μ) be the pair determined by $(\tilde{h}, v)$.
The pair (ψ, μ) that we obtain in this way is unique up to $\mathbb{F}_q^*$. In particular, the isomorphy class of (ψ, μ) is uniquely determined in this way.

Remark 5.5.4. To make the choice of an element in $\mathbb{F}_q^*$ in the above construction 'visible', we consider the set of elements z in $\varphi[f](K_V)\backslash\varphi[f](V)$ with

$$w_f(z, e_\Lambda(\mu(1))) \in \mathbb{F}_q^* \cdot w_f(\lambda(1,0), \lambda(0,1)).$$

The set of these elements equals $\mathbb{F}_q^* \cdot z'$ where z' is any element in this set. Fix one of the elements those elements z. We can now equip ψ with the unique μ such that

$$w_f(z, e_\Lambda(\mu(1))) = w_f(\lambda(1,0), \lambda(0,1)).$$

In this way, we can associate to a triple (φ, λ, z) a unique triple (ψ, μ, Λ).

Our next goal is to prove Theorem 5.5.8. Drinfeld's construction gives an A-lattice Λ of rank 1 such that the pair (ψ, Λ) determines φ and vice versa.
As in Drinfeld's proof, we can consider ψ_a as a map

$$\psi_a : K_V^{\text{sep}} \longrightarrow K_V^{\text{sep}}$$

for every $a \in A$. Because $f \notin \ker(A_f \longrightarrow K_V)$, it follows that all roots of the equation $\psi_a(X) = \alpha$ lie in K_V^{sep} for all $\alpha \in \Lambda$. And thus we have

$$(\psi_f)^{-1}\Lambda \subset K_V^{\text{sep}},$$

as in Drinfeld's proof.

Lemma 5.5.5. *If the f-torsion of φ is K_V-rational, then $(\psi_f)^{-1}\Lambda \subset K_V$. In particular $\Lambda \subset K_V$.*

Proof. Cf. Lemma 2.4 in [3]. By definition Λ is invariant under G_{K_V}. Consequently, also $(\psi_f)^{-1}\Lambda$ is invariant under G_{K_V} as the coefficients of ψ_f lie in V. In fact, the action of G_{K_V} on $(\psi_f)^{-1}\Lambda$ splits according to the following splitting exact sequence of A-modules:

$$0 \longrightarrow \psi[f](K_V) \longrightarrow (\psi_f)^{-1}\Lambda \longrightarrow ((\psi_f)^{-1}\Lambda)_{\text{proj}} \longrightarrow 0.$$

The A-module $((\psi_f)^{-1}\Lambda)_{\text{proj}}$ is the projective part of $(\psi_f)^{-1}\Lambda$, which is isomorphic to Λ. By Drinfeld's construction there is an analytic map e_Λ with the commuting property as in diagram (5.3), and this map commutes with the action of G_{K_V}. Moreover, via this map $\varphi[f]$ is isomorphic to $(\psi_f)^{-1}\Lambda/\Lambda$. By the assumption that $\varphi[f]$ is K_V-rational, it follows that G_{K_V} acts trivially on $(\psi_f)^{-1}\Lambda/\Lambda$ and thus on $\psi[f]$. The latter fact implies that the only action of G_{K_V} on $(\psi_f)^{-1}\Lambda$ is on the projective part of $(\psi_f)^{-1}\Lambda$, hence this action gives a subgroup G of $\mathrm{Gl}_1(A) = \mathbb{F}_q^*$. On the other hand, this subgroup G maps injectively into $\mathrm{Aut}_A((\psi_f)^{-1}\Lambda/\Lambda)$. But G_{K_V} acts trivially on $(\psi_f)^{-1}\Lambda/\Lambda$, hence G is trivial. ☐

So we see that we can associate to a pair (φ, λ) with $P'(\varphi)$ a triple (ψ, μ, Λ) such that (ψ, μ) is defined over V and Λ is an A-lattice with $(\psi_f)^{-1}\Lambda \subset K_V$. This triple is unique up to $\mathbb{F}_q^*$. I.e., the pair (φ, λ) determines a unique element in $\{(\psi, \Lambda, \mu)\}/\mathbb{F}_q^*$.

Note, however, that the level f-structure λ is not the unique level structure such that (φ, λ) is mapped to this unique element in $\{(\psi, \mu, \Lambda\}/\mathbb{F}_q^*$. In fact, if $\sigma \in \mathrm{Gl}_2(A/fA)$, then $(\varphi, \lambda \circ \sigma) \mapsto \{(\psi, \mu, \Lambda)\}/\mathbb{F}_q^*$ if and only if $\det(\sigma) \in \mathbb{F}_q^*$. This is due to the fact that the Weil pairing is $\mathrm{Gl}_2(A/fA)$-equivariant. Therefore, if one changes λ by σ, then the morphism h induced by the Weil pairing w_f changes by $\det(\sigma)$. Consequently, the triple (ψ, Λ, μ) changes by $\det(\sigma)$. Let Σ denote the subgroup of $\mathrm{Gl}_2(A/fA)$ given by

$$\Sigma = \{\sigma \in \mathrm{Gl}_2(A/fA) \mid \det(\sigma) \in \mathbb{F}_q^*\}.$$

The previous shows that the map

$$\{(\varphi, \lambda)\}/\Sigma \longrightarrow \{(\psi, \mu, \Lambda)\}/\mathbb{F}_q^*$$

is injective.

Remark 5.5.6. Again, once we have chosen an element z as in Remark 5.5.4, we get an injective map

$$\{(\varphi,\lambda,z)\}/\mathrm{Sl}_2(A/fA) \longrightarrow \{(\psi,\mu,\Lambda)\}.$$

On the other hand, let a triple (ψ,μ,Λ) be given. We show that there exists a pair (φ,λ) such that under the above construction (φ,λ) is mapped to the class $\{(\psi,\mu,\Lambda)\}/\mathbb{F}_q^*$. The triple (ψ,μ,Λ) gives rise to the following:

(1) a morphism $\tilde{h} : R \longrightarrow V$ which induces (ψ,μ) on V;

(2) the pair (ψ,Λ) gives a Drinfeld module φ of rank 2;

(3) as the f-torsion of φ comes from $(\psi_f)^{-1}\Lambda/\Lambda$, the f-torsion of φ is K_V-rational, and thus φ has $P'(\varphi)$.

We equip φ with a level f-structure as follows: define $\lambda(0,1) := e_\Lambda(\mu(1))$ and let $\lambda(1,0) = z$ for some element $z \in \varphi[f](K_V)\backslash\varphi[f](V)$. Any z gives rise to a ring homomorphism $h_z : R \longrightarrow V$ induced by the Weil pairing w_f. As before, there exists an A_f-automorphism g_z of R with $\tilde{h}\circ g_z = h$, and g_z is given by a pair $(\mathfrak{m},\sigma) \in \mathrm{Cl}(A)\times(A/fA)^*$. We choose an element z for which σ is the identity. As before, z is unique up to a choice of $\mathbb{F}_q^*$. Clearly, the pair (φ,λ) is mapped to the class $\{(\psi,\mu,\Lambda)\}/\mathbb{F}_q^*$.
This shows that we have a bijection between the set

$$\{\text{all pairs } (\varphi,\lambda) \text{ with } P'(\varphi)\}/\Sigma$$

and the set

$$\{\text{all triples } (\psi,\mu,\Lambda) \text{ with } (\psi,\mu) \text{ over } V \text{ and } (\psi_f)^{-1}\Lambda \subset K_V\}/\mathbb{F}_q^*.$$

Remark 5.5.7. Similarly, the above argument shows that the injective map of Remark 5.5.6 is a bijection.

This bijection can be rephrased in terms of isomorphy classes of pairs (φ,λ) and triples (ψ,μ,Λ) as follows. We say that a triple (ψ,μ,Λ) is defined over V if the pair (ψ,μ) is defined over V. Two triples (ψ,μ,Λ) and (ψ',μ',Λ') over V are called *isomorphic* if there exists an element $v \in V^*$ with

$$(v\psi v^{-1}, v\mu, v\Lambda) = (\psi',\mu',\Lambda').$$

Note that

$$\{(\psi,\mu,\Lambda) \text{ over } V\}/V^* \overset{\text{bij}}{\cong} \{(\psi,\mu,\Lambda)_{K_V}\}/K_V^*. \tag{5.4}$$

Moreover, if $v \in V^*$, then

$$v : (\varphi,\lambda) \mapsto (v\varphi v^{-1}, v\lambda)$$

and

$$v : (\psi,\mu,\Lambda) \mapsto (v\psi v^{-1}, v\mu, v\Lambda).$$

Dividing out the action of V^* and considering the bijections (5.2) and (5.4) gives the following theorem.

Theorem 5.5.8. *Let $\Sigma = \{\sigma \in \mathrm{Gl}_2(A/fA) \mid \det(\sigma) \in \mathbb{F}_q^*\}$. There is a bijection between the following two sets:*

(1) Isomorphy classes of pairs (φ, λ) over K_V modulo Σ where φ is a Drinfeld module of rank 2 with stable reduction of rank 1 and λ is a full level f-structure over K_V.

(2) Isomorphy classes of triples (ψ, μ, Λ) over K_V, where ψ is a rank 1 Drinfeld module over K_V, μ is a full level f-structure over K_V and Λ is an A-lattice of rank 1, such that $(\psi_f)^{-1}\Lambda \subset K_V$.

Proof. This theorem follows from the bijection that we have given above. □

5.6 Tate-Drinfeld modules

We follow the approach of [3, 2.2] and [56] to construct the Tate-Drinfeld module of type $\mathfrak{m}$. The Tate-Drinfeld module describes the formal neighbourhood of the cusps of the moduli scheme. At the cusps the universal Drinfeld module with level structure degenerates into a Drinfeld module with stable reduction. Therefore, to define the Tate-Drinfeld module, we use the description of the stable reduction modules as given in the previous section.
Let (ψ, μ) be the universal Drinfeld module of rank 1 with level f-structure over R. Because f is invertible in R, we may assume that the generator $\mu(1)$ of the f-torsion of ψ is an invertible element in R. So we may and will assume that $\mu(1) = 1$. Then by push-forward via the embeddings

$$R \longrightarrow R[\![x]\!] \longrightarrow R(\!(x)\!)$$

one has a Drinfeld module of rank 1 with level f-structure on both $R[\![x]\!]$ and $R(\!(x)\!)$. For an element $y = \sum_{i \geq k} r_i x^i \in R(\!(x)\!)$ with $r_k \neq 0, k \in \mathbb{Z}$, we define its valuation $v_x(y)$ to be the x-valuation considered as element in $K_R(\!(x)\!)$, and $v_x(y) = k$.

To construct a Drinfeld module of rank 2 over $R(\!(x)\!)$, we first construct a lattice $\Lambda_{\mathfrak{m}} \subset K_R(\!(x)\!)$, where $\mathfrak{m} \subset A$ is an ideal. This lattice turns out to depend only on the class of $\mathfrak{m}$ in the class group of A. As before, we can consider ψ_f as a map

$$\psi_f : K_R(\!(x)\!)^{\mathrm{sep}} \longrightarrow K_R(\!(x)\!)^{\mathrm{sep}}.$$

The lattice $\Lambda_{\mathfrak{m}}$ will be constructed in such a way that $(\psi_f)^{-1}\Lambda_{\mathfrak{m}} \subset K_R(\!(x)\!)$. Applying Theorem 5.5.8 to $(\psi, \mu, \Lambda_{\mathfrak{m}})$ will give us the Tate-Drinfeld module φ.

5.6.1 The construction of the lattice

Let $\mathfrak{m} \subset A$ be an ideal. To prepare the construction of the lattice, note the following:
1. There is a unique monic skew polynomial $P \in K_R\{\tau\}$ with minimal degree such that

$$\ker(P)(K_R^{\mathrm{sep}}) = \psi[\mathfrak{m}](K_R^{\mathrm{sep}}).$$

In fact, $P \in R\{\tau\}$ because the elements in $\ker(P)(K_R^{\mathrm{sep}})$ are integral over R and R is integrally closed.

2. Because $A_f \hookrightarrow K_R$, the extension $K_R(\psi[\mathfrak{m}])/K_R$ is Galois. Moreover, because $\psi[\mathfrak{m}](K_R^{\text{sep}}) \cong A/\mathfrak{m}$, there is an injective representation

$$\text{Gal}(K_R(\psi[\mathfrak{m}])/K_R) \longrightarrow (A/\mathfrak{m})^*.$$

So the Galois group of this extension is a subgroup of $(A/\mathfrak{m})^*$.
3. The field $K_R(\psi[\mathfrak{m}])((y))$ is the splitting field of the equation $P(\frac{1}{y}) = \frac{1}{x}$ over $K_R((x))$. Then

$$K_R(\psi[\mathfrak{m}])((y))/K_R(\psi[\mathfrak{m}])((x))$$

is a Galois extension which is totally ramified and its Galois group is isomorphic to $A/\mathfrak{m}$. The Galois action is given by $y \mapsto y + \alpha$ with $\alpha \in \psi[\mathfrak{m}](K_R(\psi[\mathfrak{m}]))$.

Let l be the following A-module homomorphism:

$$l : f^{-1}A \longrightarrow K_R(\psi[\mathfrak{m}])((y)) \quad \text{by} \quad f^{-1}a \mapsto \psi_a\left(\frac{1}{y}\right).$$

We use l to define the lattice $\Lambda_\mathfrak{m}$.

Lemma 5.6.1. *The A-module $l(f^{-1}\mathfrak{m})$ lies inside $R((x))$.*

Proof. For every $m \in \mathfrak{m}$, there exists a skew polynomial $Q \in R\{\tau\}$ such that $\psi_m = Q \cdot P$. Note that we use here that one has division with remainder in the skew ring $R\{\tau\}$, because the leading coefficients of both P and ψ_m are in R^*. Consequently, $\psi_m(\frac{1}{y}) = Q(\frac{1}{x}) \in R((x))$. □

Remark 5.6.2. Let m_1, m_2 generate $\mathfrak{m}$, then there are elements $Q_i \in R\{\tau\}$ with $Q_i \circ P = \psi_{m_i}$. We will use this in the following a few times without further mentioning it.

Define the lattice Λ as follows:

$$\Lambda := l(\mathfrak{m}), \quad \text{then} \quad (\psi_f)^{-1}\Lambda = l(f^{-1}\mathfrak{m}) + \psi[f](R) \subset R((x)).$$

Lemma 5.6.3. *The lattice Λ only depends on the class of $\mathfrak{m}$ in $\text{Cl}(A)$.*

Proof. Suppose that $\mathfrak{m}' \subset A$ is another ideal representing the same class as $\mathfrak{m}$ in $\text{Cl}(A)$. Then there exist elements $b, b' \in A$ with $b'\mathfrak{m}' = b\mathfrak{m}$. So we may reduce to case where $\mathfrak{m}' = b\mathfrak{m}$ for some $b \in A$. The extension $K_R(\psi[\mathfrak{m}'])/K_R(\psi[\mathfrak{m}])$ is given by adding the roots of the polynomial $\psi_b(X)$ to $K_R(\psi[\mathfrak{m}])$. The extension

$$K_R(\psi[\mathfrak{m}'])((y'))/K_R(\psi[\mathfrak{m}'])((y))$$

is given by adding the roots of the equation $\psi_b(\frac{1}{y'}) = \frac{1}{y}$. It is not difficult to see that $\mathfrak{m}$ and $\mathfrak{m}'$ give the same Λ. □

By this lemma it makes sense to talk about the type $\mathfrak{m}$ in $\text{Cl}(A)$ of the lattice. We will denote $\mathfrak{m}$ for both the ideal in A as for the ideal class in $\text{Cl}(A)$. Moreover, this gives us some freedom in choosing $\mathfrak{m}$ to construct a lattice of type $\mathfrak{m}$:

Lemma 5.6.4. *Let $\mathfrak{a}, \mathfrak{m} \subset A$ be non-zero ideals, then there is an element $x \in K_A$, the quotient field of A, such that $x\mathfrak{m} + \mathfrak{a} = A$.*

Proof. By Proposition VII.5.9 in [4], there is an element $x \in K_A$ with $v_\mathfrak{p}(x) = -v_\mathfrak{p}(\mathfrak{m})$ for all primes $\mathfrak{p} \subset A$ dividing $\mathfrak{a}$ and $v_\mathfrak{p}(x) \geq 0$ for all other primes $\mathfrak{p}$ of A. Consequently, $x\mathfrak{m} \subset A$, and there is no prime ideal $\mathfrak{p}$ of A dividing both $x\mathfrak{m}$ and $\mathfrak{a}$. □

This lemma shows that we can choose a representative $\mathfrak{m}$ of the class type $[\mathfrak{m}]$ of the lattice such that this representative is relatively prime to some chosen ideal $\mathfrak{a} \subset A$. This gives some help in technical parts of some proofs later on.

We write $\Lambda_\mathfrak{m}$ for the lattice Λ of type $\mathfrak{m}$ that we constructed above. Note that $K_R(\!(x)\!)$ is the quotient field of the complete discrete valuation ring $K_R[\![x]\!]$. By the constructions in Section 5.5 and Theorem 5.5.8, we may associate to the triple $(\psi, \mu, \Lambda_\mathfrak{m})$ a unique Drinfeld module φ of rank 2 over $K_R(\!(x)\!)$ with stable reduction of rank 1. Moreover, the f-torsion of φ is $K_R(\!(x)\!)$-rational.

In fact, φ is a Drinfeld module over $R(\!(x)\!)$, as we will now show. Using Theorem 5.5.3, the corresponding exponential map is

$$e_{\Lambda_\mathfrak{m}}(z) := z \prod_{\alpha \in \Lambda_\mathfrak{m} \setminus \{0\}} \left(1 - \frac{z}{\alpha}\right),$$

and φ is determined by the following diagram, which commutes for all $a \in A$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda_\mathfrak{m} & \longrightarrow & K_R(\!(x)\!) & \xrightarrow{e_{\Lambda_\mathfrak{m}}} & K_R(\!(x)\!) \\ & & \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \varphi_a} \\ 0 & \longrightarrow & \Lambda_\mathfrak{m} & \longrightarrow & K_R(\!(x)\!) & \xrightarrow{e_{\Lambda_\mathfrak{m}}} & K_R(\!(x)\!). \end{array} \tag{5.5}$$

Note that by construction $\Lambda_\mathfrak{m} \subset R(\!(x)\!)$, each non-zero element of $\Lambda_\mathfrak{m}$ has negative x-valuation and the leading coefficient of this non-zero element is in R^*. Consequently, the map $e_{\Lambda_\mathfrak{m}}(z) = 1 + \sum_{i \geq 1} s_i z^i$ has all its coefficients $s_i \in R[\![x]\!]$. So its inverse exists in $R[\![x]\!][\![z]\!]$ and

$$\varphi_a = e_{\Lambda_\mathfrak{m}} \circ \psi_a \circ e_{\Lambda_\mathfrak{m}}^{-1}.$$

Therefore, φ_a has its coefficients in $R[\![x]\!]$ for all $a \in A$.

Lemma 5.6.5. *The ring homomorphism φ is a Drinfeld module over $R(\!(x)\!)$.*

Proof. We only need to prove that the leading coefficient of φ_a is an element of $R(\!(x)\!)^*$. To prove this, we simply copy the computation of Lemma 2.10 in [3]. Note that

$$\varphi_a(z) = az \prod_{\alpha \in ((\psi_a)^{-1}\Lambda_\mathfrak{m}/\Lambda_\mathfrak{m}) \setminus \{0\}} \left(1 - \frac{z}{e_{\Lambda_\mathfrak{m}}(\alpha)}\right).$$

So we need to show that

$$(*) \qquad \prod_{\alpha \in ((\psi_a)^{-1}\Lambda_\mathfrak{m}/\Lambda_\mathfrak{m}) \setminus \{0\}} e_{\Lambda_\mathfrak{m}}(\alpha) = a \cdot u$$

with $u \in R(\!(x)\!)^*$

Every $\alpha \in ((\psi_a)^{-1}\Lambda_\mathfrak{m}/\Lambda_\mathfrak{m}) \setminus \{0\}$ which is not in $\psi[a]$ can be written uniquely as $\alpha = \alpha_1 + \alpha_2$

where α_1 runs through $\psi[a]$ and α_2 runs through a set of representatives in $\psi_a^{-1}\Lambda_{\mathfrak{m}}/\Lambda_{\mathfrak{m}}$ of the non-zero elements of $(\psi_a)^{-1}\Lambda_{\mathfrak{m}}/(\Lambda_{\mathfrak{m}}+\psi[a])$. This set is denoted by S_1. The remaining elements α can be written as $\alpha = \alpha_1$ where α_1 runs through $\psi[\mathfrak{m}]\backslash\{0\}$. This set is denoted by S_2.
By definition

$$e_{\Lambda_{\mathfrak{m}}}(\alpha) = \alpha \prod_{\beta\in\Lambda_{\mathfrak{m}}\backslash\{0\}} \left(1-\frac{\alpha}{\beta}\right).$$

Using the rule

$$\psi_a(z) = \prod_{\alpha_1\in\psi[a]} (z-\alpha_1),$$

we see that

$$\prod_{S_1} e_{\Lambda_{\mathfrak{m}}}(\alpha) = \prod_{\alpha_2\neq 0}\left(\psi_a(\alpha_2)\prod_{\beta\in\Lambda_{\mathfrak{m}}\backslash\{0\}}\left(\frac{\psi_a(\beta-\alpha_2)}{\beta^{\#A/(a)}}\right)\right).$$

This is in fact an element in $R(\!(x)\!)^*$, which can be seen as follows. Clearly, $0\neq\psi_a(\alpha_2)\in\Lambda_{\mathfrak{m}}$, so this element is in $R(\!(x)\!)^*$. Moreover, the element $\psi_a(\beta-\alpha_2)\in\Lambda_{\mathfrak{m}}$ cannot be 0: if it were 0, then any representative of α_2 would lie in $\beta+\psi[a]\subset\Lambda_{\mathfrak{m}}+\psi[a]$, i.e., the class $\alpha_2 = 0$, contradicting the definition of the set S_2. So also $\psi_a(\beta-\alpha_2)\in R(\!(x)\!)^*$. Finally, for almost all β we have that

$$\frac{\psi_a(\beta-\alpha_2)}{\beta^{\#A/(a)}}\in R[\![x]\!]^*.$$

So the product exists, and

$$\prod_{S_1} e_{\Lambda_{\mathfrak{m}}}(\alpha)\in R(\!(x)\!)^*.$$

On the other hand, using the rule

$$h(z) := \frac{\psi_a(z)}{z} = \prod_{\alpha_1\in S_2}(z-\alpha_1)$$

and recalling that $h(0) = a$, we see that

$$\prod_{S_2} e_{\Lambda_{\mathfrak{m}}}(\alpha) = a\cdot\prod_{\beta\in\Lambda_{\mathfrak{m}}\backslash\{0\}}\left(\frac{h(\beta)}{\beta^{\#A/(a)-1}}\right),$$

with each $\frac{h(\beta)}{\beta^{\#A/(a)-1}}$ is in $R[\![x]\!]^*$.
Finally,

$$\prod_{\alpha\in((\psi_a)^{-1}\Lambda_{\mathfrak{m}}/\Lambda_{\mathfrak{m}})\backslash\{0\}} e_{\Lambda_{\mathfrak{m}}}(\alpha) = \prod_{S_1} e_{\Lambda_{\mathfrak{m}}}(\alpha)\cdot\prod_{S_2} e_{\Lambda_{\mathfrak{m}}}(\alpha).$$

This finishes the proof. □

From the construction of the Drinfeld module φ with θ over $R(\!(x)\!)$, coming from the triple $(\psi,\mu,\Lambda_{\mathfrak{m}})$, we can deduce at once the following list of properties:

(1) $\varphi : A \longrightarrow R[\![x]\!]\{\tau\}$ is a ring homomorphism;

(2) the f-torsion of φ is $R(\!(x)\!)$-rational;

(3) there is an isomorphism $A/fA \longrightarrow \varphi[f](R[\![x]\!])$ given by $1 \mapsto e_{\Lambda_{\mathfrak{m}}}(\mu(1))$.

(4) $\varphi \bmod xR[\![x]\!] = \psi$, because $e_{\Lambda_{\mathfrak{m}}}(z) \bmod xR[\![x]\!]$ is the identity map.

As in Section 5.5, the triple $(\psi, \mu, \Lambda_{\mathfrak{m}})$ induces on φ a level f-structure λ with $\lambda(0,1) = e_{\Lambda_{\mathfrak{m}}}(\mu(1))$ and $\lambda(1,0)$ is determined up to $\mathbb{F}_q^*$ by the Weil pairing.

In this way, we get for every element $\mathfrak{m} \in \mathrm{Cl}(A)$ a pair $(\varphi^{\mathfrak{m}}, \lambda^{\mathfrak{m}})$. The action of $\sigma \in \mathrm{Gl}_2(A/fA)$ on this pair is given by

$$\sigma : (\varphi^{\mathfrak{m}}, \lambda^{\mathfrak{m}}) \longrightarrow (\varphi^{\mathfrak{m}}, \lambda^{\mathfrak{m}} \circ \sigma).$$

The action of the class group of A is described in the following lemma. Let $\mathfrak{n} \in \mathrm{Cl}(A)$ and let $g_{\mathfrak{n}}$ denote the A_f-linear automorphism of R which describes the action of $\mathfrak{n}$ on R. Let $\Lambda_{\mathfrak{n}^{-1}\mathfrak{m}}$ denote the lattice of type $\mathfrak{n}^{-1}\mathfrak{m}$.

Lemma 5.6.6. *Using the above notations, the element* $\mathfrak{n}$ *maps the triple* $(\psi, \mu, \Lambda_{\mathfrak{m}})$ *to*

$$(g_{\mathfrak{n}}(\psi), g_{\mathfrak{n}}(\mu), g_{\mathfrak{n}}(\Lambda_{\mathfrak{n}^{-1}\mathfrak{m}})).$$

Proof. We choose representatives $\mathfrak{m}, \mathfrak{n} \subset A$ of the classes of $\mathfrak{m}$ and $\mathfrak{n}$ in $\mathrm{Cl}(A)$ such that $\mathfrak{n}^{-1}\mathfrak{m} \subset A$, and $\mathfrak{n}$ and $\mathfrak{m}$ are relatively prime to f.
Write $\varphi = \varphi^{\mathfrak{m}}$. The action of $\mathfrak{n}$ on φ is given by a unique monic skew polynomial Q with minimal degree such that

$$\ker(Q)(K_R(\!(x)\!)^{\mathrm{sep}}) = \varphi[\mathfrak{n}](K_R(\!(x)\!)^{\mathrm{sep}}).$$

Let φ' be the image under Q:

$$\varphi \xrightarrow{Q} \varphi'.$$

Writing $\Lambda_{\mathfrak{m}} = l(\mathfrak{m})$ as before, it is not difficult to see that

$$\ker(Q \circ e_{\Lambda_{\mathfrak{m}}})(K_R(\!(x)\!)^{\mathrm{sep}}) = l(\mathfrak{n}^{-1}\mathfrak{m}) + \psi[\mathfrak{n}](K_R(\!(x)\!)^{\mathrm{sep}}).$$

The action of $\mathfrak{n}$ on R, denoted by $g_{\mathfrak{n}}$, corresponds to a skew polynomial $P \in R\{\tau\}$ with $\ker(P)(K_R^{\mathrm{sep}}) = \psi[\mathfrak{n}](K_R^{\mathrm{sep}})$. Let Λ' be the lattice given by

$$\Lambda' := P \circ \ker(Q \circ e_{\Lambda_{\mathfrak{m}}})(K_R(\!(x)\!)^{\mathrm{sep}}) = P \circ l(\mathfrak{n}^{-1}\mathfrak{m}) \overset{\mathrm{def}}{=} \{g_{\mathfrak{n}}(\psi_{fa})P(\frac{1}{y}) \mid a \in \mathfrak{n}^{-1}\mathfrak{m}\}.$$

Then $\Lambda' = g_{\mathfrak{n}}(\Lambda_{\mathfrak{n}^{-1}\mathfrak{m}})$, and it is not difficult to see that φ' corresponds to the pair $(g_{\mathfrak{n}}(\psi), g_{\mathfrak{n}}(\Lambda_{\mathfrak{n}^{-1}\mathfrak{m}}))$. □

As a corollary to this lemma, we see that $\mathfrak{n}$ maps $\varphi^{\mathfrak{m}}$ to $g_{\mathfrak{n}}(\varphi^{\mathfrak{n}^{-1}\mathfrak{m}})$. As the definition of $\lambda^{\mathfrak{m}}$ and $\lambda^{\mathfrak{n}^{-1}\mathfrak{m}}$ depend on a choice of $\mathbb{F}_q^*$, we cannot say that $\mathfrak{n}$ maps $(\varphi^{\mathfrak{m}}, \lambda^{\mathfrak{m}})$ to $(\varphi^{\mathfrak{n}^{-1}\mathfrak{m}}, \lambda^{\mathfrak{n}^{-1}\mathfrak{m}})$. Bearing this in mind, we propose for every type $\mathfrak{m}$ the following definition of the pairs $(\varphi^{\mathrm{td}}_{\mathfrak{m}}, \lambda^{\mathrm{td}}_{\mathfrak{m}})$. For $\mathfrak{m} = A$ we define $(\varphi^{\mathrm{td}}_{\mathfrak{m}}, \lambda^{\mathrm{td}}_{\mathfrak{m}}) := (\varphi^{\mathfrak{m}}, \lambda^{\mathfrak{m}})$. This pair is unique up to $\mathbb{F}_q^*$. For the rest of this chapter, we keep this choice is fixed. In general, $\mathfrak{m}^{-1}$ gives rise to an automorphism $g_{\mathfrak{m}^{-1}}$ of R. We extend this to an automorphism of $R[\![x]\!]$ by letting it act trivially on x. We define $(\varphi^{\mathrm{td}}_{\mathfrak{m}}, \lambda^{\mathrm{td}}_{\mathfrak{m}})$ to be the image of $(\varphi^{\mathrm{td}}_A, \lambda^{\mathrm{td}}_A)$ under $\mathfrak{m}^{-1}$.

Definition 5.6.7. The pair $(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda_{\mathfrak{m}}^{\mathrm{td}})$ is called the *standard Tate-Drinfeld module of rank* 2 *and type* $\mathfrak{m}$ *with level f-structure.* A pair $(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda)$, where λ is any level f-structure over $R(\!(x)\!)$ is called a *Tate-Drinfeld module with level f-structure.*

An isomorphy class of $(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda)$ consists of all pairs $(\xi\varphi_{\mathfrak{m}}^{\mathrm{td}}\xi^{-1}, \xi\lambda)$, with $\xi \in R[\![x]\!]^*$. Every isomorphy class of Tate-Drinfeld modules with level f-structure comes from a unique morphism

$$\mathrm{Spec}(R(\!(x)\!)) \longrightarrow M^2(f).$$

Because we always assume that $\mu(1) = 1$, the pair $(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda_{\mathfrak{m}}^{\mathrm{td}})$ is fixed in its isomorphy class. For every λ there is a unique $\sigma \in \mathrm{Gl}_2(A/fA)$ with $\lambda = \lambda_{\mathfrak{m}}^{\mathrm{td}} \circ \sigma$. Therefore, the pair $(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda)$ is fixed in its isomorphy class.

5.7 The universal Tate-Drinfeld module

In this section we introduce the universal Tate-Drinfeld module. This is a scheme $\mathcal{Z}$ consisting of the coproduct of a number of copies of $\mathrm{Spec}(R[\![x]\!])$ and which is equipped with a Tate-Drinfeld structure $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$. Let $N \subset \mathrm{Gl}_2(A/fA)$ be the subgroup

$$N = \begin{pmatrix} \mathbb{F}_q^* & A/fA \\ 0 & (A/fA)^* \end{pmatrix}.$$

Choose representatives $\sigma_1, \ldots, \sigma_n$ of the cosets of $N\sigma_i$ in $\mathrm{Gl}_2(A/fA)$ and let σ_1 be the identity. As $\det(N) = (A/fA)^*$, we may assume that the representatives σ_i are elements of $\mathrm{Sl}_2(A/fA)$. Set

$$\mathcal{Z} = \mathrm{Spec}\left(\bigoplus_{(\mathfrak{m},\sigma_i)} R[\![x]\!]_{(\mathfrak{m},\sigma_i)}\right),$$

The pair $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$ is the Drinfeld module φ^{td} of rank 2 with level f-structure λ^{td} over $\mathrm{Spec}\left(\bigoplus_{(\mathfrak{m},\sigma_i)} R(\!(x)\!)_{(\mathfrak{m},\sigma_i)}\right)$ such that the restriction of $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$ to $R(\!(x)\!)_{(\mathfrak{m},\sigma_i)}$ is equal to $(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda_{\mathfrak{m}}^{\mathrm{td}} \circ \sigma_i)$ for every pair $(\mathfrak{m}, \sigma_i)$.

Definition 5.7.1. The *universal Tate-Drinfeld module of rank* 2 *with level f-structure* is the scheme $\mathcal{Z}$ together with the pair $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$.

Remark 5.7.2. As before, a pair $(\mathfrak{m}, \sigma_i) \in \mathrm{Cl}(A) \times \mathrm{Gl}_2(A/fA)$ determines a unique action on $(\varphi_A^{\mathrm{td}}, \lambda_A^{\mathrm{td}})$. And by definition we have

$$(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda_{\mathfrak{m}}^{\mathrm{td}} \circ \sigma_i) = (\mathfrak{m}^{-1}, \sigma_i)(\varphi_A^{\mathrm{td}}, \lambda_A^{\mathrm{td}}).$$

The action of $\mathrm{Cl}(A) \times \mathrm{Gl}_2(A/fA)$ on the universal Tate-Drinfeld module is given by this action of $(\mathfrak{m}, \sigma_i)$ and the fact that N acts trivially.

Clearly, we would like to have a certain universal property for this universal Tate-Drinfeld module. The weak versions of the universal property that we need can be found in Theorems 5.7.8 and 5.7.4. In Proposition 5.7.3 the main work is done for Theorem 5.7.4. This proposition explains the subgroup N.

Proposition 5.7.3. *Let $\sigma \in \mathrm{Gl}_2(A/fA)$. There exists an A_f-linear ring homomorphism $h_\sigma : R[\![x]\!] \longrightarrow R[\![x]\!]$ such that*

$$h_\sigma(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda_{\mathfrak{m}}^{\mathrm{td}}) \cong (\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda_{\mathfrak{m}}^{\mathrm{td}} \circ \sigma)$$

if and only if $\sigma \in N$. If $\sigma \in N$, then h_σ is given by $\det(\sigma) : R \longrightarrow R$ and $x \mapsto \delta x$ for some $\delta \in R[\![x]\!]^*$. Moreover, for any $\sigma \in N$ the image of $(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda_{\mathfrak{m}}^{\mathrm{td}} \circ \sigma_i)$ under h_σ is isomorphic to $(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda_{\mathfrak{m}}^{\mathrm{td}} \circ \sigma \circ \sigma_i)$.

Proof. We write $\sigma = (\sigma_{i,j})$. First suppose that h_σ exists. Because h_σ is determined by $\det(\sigma)$ and $h_\sigma(x)$, it must respect the ordering of the x-valuation. This implies that $h_\sigma(\lambda(1,0))$ must have minimal negative valuation and $h_\sigma(\lambda(0,1))$ must have valuation 0. Hence the only σ's whose action may come from an A_f-linear ring homomorphism h_σ are $\sigma \in N$. This shows the 'only if'.
We prove the 'if'-part in two steps. Let $\sigma = (\sigma_{i,j}) \in N$.
1. First suppose that $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ with $\alpha \in (A/fA)^*$. The action of σ induces an action

$$\alpha = \det(\sigma) : M^1(f) \longrightarrow M^1(f),$$

i.e., there is an A_f-linear ring homomorphism $h_\alpha : R \longrightarrow R$ such that

$$(h_\alpha(\psi), h_\alpha(\mu)) = (\xi\psi\xi^{-1}, \xi\mu\alpha) \cong (\psi, \mu\alpha),$$

for some $\xi \in R^*$. We use the notations of Section 5.6. Recall that the element $\frac{1}{y}$ was used to define the lattice $\Lambda_{\mathfrak{m}}^{\mathrm{td}}$ and that $\frac{1}{x} = P(\frac{1}{y})$. $P \in R\{\tau\}$ is the skew polynomial of minimal degree with

$$\ker(P)(K_R^{\mathrm{sep}}) = \psi[\mathfrak{m}](K_R^{\mathrm{sep}}).$$

The map $\frac{1}{y} \mapsto \xi\frac{1}{y}$ induces, by applying P,

$$\frac{1}{x} \mapsto h_\alpha(P)(\xi\frac{1}{y}) = \delta^{-1}\frac{1}{x} \quad \text{for some } \delta^{-1} \in R[\![x]\!]^*.$$

To see this, note that $h_\alpha(P) = \zeta P \xi^{-1}$ for some $\zeta \in R[\![x]\!]^*$. (In fact, ζ is determined by the fact that $h_\alpha(P)$ is monic.)
The map h_α is extended to a ring homomorphism

$$h_\sigma : R[\![x]\!] \longrightarrow R[\![x]\!], \quad x \mapsto \delta x, \quad h_\sigma(r) = h_\alpha(r) \quad \forall r \in R.$$

An easy computation shows that

$$h_\sigma(\Lambda_{\mathfrak{m}}^{\mathrm{td}}) = \xi\Lambda_{\mathfrak{m}}^{\mathrm{td}}, \quad \text{and thus} \quad h_\sigma(\varphi_{\mathfrak{m}}^{\mathrm{td}}) = \xi\varphi_{\mathfrak{m}}^{\mathrm{td}}\xi^{-1}.$$

Using that there is an element $m \in \mathfrak{m}$ with $\lambda(1,0) = e_{\Lambda_{\mathfrak{m}}^{\mathrm{td}}}(\psi_m(\frac{1}{y}))$, we see that

$$h_\sigma \begin{pmatrix} \lambda(1,0) \\ \lambda(0,1) \end{pmatrix} = \begin{pmatrix} e_{\xi\Lambda_{\mathfrak{m}}^{\mathrm{td}}}(\xi\psi_m(\frac{1}{y})) \\ e_{\xi\Lambda_{\mathfrak{m}}^{\mathrm{td}}}(\xi\mu(1)\alpha) \end{pmatrix} = \begin{pmatrix} \xi\lambda(1,0) \\ \xi\lambda(0,1)\alpha \end{pmatrix}.$$

And so indeed,

$$h_\sigma(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda) = (\xi\varphi_{\mathfrak{m}}^{\mathrm{td}}\xi^{-1}, \xi\lambda \circ \sigma) \cong (\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda \circ \sigma).$$

2. Having dealt with the first case, we may assume that $\sigma \in N$ and $\det(\sigma) \in \mathbb{F}_q^*$. As a consequence, the map h_σ that we are looking for is R-linear and in particular, if (ψ, μ, Λ) is the triple associated to $(\varphi_{\mathfrak{m}}^{\mathrm{td}}, \lambda)$, then $h_\sigma(\psi) = \psi$ and $h_\sigma(\mu) = \mu$.
As always, the action of $\mathbb{F}_q^* \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is trivial, so we may assume that $\sigma_{2,2} = 1$. We first give a proof in the simple case $\Lambda \cong A$ and then prove it for general Λ.
i. Suppose $\Lambda \cong A$. Note that in that case, the elements $e_\Lambda(\frac{1}{x})$ and $e_\Lambda(\mu(1))$ generate the f-torsion of φ^{td}. In fact, there is a basis transformation

$$\alpha = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \alpha \begin{pmatrix} \lambda(1,0) \\ \lambda(0,1) \end{pmatrix} = \begin{pmatrix} e_\Lambda(\frac{1}{x}) \\ e_\Lambda(\mu(1)) \end{pmatrix}$$

and with $\alpha_{1,1} \in (A/fA)^*$, $\alpha_{1,2} \in A/fA$.
Let $\rho = \alpha\sigma\alpha^{-1}$, then $\rho = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} \\ 0 & 1 \end{pmatrix}$ with $\rho_{1,1} \in \mathbb{F}_q^*$, $\rho_{1,2} \in A/fA$. We set

$$\delta^{-1} = \rho_{1,1} + \rho_{1,2}(\mu(1)) \cdot x \in R[\![x]\!]^*,$$

and we define h_σ to be the R-linear map given by

$$h_\sigma : x \mapsto \delta \cdot x.$$

Because

$$\psi_f(\delta^{-1}\frac{1}{x}) = \rho_{1,1}\psi_f(\frac{1}{x}),$$

it follows that $h_\sigma(\Lambda) = \Lambda$, and thus h_σ commutes with e_Λ. Moreover

$$h_\sigma(\psi, \mu, \Lambda) = (\psi, \mu, \Lambda).$$

Therefore

$$h_\sigma(\varphi_{\mathfrak{m}}^{\mathrm{td}}) = \varphi_{\mathfrak{m}}^{\mathrm{td}}.$$

So h_σ commutes with the A-action and with e_Λ. To see what happens on the level structure λ is now an easy computation.

$$h_\sigma \begin{pmatrix} \lambda(1,0) \\ \lambda(0,1) \end{pmatrix} = h_\sigma \alpha^{-1} \begin{pmatrix} e_\Lambda(\frac{1}{x}) \\ e_\Lambda(\mu(1)) \end{pmatrix} =$$

$$\alpha^{-1}\rho \begin{pmatrix} e_\Lambda(\frac{1}{x}) \\ e_\Lambda(\mu(1)) \end{pmatrix} = \sigma \begin{pmatrix} \lambda(1,0) \\ \lambda(0,1) \end{pmatrix}.$$

We conclude that

$$h_\sigma(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}}) = (\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}} \circ \sigma).$$

ii. In general, let $\Lambda \cong \mathfrak{m}$ and recall the construction of Λ in Section 5.6. Using the same notations as in Section 5.6, let $m \in \mathfrak{m}$ be an element such that the image of $\frac{1}{z} := \psi_m(\frac{1}{y})$ and $\mu(1)$ under e_Λ generate the f-torsion of $\varphi_{\mathfrak{m}}^{\mathrm{td}}$. We let α be as in i, i.e.,

$$\alpha \begin{pmatrix} \lambda(1,0) \\ \lambda(0,1) \end{pmatrix} = \begin{pmatrix} e_\Lambda(\frac{1}{z}) \\ e_\Lambda(\mu(1)) \end{pmatrix}$$

and $\rho = \alpha\sigma\alpha^{-1}$. As in the previous case, we look for an h_σ such that

$$\frac{1}{z} \mapsto (\rho_{1,1} + \rho_{1,2}\mu(1)z) \cdot \frac{1}{z}.$$

This can be done as follows. We start by assuming that $m \in (A/fA)^*$ - we will show below that we may assume this in general. Let $b \in A$ such that $b \equiv m^{-1}\rho_{1,2} \in A/fA$. Let

$$\frac{1}{y} \mapsto \rho_{1,1}\frac{1}{y} + \psi_b\mu(1).$$

Applying P gives our candidate for h_σ:

$$\frac{1}{x} \mapsto \delta^{-1}\frac{1}{x} \quad \text{with } \delta^{-1} = \rho_{1,1} + P(\psi_b(\mu(1))) \cdot x.$$

By construction

$$h_\sigma : \frac{1}{z} \mapsto \rho_{1,1}\frac{1}{z} + \psi_m(\psi_b(\mu(1))) = \rho_{1,1}\frac{1}{z} + \rho_{1,2}\mu(1).$$

Note that all elements of Λ have the form $\psi_{f\tilde{m}}(\frac{1}{y})$, with $\tilde{m} \in \mathfrak{m}$. And clearly,

$$h_\sigma(\psi_{f\tilde{m}}(\frac{1}{y})) = \rho_{1,1}\psi_{f\tilde{m}}(\frac{1}{y}).$$

Because $\rho_{1,1} \in \mathbb{F}_q^*$, we see that $h_\sigma(\Lambda) = \Lambda$.
We can conclude the proof in the same way as in the case of $\Lambda \cong A$.
Finally, it remains to be shown that m, which we used to define $\frac{1}{z} = \psi_m(\frac{1}{y})$, is an element of $(A/fA)^*$. By Lemma 5.6.4 we may assume that $\mathfrak{m}$ and (f) are relatively prime. Furthermore, because $\frac{1}{z}$ generates one direct summand of the f-torsion, one has $\psi_b(\frac{1}{z}) \in \Lambda$ if and only if $b \in fA$, i.e., $bm \in f\mathfrak{m}$ if and only if $b \in fA$. Consequently, $(f) + (m) = A$.
The 'moreover'-part of the proposition is obvious. □

Using Proposition 5.7.3, we can immediately prove one weak form of the universal property of the universal Tate-Drinfeld module.

Theorem 5.7.4. *For every Tate-Drinfeld module $(\varphi^{\mathrm{td}}_{\mathfrak{m}}, \lambda)$ there is a unique ring homomorphism*

$$h : \bigoplus_{(\mathfrak{m}',\sigma_i)} R[\![x]\!]_{(\mathfrak{m}',\sigma_i)} \longrightarrow R[\![x]\!]$$

such that

$$h(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}}) \cong (\varphi^{\mathrm{td}}_{\mathfrak{m}}, \lambda).$$

Proof. Let $\sigma \in \mathrm{Gl}_2(A/fA)$ such that $\lambda^{\mathrm{td}}_{\mathfrak{m}} \circ \sigma = \lambda$ and let $\sigma_k \in \{\sigma_1, \ldots, \sigma_n\}$ such that $\sigma \in \sigma_k N$. The map h is defined as follows: h is the zero-map on $R[\![x]\!]_{(\mathfrak{m}',\sigma_i)}$ if $\mathfrak{m}' \neq \mathfrak{m}$ in the class group of A or if $\sigma_k \neq \sigma_i$. On $R[\![x]\!]_{(\mathfrak{m},\sigma_k)}$ it is the map defined in Proposition 5.7.3. To show uniqueness, note that any A_f-linear ring homomorphism

$$h : R[\![x]\!]_{(\mathfrak{m}',\sigma_i)} \longrightarrow R[\![x]\!]$$

induces on $R[\![x]\!]$ a Tate-Drinfeld module whose corresponding lattice has type $\mathfrak{m}'$. Hence, any such ring homomorphism keeps the type of the Tate-Drinfeld module fixed. Moreover, if there is an A_f-morphism

$$R[\![x]\!]_{(\mathfrak{m},\sigma_i)} \longrightarrow R[\![x]\!]_{(\mathfrak{m},\sigma_j)}$$

which induces the Tate-Drinfeld structure, then there is a morphism

$$R[\![x]\!]_{(\mathfrak{m},\sigma_1)} \longrightarrow R[\![x]\!]_{(\mathfrak{m},\sigma_j\sigma_i^{-1})},$$

and thus by Proposition 5.7.3 we see that $\sigma_j\sigma_i^{-1} \in N$. So $\sigma_i = \sigma_j$. □

5.7.1 The universal property of $\mathcal{Z}$

Let, as in Section 5.5, V be a complete discrete valuation A_f-ring π a generator of its maximal ideal, and K_V its field of fractions. Let (φ, λ) be a Drinfeld module φ of rank 2 over K_V with level f-structure λ such that φ has stable reduction of rank 1 at π. In this subsection we discuss the other weak version of the universal property of the universal Tate-Drinfeld module, which we need in the next section. We prove that there exists a unique ring homomorphism

$$h_{\varphi,\lambda} : \bigoplus_{(\mathfrak{m},\sigma_i)} R[\![x]\!] \longrightarrow V$$

such that

$$h_{\varphi,\lambda}(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}}) \cong (\varphi, \lambda).$$

In Theorem 5.5.8 we showed that each triple (φ, λ, z) corresponds modulo Σ to a unique triple (ψ, μ, Λ). Let $\mathfrak{m}$ be the type of Λ. In trying to avoid confusion, we will write $(\psi^{\mathrm{un}}, \mu^{\mathrm{un}}, \Lambda^{\mathrm{td}}_{\mathfrak{m}})$ for the triple used in defining the Tate-Drinfeld module of type $\mathfrak{m}$ over $R(\!(x)\!)$, where indeed $(\psi^{\mathrm{un}}, \mu^{\mathrm{un}})$ is the universal Drinfeld module with level f-structure of rank 1 over R.
The pair (ψ, μ) over V comes from an A_f-linear ring homomorphism, which we called $\tilde{h}$:

$$\tilde{h} : R \longrightarrow V.$$

We have $\tilde{h}(\psi^{\mathrm{un}}, \mu^{\mathrm{un}}) = (\psi, \mu)$.

We show that there exists an extension of $\tilde{h}$ to $R[\![x]\!]$

$$h_{\psi,\mu} : R[\![x]\!] \longrightarrow V,$$

where $R[\![x]\!]$ comes equipped with $\varphi^{\mathrm{td}}_{\mathfrak{m}}$, such that $h_{\psi,\mu}(\varphi^{\mathrm{td}}_{\mathfrak{m}}) = \varphi$; cf. Proposition 5.7.6. The main point is showing that there exists a ring homomorphism $h_{\psi,\mu}$ such that

$$h_{\psi,\mu}((\psi^{\mathrm{un}}_f)^{-1}\Lambda^{\mathrm{td}}_{\mathfrak{m}}) = (\psi_f)^{-1}\Lambda.$$

Lemma 5.7.5. *Let m_1 and m_2 generate the ideal $\mathfrak{m}$. Then there exists an element $\zeta \in \overline{K}_V$ such that the projective part of $(\psi_f)^{-1}\Lambda$ is generated as A-module by the elements $\psi_{m_1}(\zeta)$ and $\psi_{m_1}(\zeta)$.*

Proof. Let $M = \overline{K}_V$ be the algebraic closure of K_V, and let M_{tor} be the set of A-torsion points in M; the A-action is given by ψ.
1. The A-module M_{tor} is divisible, i.e., for all $a \in A\backslash\{0\}$ the map

$$\psi_a : M_{\text{tor}} \longrightarrow M_{\text{tor}}$$

is surjective. Namely, if $x \in M_{\text{tor}}$, then the equation $\psi_a(z) = x$ has solutions in M_{tor}. Consequently, M_{tor} is an injective module. Cf. Theorem 7.1 in [31]], where it is shown that divisibility is the same as injectivity for modules over a PID; it is not difficult to extend this to a theorem over Dedekind domains.
2. M/M_{tor} has a natural K_A-module structure. The following sequence of A-modules is exact:

$$0 \longrightarrow M_{\text{tor}} \longrightarrow M \longrightarrow M/M_{\text{tor}} \longrightarrow 0.$$

Note that Λ is torsion-free, hence $\Lambda \oplus M_{\text{tor}} \hookrightarrow M$. Consider the projection map

$$s : \Lambda \oplus M_{\text{tor}} \longrightarrow M_{\text{tor}}, \quad \text{by } \alpha \oplus m \mapsto m.$$

Because M_{tor} is an injective module, it follows that s extends to a map

$$s : M \longrightarrow M_{\text{tor}}.$$

So the exact sequence splits according to s and $M \cong M_{\text{tor}} \oplus M/M_{\text{tor}}$.
3. According to 2., we may write $(\psi_f)^{-1}\Lambda = N_1 \oplus N_2$ with $N_1 = \psi[f](V)$ being the torsion part and $N_2 \cong \mathfrak{m}$ being the projective part of $(\psi_f)^{-1}\Lambda$. Let e_1, e_2 be generators of N_2 such that $\psi_{m_2} e_1 = \psi_{m_1} e_2$. Then

$$(\psi_{m_1})^{-1} e_1 \equiv (\psi_{m_2})^{-1} e_2 \mod M_{\text{tor}}.$$

Let $\zeta_i \in M$ be the unique element with $\zeta_i \mapsto (\psi_{m_i})^{-1} e_i \in M/M_{\text{tor}}$ and $s(\zeta_i) = 0$. Then $\zeta_1 - \zeta_2 \mapsto 0 \in M/M_{\text{tor}}$ and $s(\zeta_1 - \zeta_2) = 0$. Consequently, $\zeta := \zeta_1 = \zeta_2$ is the element we are looking for. □

Proposition 5.7.6. *Every rank 2 Drinfeld module φ over V with K_V-rational f-torsion and with stable reduction of rank 1 and type $\mathfrak{m}$ is induced by $\varphi_{\mathfrak{m}}^{\text{td}}$ via the ring homomorphism*

$$h_{\psi,\mu} : R[\![x]\!] \longrightarrow V,$$

i.e., $h_{\psi,\mu}(\varphi_{\mathfrak{m}}^{\text{td}}) \cong_V \varphi$. Extending $h_{\psi,\mu}$ to

$$R(\!(x)\!) \longrightarrow K_V$$

maps the f-torsion of $\varphi_{\mathfrak{m}}^{\text{td}}$ isomorphically to the f-torsion of φ.

Proof. In Section 5.6 we introduced skew polynomials $P, Q_1, Q_2 \in R\{\tau\}$ with $Q_i \circ P = \psi_{m_i}^{\text{un}}$. In particular, we see that $\tilde{h}(P)$ divides $P' := \gcd(\psi_{m_1}, \psi_{m_2}) \in K_V\{\tau\}$. We may assume that $\mathfrak{m}$ is not contained in the kernel of $A_f \longrightarrow V$. Therefore, $\deg_\tau P' = \deg_\tau h_{\psi,\mu}(P)$. Consequently, there exist elements $\beta_i \in K_V\{\tau\}$ with

$$\tilde{h}(P) = \beta_1 \psi_{m_1} + \beta_2 \psi_{m_2}.$$

Let ζ be the element from Lemma 5.7.5, and define $\frac{1}{z} := \tilde{h}(P)(\zeta)$. As $\psi_{m_1}(\zeta)$ and $\psi_{m_1}(\zeta)$ generate the projective part of $(\psi_f)^{-1}\Lambda \subset K_V$, we see that $\frac{1}{z} \in K_V$ and $z \in V$.
We extend $\tilde{h}$ to

$$h_{\psi,\mu} : R[\![x]\!] \longrightarrow V \quad \text{by} \quad x \mapsto z.$$

One can easily verify that

$$h_{\psi,\mu}((\psi_f^{\mathrm{un}})^{-1}\Lambda_{\mathfrak{m}}^{\mathrm{td}}) = (\psi_f)^{-1}\Lambda.$$

□

Remark 5.7.7. The proof of Proposition 2.5 in [56] is not entirely complete. One way of completing it, is by adding the construction of the lattice as is done here. This makes sure that the morphism h in Proposition 2.5 in [56] indeed exists.

Theorem 5.7.8. *Every pair (φ, λ) consisting of a rank 2 Drinfeld module φ over V with stable reduction of rank 1 with level f-structure λ is induced by $(\varphi^{\mathrm{td}}, \lambda^{\mathrm{td}})$ via the unique ring homomorphism $h_{\varphi,\lambda}$.*

Proof. Let (ψ, μ, Λ) be the triple associated to (φ, λ), and let $\mathfrak{m}$ be the type of Λ. Let $h_{\psi,\mu} : R[\![x]\!] \longrightarrow V$ be the morphism such that $h_{\psi,\mu}(\varphi_{\mathfrak{m}}^{\mathrm{td}}) = \varphi$. Let $\sigma \in \mathrm{Gl}_2(A/fA)$ such that $h_{\psi,\mu}(\lambda_{\mathfrak{m}}^{\mathrm{td}} \circ \sigma) = \lambda$. The element σ lies in a unique class $N\sigma_i$. Write $\sigma = \tau\sigma_i$ with $\tau \in N$. Let h_τ be the ring homomorphism which is defined in Proposition 5.7.3. Define

$$h_{\varphi,\lambda} : \oplus R[\![x]\!]_{(\mathfrak{m}',\sigma_j)} \longrightarrow V$$

as follows: $h_{\varphi,\lambda}$ equals $h_{\psi,\mu} \circ h_\tau$ on $R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$ and is zero on the other copies of $R[\![x]\!]$.
The uniqueness follows from the construction and from Theorem 5.7.4. □

5.8 The compactification of $M^2(f)$

In this section we describe a compactification $\overline{M}^2(f)$ of $M^2(f)$, which is analogous to the compactification of the classical modular curves given by Katz and Mazur Section 8 of [33]. We define the scheme of cusps, which we call *Cusps*. This is a closed subscheme of $\overline{M}^2(f)$. Moreover, we consider the formal scheme $\widehat{\mathit{Cusps}}$, which is the completion of $\overline{M}^2(f)$ along *Cusps*. In the following section we will use the universal Tate-Drinfeld module as defined in the previous sections to describe the scheme of cusps.

5.8.1 The morphism j_a

Let $a \in A \backslash \mathbb{F}_q$. Let (φ, λ) be the universal Drinfeld module of rank 2 with level f-structure over $M^2(f)$. Let B be the ring with $\mathrm{Spec}(B) = M^2(f)$ and write

$$\varphi_a = \sum_{i=0}^{2\deg(a)} b_i \tau^i \quad \text{with } b_i \in B \text{ for all } i \text{ and } b_{2\deg(a)} \in B^*.$$

Let $j_a : M^2(f) \longrightarrow \mathbb{A}^1_{A_f}$ be the morphism given by

$$j_a^\# : A_f[j] \longrightarrow B, \quad j \mapsto b_{\deg(a)}^{q^{\deg(a)}+1} / b_{2\deg(a)};$$

cf. [37, 4.2]. Clearly, j_a factors over $M^2(1)$.

Lemma 5.8.1. *The morphism j_a is finite and flat.*

Proof. (This is the proof of Proposition 4.2.3 in [37].) The morphism j_a is of finite type, hence we may use the valuative criterion to prove properness. Suppose that V is a discrete valuation ring, let K_V be its quotient field, and suppose that there are morphisms given that make the following diagram commutative.

$$\begin{array}{ccc} \mathrm{Spec}(K_V) & \longrightarrow & M^2(f) \\ \big\downarrow & & \big\downarrow{\scriptstyle j_a} \\ \mathrm{Spec}(V) & \longrightarrow & \mathbb{A}^1_{A_f}. \end{array}$$

Note that the upper horizontal map gives rise to a (unique) map

$$\mathrm{Spec}(V) \longrightarrow M^2(f)$$

if and only if the pull-back of the pair (φ, λ) via the upper horizontal map has good reduction at the maximal ideal of V. So j_a is proper if and only if $j_a(x) \in V$ implies that the pull-back (φ', λ') over K_V has good reduction. If this pull-back does not have good reduction, it has stable reduction of rank 1. Suppose that (φ, λ) has stable reduction of rank 1, then there exists an element $s \in K_V^*$ such that $s\varphi_a s^{-1}$ has all coefficients in V, the $\deg(a)$'th coefficient has valuation 0, and the $2\deg(a)$'th coefficient has strictly positive valuation. This means that the image of $j_a(x)$ is not in V. We conclude that j_a is proper.
Because each connected component of $M^2(f)$ is an affine variety over $\mathbb{F}_q$, it follows by [27, ex. II.4.6] that j_a restricted to such a connected component is finite. And thus j_a is finite.
The finite ring homomorphism $j_a^\#$ is injective. Let $\mathfrak{P} \subset B$ be a prime ideal lying above $\mathfrak{p} \subset A_f[j]$. Then both local rings $B_\mathfrak{P}$ and $A_f[j]_\mathfrak{p}$ are regular and of equal dimension. By the finiteness of j_a it follows that $B_\mathfrak{P}$ is a free $A_f[j]_\mathfrak{p}$-module, cf. Corollary IV.22 in [50]. Hence, j_a is flat. □

5.8.2 The compactification

The ring B is a finite $A_f[j]$-algebra via $j_a^\#$. Let C denote the normalization of $A_f[\frac{1}{j}]$ inside the quotient ring of B. Then C is finite over $A_f[\frac{1}{j}]$; cf. Corollary 13.13 in [15].
The compactification $\overline{M}^2(f)$ of $M^2(f)$ is defined as the scheme obtained by glueing $\mathrm{Spec}(B)$ and $\mathrm{Spec}(C)$ along their intersection. We obtain a finite morphism

$$\bar{j}_a : \overline{M}^2(f) \longrightarrow \mathbb{P}^1_{A_f}.$$

The following diagram is cartesian

$$\begin{array}{ccc} M^2(f) & \longrightarrow & \overline{M}^2(f) \\ {\scriptstyle j_a}\big\downarrow & & \big\downarrow{\scriptstyle \bar{j}_a} \\ \mathbb{A}^1_{A_f} & \longrightarrow & \mathbb{P}^1_{A_f}. \end{array}$$

Remark 5.8.2. By Proposition 5.9.1 it follows that $\overline{j}_a$ is flat in the points of the boundary of $\overline{M}^2(f)$; therefore, $\overline{j}_a$ is flat.

Lemma 5.8.3. *The scheme $\overline{M}^2(f)$ is independent of the chosen element a.*

Proof. Let B' be a connected component of B. Let $a_1, a_2 \in A\backslash\mathbb{F}_q$ and consider the maps $j_{a_i} : A_f[j_i] \longrightarrow B'$. Let C_i be the integral closure of $A_f[\frac{1}{j_i}]$ inside $K_{B'}$. Let X_i be the scheme obtained by glueing $\mathrm{Spec}(C_i)$ and $\mathrm{Spec}(B')$ along their intersection.
Let $\mathfrak{p} \in X_1\backslash\mathrm{Spec}(B')$ be any prime of height one of C_1, then $\frac{1}{j_1} \in \mathfrak{p}$. Let $v_\mathfrak{p}$ be the valuation of $K_{B'}$ given by $\mathfrak{p}$. If $v_\mathfrak{p}(\frac{1}{j_2}) \leq 0$, then $\mathfrak{p}$ would correspond to a valuation of $K_A[j_2]$ and therefore to a valuation of B'; cf. Section VII.9 in [4]. Consequently, $v_\mathfrak{p}(\frac{1}{j_2}) > 0$. The same is true if we interchange j_1 and j_2. We conclude that the set of valuations v of K_B' with $v(j_i) > 0$ does not depend on i. Therefore, $X_1 = X_2$. □

5.8.3 The scheme of cusps

To describe the boundary of $\overline{M}^2(f)$, we introduce the *scheme of cusps*, which we call *Cusps*. Let $\mathfrak{r}$ be the intersection of all height 1 primes $\mathfrak{p}$ containing $\frac{1}{j}$, i.e., $\mathfrak{r} = \mathrm{rad}(\frac{1}{j})$. And $V(\mathfrak{r}) = \overline{M}^2(f)\backslash M^2(f)$. Let $\widehat{C} := \varprojlim C/\mathfrak{r}^n$.

Lemma 5.8.4. *The ring $\hat{C}$ is normal and a finite $A_f[\![\frac{1}{j}]\!]$-algebra.*

Proof. The ring B is regular. So $C = \oplus_i C_i$ where each C_i is an integrally closed domain. The ring C is excellent. By [24, 7.8.3.vii] it follows that $\widehat{C}$ is normal. As C' is a finite $A_f[\frac{1}{j}]$-algebra, it follows that $\widehat{C}$ is a finite $A_f[\![\frac{1}{j}]\!]$-algebra. □

We denote $A_f(\!(\frac{1}{j})\!) := A_f[\![\frac{1}{j}]\!][j]$. Furthermore, we define the formal scheme

$$\widehat{Cusps} := \mathrm{Spf}(\widehat{C}),$$

which is the formal neighbourhood of *Cusps*. Let $\mathcal{O}$ deote the structure sheaf of $\widehat{Cusps}$. The scheme of cusps is defined as

$$Cusps := (\widehat{Cusps}, \mathcal{O}/\mathfrak{r}) = \mathrm{Spec}(C/\mathfrak{r}).$$

Theorem 5.8.5. *The A_f-morphism w_f given by Theorem 5.4.1 can be extended to an A_f-morphism*

$$w_f : \overline{M}^2(f) \longrightarrow M^1(f).$$

Its restriction to the scheme of cusps gives a finite A_f-morphism

$$w_f : Cusps \longrightarrow M^1(f).$$

Proof. The Weil pairing gives an R-algebra structure $R \longrightarrow B$. Because R is integral over A_f, B is the integral closure of $A_f[j]$ in the quotient ring of B and C is the integral closure of $A_f[\frac{1}{j}]$ in this quotient ring, it follows immediately, that the ring homomorphism $R \longrightarrow B$ gives a ring homomorphism $R \longrightarrow C$. These two maps glue to

$$w_f : \overline{M}^2(f) \longrightarrow M^1(f).$$

The restriction of w_f to *Cusps* is given by $R \longrightarrow C \longrightarrow C/\mathfrak{r}$. As C is finite over $A_f[\frac{1}{j}]$, it follows that $C/\mathfrak{r}$ is finite over A_f. As R is finite over A_f, we may conclude that w_f restricted to *Cusps* is finite. □

5.9 The cusps and the Tate-Drinfeld module

In the previous Section 5.8, we defined the scheme of cusps and the formal scheme $\widehat{Cusps} = \mathrm{Spf}(\widehat{C})$. In this section, we will relate these schemes to the universal Tate-Drinfeld module, which we introduced in Section 5.7. In fact, using the universal property of the universal Tate-Drinfeld module and the $\mathrm{Cl}(A) \times \mathrm{Gl}_2(A/fA)$-equivariance of the Weil-pairing, we will be able to prove the following proposition. If we write '$\oplus_\mathfrak{p}$', we mean the direct sum over all minimal primes $\mathfrak{p}$ containing $\frac{1}{j}$.

Proposition 5.9.1. *There exists an* $R[\![\frac{1}{j}]\!]$*-linear isomorphism*

$$\widehat{C} \cong \oplus_\mathfrak{p} \varprojlim C/\mathfrak{p}^n \stackrel{\sim}{\longrightarrow} \oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)},$$

such that

$$Cusps \stackrel{\sim}{\longrightarrow} \oplus R_{(\mathfrak{m},\sigma_i)}.$$

From this proposition we can derive the following theorem:

Theorem 5.9.2. *The compactification* $\overline{M}^2(f)$ *of* $M^2(f)$ *is regular, and even smooth over* $\mathrm{Spec}(A_f)$*. Furthermore, the scheme of cusps is isomorphic to*

$$Cusps \cong \coprod_{(\mathfrak{m},\sigma_i)} M^1(f),$$

where $\mathfrak{m}$ *runs through* $\mathrm{Cl}(A)$ *and* σ_i *runs through the cosets of* $N\backslash \mathrm{Gl}_2(A/fA)$ *where*

$$N = \begin{pmatrix} \mathbb{F}_q^* & A/fA \\ 0 & (A/fA)^* \end{pmatrix} \subset \mathrm{Gl}_2(A/fA).$$

Consequently, the scheme Cusps consists of $\frac{h(A)\cdot \#\mathrm{Sl}_2(A/fA)}{\#(A/fA)\cdot(q-1)}$ *copies of* $M^1(f)$*.*

Proof. By Proposition 5.9.1 the ring C is regular in the points above $\frac{1}{j}$ and thus C is regular. Consequently, $\overline{M}^2(f)$ is regular. The description of *Cusps* and the number of its components follows from Proposition 5.9.1.
To prove smoothness over $\mathrm{Spec}(A_f)$, note that by the corollary to Proposition 5.4 in [11], the morphism $M^2(f) \longrightarrow \mathrm{Spec}(A_f)$ is smooth. So we only need to prove smoothness in the closed points of *Cusps*. We have

$$\widehat{C} \cong \oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)},$$

so $\widehat{C}$ is formally smooth over A_f. This implies by 17.5.1 and 17.5.3 in [25] that the morphism $\overline{M}^2(f) \longrightarrow \mathrm{Spec}(A_f)$ is smooth in the closed points of *Cusps*. □

5.9.1 The proof of Proposition 5.9.1

The rest of this section is devoted to proving Proposition 5.9.1. The universal Tate-Drinfeld module over $\mathcal{Z}$ gives rise to an A_f-morphism

$$\mathcal{Z}_{\text{open}} \longrightarrow M^2(f),$$

where $\mathcal{Z}_{\text{open}}$ denotes the localization of $\mathcal{Z}$ at (x), i.e.,

$$\mathcal{Z}_{\text{open}} = \operatorname{Spec}(\oplus R(\!(x)\!)_{(\mathfrak{m},\sigma_i)}).$$

It follows from Remark 5.7.2 that this morphism is $\mathrm{Cl}(A) \times \mathrm{Gl}_2(A/fA)$-equivariant.

Let $\mathcal{Z}_{x=0}$ denote the scheme

$$\operatorname{Spec}\left(\oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)}/(x)\right).$$

The line of argument is as follows: in Lemma 5.9.3 we show how to relate the Tate-Drinfeld module to the study of the cusps, and in Lemma 5.9.4 we describe the scheme *Cusps*. This latter lemma enables us to lift the isomorphism *Cusps* $\longrightarrow \mathcal{Z}_{x=0}$ to an isomorphism $\widehat{C} \longrightarrow \mathcal{Z}$. Let $\widehat{C}_{\mathfrak{p}}$ denote the completion of the local ring of C at $\mathfrak{p}$.

Lemma 5.9.3. *Every $R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$ is a finite $A_f[\![\frac{1}{j}]\!]$-algebra. Consequently, the morphism $\mathcal{Z}_{\text{open}} \longrightarrow M^2(f)$ comes from a morphism*

$$h_1 : \mathcal{Z} \longrightarrow \operatorname{Spec}(C) \subset \overline{M}^2(f).$$

Moreover, the universal property of the Tate-Drinfeld module gives rise to a morphism

$$h_2 : \operatorname{Spec}(\oplus_{\mathfrak{p}} \widehat{C}_{\mathfrak{p}}) \longrightarrow \mathcal{Z}.$$

The composition $h_1 \circ h_2$ is the natural morphism.

Proof. Write $R[\![x]\!] = R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$ for some pair $(\mathfrak{m}, \sigma_i)$ and write $(\varphi^{\text{td}}_{\mathfrak{m}})_a = \sum_i c_i\tau^i$ for the element $a \in A$ which is used to define j_a. Then $c_{2\deg(a)} \in R(\!(x)\!)^*$, because $\varphi^{\text{td}}_{\mathfrak{m}}$ is a Drinfeld module over $R(\!(x)\!)$. Moreover, $\varphi^{\text{td}}_{\mathfrak{m}} \mod (x) = \psi$. So the coefficient $c_{\deg(a)} \in R[\![x]\!]^*$. This implies by definition, that $\frac{1}{j}$ is mapped to $\alpha \cdot x^k \in R[\![x]\!]$, with $\alpha \in R[\![x]\!]^*$ and $k \in \mathbb{Z}_{>0}$. From this it follows that $R[\![x]\!]$ is a finite $R[\![\frac{1}{j}]\!]$-module. The morphism

$$\mathcal{Z}_{\text{open}} \longrightarrow M^2(f)$$

comes from a ring homomorphism

$$C[j] \longrightarrow \left(\oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)}\right) \otimes_{A_f[\![\frac{1}{j}]\!]} A_f(\!(\frac{1}{j})\!),$$

Because C is finite over $A_f[\frac{1}{j}]$ and $R[\![x]\!]$ is finite over $A_f[\![\frac{1}{j}]\!]$, it follows that the image of C under this ring homomorphism lies in $\oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$.
For the 'moreover'-part, let $\mathfrak{p} \subset C$ be a minimal prime ideal containing $\frac{1}{j}$. The ring $\widehat{C}_{\mathfrak{p}}$

is a complete discrete valuation ring and comes equipped with a Tate-Drinfeld structure via the morphism

$$\mathrm{Spec}(K_{\widehat{C}_{\mathfrak{p}}}) \longrightarrow M^2(f).$$

By Theorem 5.7.8 there exists a unique ring homomorphism

$$\oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)} \longrightarrow \widehat{C}_{\mathfrak{p}}$$

which induces on $\widehat{C}_{\mathfrak{p}}$ this Tate-Drinfeld structure. This can be done for every minimal prime $\mathfrak{p}$ containing $\frac{1}{j}$. □

Lemma 5.9.4. *The morphism h_1 induces an isomorphism*

$$\mathcal{Z}_{x=0} \xrightarrow{\sim} \mathit{Cusps}.$$

Every pair $(\mathfrak{m}, \sigma_i)$ corresponds via this isomorphism to one and only one minimal prime $\mathfrak{p} \subset C$ containing $\frac{1}{j}$. Consequently,

$$\widehat{C} = \oplus_{\mathfrak{p}} \varprojlim C/\mathfrak{p}^n.$$

Proof. We will first prove that the number of irreducible components of *Cusps* equals the number of irreducible components of $\mathcal{Z}_{x=0}$. Subsequently, we will show that these components of *Cusps* intersect nowhere.
Because $\mathit{Cusps} \cong \mathrm{Spec}(C/(\cap\mathfrak{p}))$, the irreducible components of *Cusps* are in a one-to-one corresponcende to the minimal primes containing $\frac{1}{j}$. The morphisms h_1 and h_2 introduced in Lemma 5.9.3 give rise to the following maps on the sets of irreducible components.

$$\{\text{irr. comp. of } \mathit{Cusps}\} \xrightarrow{h_2} \{\text{irr. comp. of } \mathcal{Z}_{x=0}\} \xrightarrow{h_1} \{\text{irr. comp. of } \mathit{Cusps}\}.$$

As a morphism of schemes $h_1 \circ h_2$ is the natural map. Therefore, the composition $h_1 \circ h_2$ on the set of irreducible components is the identity. Consequently, h_2 on the irreducible components is injective.
Moreover, the set of irreducible components of $\mathcal{Z}_{x=0}$ is by definition one orbit under the elements $(\mathfrak{m}, \sigma_i)$. Clearly, the map h_1 on the set of connected components is equivariant under this group action, and as the image of h_2 is not empty, it follows that the first map on the irreducible components is also surjective. So we may conclude that the number of irreducible components of *Cusps* equals the number of irreducible components of $\mathcal{Z}_{x=0}$.
The irreducible components of $\mathcal{Z}_{x=0}$ intersect nowhere. We will prove that this is also the case for the irreducible components of *Cusps*. By the extension of the Weil pairing to $\overline{M}^2(f)$, the ring C comes equipped with an R-algebra structure. Let

$$\zeta : R \longrightarrow \oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$$

denote the composition $R \xrightarrow{w_f^{\#}} C \xrightarrow{h_1^{\#}} \oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)}$.
Choose any maximal ideal $\mathfrak{n} \subset R$ and let $\mathfrak{q}$ run over all minimal primes of $C/\mathfrak{n}C$ containing $\frac{1}{j}$. We write

$$\widehat{C/\mathfrak{n}C} = \varprojlim (C/\mathfrak{n}C)/(\frac{1}{j})^n,$$

and $(\widehat{C/\mathfrak{n}C})_{\mathfrak{q}}$ for the completion along $\mathfrak{q}$ of the local ring $(C/\mathfrak{n}C)_{\mathfrak{q}}$.
In this case we have analogues of the morphisms h_1, h_2, namely, R-algebra homomorphisms

$$\widehat{C/\mathfrak{n}C} \xrightarrow{\tilde{h}_1} \oplus R/\zeta(\mathfrak{n})[\![x]\!]_{(\mathfrak{m},\sigma_i)} \xrightarrow{\tilde{h}_2} \oplus \widehat{C/\mathfrak{n}C}_{\mathfrak{q}}.$$

And now, as before, we define maps on the sets of irreducible components:

$$\tilde{h}_2 : \{\text{irr. comp. of } \mathit{Cusps} \times \mathrm{Spec}(R/\mathfrak{n})\} \longrightarrow \{\text{irr. comp. of } \mathcal{Z}_{x=0} \times \mathrm{Spec}(R/\mathfrak{n})\},$$

and

$$\tilde{h}_1 : \{\text{irr. comp. of } \mathcal{Z}_{x=0} \times \mathrm{Spec}(R/\mathfrak{n})\} \longrightarrow \{\text{irr. comp. of } \mathit{Cusps} \times \mathrm{Spec}(R/\mathfrak{n})\}.$$

The composition of these two maps on the set of irreducible components is the identity. Namely, the composition $\tilde{h}_1 \circ \tilde{h}_2$ is the natural map on the rings. Using the same argument as above shows that $\tilde{h}_2$ on the irreducible components is a bijection.
We conclude that for every prime $\mathfrak{n} \subset R$ the number of irreducible components of

$$\mathit{Cusps} \times \mathrm{Spec}(R/\mathfrak{n})$$

equals the number of irreducible components of *Cusps*. Recall that by Theorem 5.8.5 the morphism $w_f : \mathit{Cusps} \longrightarrow M^1(f)$ is finite. Therefore, if the irreducible components would intersect above some prime ideal $\mathfrak{n} \subset R$, then $\mathit{Cusps} \times \mathrm{Spec}(R/\mathfrak{n})$ would have less irreducible components. As this is not the case, we conclude that the irreducible components of *Cusps* intersect nowhere.
We write

$$\mathit{Cusps} = \mathrm{Spec}(\oplus S_{(\mathfrak{m},\sigma_i)})$$

where $\mathrm{Spec}(S_{(\mathfrak{m},\sigma_i)})$ are the connected components of *Cusps*. For every pair $(\mathfrak{m}, \sigma_i)$, we get R-linear ring homomorphisms on the connected components:

$$S_{(\mathfrak{m},\sigma_i)} \xrightarrow{h_1^{\#}} R \xrightarrow{h_2^{\#}} S_{(\mathfrak{m},\sigma_i)}.$$

Because the composition is the identity and $S_{(\mathfrak{m},\sigma_i)}$ is finite over R, it follows that

$$S_{(\mathfrak{m},\sigma_i)} \cong R.$$

For the latter two statements of the lemma, note that the isomorphism implies that the minimal primes $\mathfrak{p}$ are relatively prime and, consequently, $\mathfrak{r} = \prod \mathfrak{p}$. □

The next step is to lift the isomorphism from Lemma 5.9.4 to an isomorphism

$$\widehat{C} \xrightarrow{\sim} \oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)}.$$

Let

$$\mathcal{W} := \varprojlim C/\mathfrak{p}^n$$

for some minimal prime $\mathfrak{p}$ containing $\frac{1}{j}$.

Lemma 5.9.5. *The ring $\mathcal{W}$ is isomorphic to $R[\![x]\!]$.*

Proof. By Lemma 5.8.4 the ring $\mathcal{W}$ is integrally closed and a finite $A_f[\![\frac{1}{j}]\!]$-algebra, and by the isomorphism of Lemma 5.9.4, the ring $\mathcal{W}$ is a finite $R[\![\frac{1}{j}]\!]$-algebra.
The completion of the local ring $\mathcal{W}_\mathfrak{p}$ is isomorphic to $\widehat{C}_\mathfrak{p}$. The morphisms h_1 and h_2 give on the completions of the local rings injective maps

$$\widehat{C}_\mathfrak{p} \xrightarrow{h_1^\#} K_R[\![x]\!] \xrightarrow{h_2^\#} \widehat{C}_\mathfrak{p}.$$

As $h_2^\# \circ h_1^\#$ is the identity, there exists an isomorphism $K_R[\![x]\!] \cong \widehat{C}_\mathfrak{p}$.
We conclude that $\mathcal{W}$ is regular. Therefore, we may asssume that x is an element of $\mathcal{W}$ and that $\mathcal{W}$ is a finite $R[\![x]\!]$-algebra. So we get injective $R[\![x]\!]$-linear ring homomorphisms

$$R[\![x]\!] \longrightarrow \mathcal{W} \longrightarrow R[\![x]\!]$$

where the first map is the $R[\![x]\!]$-structure morphism of $\mathcal{W}$ and the second map is $h_1^\#$. We conclude that $\mathcal{W} \cong R[\![x]\!]$. □

This enables us to prove Proposition 5.9.1:

Proof of Proposition 5.9.1. By the previous lemma, it follows that

$$\widehat{C} \cong \oplus R[\![x]\!]_{(\mathfrak{m},\sigma_i)}.$$

Together with Lemma 5.9.4 the proposition follows. □

5.10 Components of $M^2(f)$

In this section we describe the geometric components of $\overline{M}^2(f)$ and prove the connectedness of $M^2(f)$. For a non-zero prime $\mathfrak{P} \subset R$ we write $\kappa(\mathfrak{P}) := R/\mathfrak{P}$. The first result is the following:

Theorem 5.10.1. *The scheme*

$$\overline{M}^2(f) \underset{A_f}{\times} M^1(f)$$

consists of $h(A) \cdot [(A/fA)^* : \mathbb{F}_q^*]$ connected components, which are all geometrically connected. Moreover, for every non-zero prime ideal $\mathfrak{P} \subset R$ the fibre at $\mathfrak{P}$

$$\overline{M}^2(f) \underset{A_f}{\times} \operatorname{Spec}(\kappa(\mathfrak{P})),$$

consists of $h(A) \cdot [(A/fA)^* : \mathbb{F}_q^*]$ connected components, which are all geometrically connected.

Proof. Let K_∞ be the completion of the quotient field of A_f along the point ∞, and let $\mathbb{C}_\infty$ denote the completion of the algebraic closure of K_∞. By the analytic theory, as is shown in [57], we know that

$$\overline{M}^2(f) \times_{A_f} \operatorname{Spec}(\mathbb{C}_\infty)$$

consists of $h(A)\cdot[(A/fA)^*:\mathbb{F}_q^*]$ components. Because R is a Galois extension of A_f with Galois group G, we have $R\otimes_{A_f}R\cong\oplus_G R$. By the Weil pairing one sees that

$$\overline{M}^2(f)\underset{A_f}{\times}M^1(f)\xrightarrow{w_f}M^1(f)\times_{A_f}M^1(f)$$

consists of $h(A)\cdot\#G$ connected components. As $\#G=[(A/fA)^*:\mathbb{F}_q^*]$, these components are geometrically connected components.
Consider the fibres over R. Let $\mathfrak{P}\subset R$ be a non-zero prime ideal and let V be the completion along $\mathfrak{P}$ of the local ring $R_{\mathfrak{P}}$. Suppose $\overline{M}^2(f)\times_R\operatorname{Spec}(\kappa(\mathfrak{P}))$ has more than one connected component, then also $\overline{M}^2(f)\times_R\operatorname{Spec}(V/\mathfrak{P}^n)$ has more than one connected component for every n and consequently, both $\overline{M}^2(f)\times\operatorname{Spec}(V)$ and $\overline{M}^2(f)\times_R\operatorname{Spec}(K_V)$ consist of more than one component. This, however, contradicts the fact that $\overline{M}^2(f)\times_R M^1(f)$ is geometrically connected. So we conclude that $\overline{M}^2(f)\times_R\operatorname{Spec}(\kappa(\mathfrak{P}))$ is geometrically connected. □

This theorem enables us to say something about the Drinfeld modular curves. Let, as before, $N=\begin{pmatrix}\mathbb{F}_q^* & A/fA\\ 0 & (A/fA)^*\end{pmatrix}$.

Theorem 5.10.2. *For every R-field K the curve $\overline{M}^2(f)\times_R\operatorname{Spec}(K)$ is a smooth, irreducible curve containing $h(A)\cdot[\mathrm{Gl}_2(A/fA):N]$ cusps.*

Proof. Clearly, the scheme *Cusps* consists of $h(A)\cdot[\mathrm{Gl}_2(A/fA):N]$ copies of R. Consequently,

$$\mathit{Cusps}\times\operatorname{Spec}(K)$$

consists of $h(A)\cdot[\mathrm{Gl}_2(A/fA):N]$ points. The irreducibility follows immediately from the proof of Theorem 5.10.1. □

5.10.1 The analogue of $X_0(N)$

The analogue in the setting of Drinfeld modular curves of the modular curve $X_0(N)$ is the curve

$$X_0(f):=\overline{M}^2(f)/H,\quad\text{where } H=\begin{pmatrix}(A/fA)^* & A/fA\\ 0 & (A/fA)^*\end{pmatrix}\subset\mathrm{Gl}_2(A/fA).$$

One may deduce from Theorem 5.10.1 the following theorem concerning the cusps and the geometric components of $\overline{M}^2(f)/H$. Define $R_0=R^{(A/fA)^*/\mathbb{F}_q^*}$, i.e., $\operatorname{Spec}(R_0)=M^1(1)$. Write Cusps_0 for the scheme of cusps of $X_0(f)$.

Theorem 5.10.3. *The Weil pairing induces an isomorphism*

$$\mathit{Cusps}_0\xrightarrow{\sim}\coprod_{(\mathfrak{m},\rho)}M^1(1)$$

where ρ runs through the double cosets $N\backslash\mathrm{Gl}_2(A/fA)/H$.
The scheme $X_0(f)$ is connected, and for any R_0-field K the scheme $X_0(f)\times\operatorname{Spec}(K)$ consists of $h(A)$ geometrically connected components.

Proof. The morphism w_f gives an isomorphism between $M^1(f)$ and any connected component of *Cusps*. Consequently, w_f gives an isomorphism $Cusps \longrightarrow \oplus_{(\mathfrak{m},\sigma_i)} M^1(f)$. Recall that N acts trivially on each copy of $M^1(f)$. Furthermore, the action of $\sigma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ with $\alpha \in (A/fA)^*$ on *Cusps* is as follows. Let $i, j \in \mathbb{N}$ such that $\sigma_i \circ \sigma \in N\sigma_j$, and consider α as an element of the Galois group $\mathrm{Gal}(K_R/K_{R_0}) \cong (A/fA)^*/\mathbb{F}_q^*$, then σ acts as

$$M^1(f)_{(\mathfrak{m},\sigma_i)} \xrightarrow{\alpha} M^1(f)_{(\mathfrak{m},\sigma_j)}.$$

Because H contains the subgroup $\left\{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \mid \alpha \in (A/fA)^*\right\}$, we see that dividing out *Cusps* by H gives an isomorphism

$$Cusps_0 \xrightarrow{\sim} \oplus_{(\mathfrak{m},\rho)} M^1(1)$$

where ρ runs through the cosets of $N\backslash \mathrm{Gl}_2(A/fA)/H$.
The number of components follows immediately from Theorem 5.10.1. □