Algorithms and moduli spaces for differential equations
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Introduction

This thesis treats several questions concerning linear differential equations. We mainly consider differential equations over \( k(z) \), with \( k \) some field of characteristic zero. Such equations are of the form \( L(y) := a_n y^{(n)} + \cdots + a_0 y = 0 \), \( a_i \in k(z) \), where \( y^{(n)} \) denotes the \( n \)-th derivative of \( y \) with respect to some differentiation on \( k(z) \), for example \( \frac{dy}{dz} \) or \( z \frac{dy}{dz} \). In this introduction we will give an overview of the problems treated in this thesis.

- Can one give explicit solutions of a second order equation over \( k(z) \)?

To answer this question, we first have to define what we mean by an explicit solution. It is quite natural to allow logarithms, exponentials and algebraic equations in the description of a solution. This leads to the notion of Liouvillian solutions. For example, the equation \( 4zy'' + 2y' = y \) has the Liouvillian solutions \( \{ e^{\sqrt{z}}, e^{-\sqrt{z}} \} \), but the Airy-equation \( y'' = zy \) has no Liouvillian solutions. The famous Kovacic algorithm [Ko86] determines if a second order equation has Liouvillian solutions. If they exist, the algorithm gives them explicitly. One problem in actually implementing the algorithm is that the solutions may involve some algebraic field extensions of the field of constants \( k \). An algorithm which determines for a given second order linear differential equation the possible extensions of the constant field, is given in Section 1.1.

Another problem is how to obtain a compact presentation of Liouvillian solutions. One way of doing this in the case of second order equations, is by writing such solutions in terms of the solutions of certain standard equations. Klein’s theorem precisely states that this is possible. We present a new proof of this. The method of representing solutions via Klein’s theorem works well because the standard equations are quite simple, and so are their solutions. In section 1.2 we give explicit formulas for the transformations involved (the
so-called pullback formulas).

One can ask whether this method can be extended to third order equations.

- Is there a variant of Klein’s theorem for third order equations?

In Section 1.3 we show that indeed there is. The difference with the second order case is that we have to allow infinitely many “standard equations”. It seems there is no obvious way to adapt the method of finding explicit formulas for pullback maps to the case of third order equations.

A possible approach in studying differential equations is to try to make families of differential equations which share certain properties. In Chapter 2 we make such families by prescribing the local behavior of the equations. We show that we actually obtain so-called fine moduli spaces (classifying spaces) in this way. To a differential equation one can associate its differential Galois group, which gives information on the complexity of the solutions.

- How does the differential Galois group vary over a family of differential equations?

This question is studied in chapter 3. More precisely, suppose we have a family of differential equations parametrized by a space $X$. Given a group $G$, we consider the subset $X(G) := \{ x \in X \mid \text{the differential equation at } x \text{ has Galois group } \subseteq G \}$. We prove that this is a closed subspace of $X$. Using this, we describe sufficient conditions on the group $G$ such that the analogous defined subset $X(=G)$ is “constructible” (cf. Section 3.1). These results are motivated by earlier work of M. F. Singer ([S93]).

Chapter 4 is concerned with the concept of monodromy. Let a differential equation $L(y) = 0$, with rational functions over $\mathbb{C}$ as coefficients, be given. Write $S := \{ s_1, \cdots, s_r \}$ for the set of singular points of $L$ on the Riemann sphere $\mathbb{P}_1^\mathbb{C}$. Roughly speaking, the monodromy gives information on how solutions of $L$ change under analytic continuation. To be more precise, let $b$ be a point in $\mathbb{P}_1^\mathbb{C} \setminus S$, and write $V$ for the local solution space of $L$ at $b$. For a loop $\lambda$ in the fundamental group $\pi_1(\mathbb{P}_1^\mathbb{C} \setminus S, b)$ analytic continuation along $\lambda$ defines a linear automorphism on $V$. The monodromy is the natural homomorphism $\pi_1(\mathbb{P}_1^\mathbb{C} \setminus S) \rightarrow \text{GL}(V)$ which arises in this way.
The classical question, posed by Hilbert as the 21\textsuperscript{st} problem of his famous list, asks whether for any homomorphism $\rho : \pi_1(\mathbb{P}_C^1 \setminus S) \rightarrow \text{GL}(V)$ there is a differential equation with monodromy given by $\rho$. For a more precise formulation, see Chapter 4. This problem is known as the Riemann-Hilbert problem. As stated here, the answer turns out to be ‘yes’. In this thesis we extend the Riemann-Hilbert problem to families, i.e.,

- For a given family of monodromy maps, does there exist a family of differential equations, such that the induced family of monodromy maps is the given one?

Under some weak conditions on the monodromy maps (cf. Section 4.2), we prove by explicit construction of a family of differential equations that the answer is again positive. The families of differential equations used here involve so-called \textit{vector bundles}. To a vector bundle on $\mathbb{P}^1$ one associates a \textit{type}. It happens to be the case that the type of the vector bundles, obtained by the mentioned construction, may vary (over the space which parametrizes the family). This leads us to a study of differential equations (or actually connections) on non-trivial vector bundles in Section 4.3. An important example in this setting is the so-called Lamé-example, which concludes the thesis.
INTRODUCTION