Chapter 4

The Riemann-Hilbert problem and examples

In this chapter we will give some examples of moduli spaces of differential equations, and we describe the connection with the Riemann-Hilbert problem.

4.1 The classical Riemann-Hilbert problem

We start by briefly recalling the classical Riemann-Hilbert problem as described in [PS03] Chapter 6.

Let $(M, \nabla)$ be a regular singular connection over $\mathbb{C}(z)$ with singular locus equal to $S = \{s_1, \ldots, s_r\} \subset \mathbb{P}_C^1$. This means that $M$ is a finite dimensional $\mathbb{C}(z)$-vector space, and $\nabla : M \to \mathbb{C}(z)dz \otimes M$ is a regular singular connection (see [PS03] 6.4.2). One defines a monodromy map associated to $(M, \nabla)$ in the following manner. Write $V := \ker(\mathbb{C}((z - b)) \otimes_{\mathbb{Q}(z)} M, \nabla)$ for the local solution space at a regular point $b \in \mathbb{P}_C^1 \setminus S$ and define $\pi_1 := \pi_1(\mathbb{P}_C^1 \setminus S, b)$. Let $\lambda \in \pi_1$ be a loop, then we can make an analytic continuation of the local solutions $V$ along $\lambda$. This analytic continuation defines a linear map on $V$, so we can associate an element of $\text{GL}(V)$ to $\lambda$. This process results in a map $\pi_1 \to \text{GL}(V)$, called the monodromy map. The question if for any given representation $\rho : \pi_1 \to \text{GL}(V)$ there is a regular singular connection with
4.2. THE RIEMANN-HILBERT PROBLEM

Theorem 6.22 of [PS03]). For general $(M, \nabla)$ this is not always the case.

We will conclude by briefly describing how a connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1_C$ with a prescribed monodromy representation can be constructed. We will use a generalization of this construction in the next section.

Write $U := \mathbb{P}^1_C \setminus S$, so $\pi_1 := \pi_1(U, 0)$. We start by constructing a regular connection on $U$ with the prescribed monodromy. For this consider the universal covering $u : \tilde{U} \to U$ of $U$. Define a connection $(\mathcal{N}, \nabla)$ on $\tilde{U}$ by $\mathcal{N} := \mathbb{C}^n \otimes \mathcal{O}_B$, and $\nabla(v \otimes f) = v \otimes f'$ for all $v \in \mathbb{C}^n$, $f \in \mathcal{O}_B$. Furthermore we define a $\pi_1$-action on $\mathcal{N}$ by $\lambda(v \otimes f) = \rho(\lambda)(v) \otimes (f \circ \lambda^{-1}) \forall \lambda \in \pi_1$. The vector bundle $\mathcal{N}$ corresponds to the geometric vector bundle $\mathbb{C}^n \times \tilde{U}$, and the corresponding $\pi_1$-action is given by $\lambda(v, \tilde{u}) = (\rho(\lambda)(v), \lambda(\tilde{u}))$, $v \in \mathbb{C}^n$, $\tilde{u} \in \tilde{U}$. Indeed in this way we get for a section $h \times id : \tilde{U} \to \mathbb{C}^n \times \tilde{U}$, $h \in \mathcal{N}(U)$, that $(\lambda(h) \times id)(\tilde{u}) = \lambda(h \times id(\lambda^{-1}(\tilde{u})))$. It is clear that the quotient $\pi_1 \setminus (\mathbb{C}^n \times \tilde{U})$ defines a geometric vector bundle on $U = \pi_1 \setminus \tilde{U}$, with corresponding vector bundle $\mathcal{M}_U := \mathcal{N}/\pi_1$. The $\pi_1$-action on $\mathcal{N}$ commutes with $\nabla$. So we find an induced connection $\nabla_U$ on $\mathcal{M}_U$. Write $\mathcal{L} := \ker(\nabla_U, \mathcal{M}_U)$. The only thing left to show is that $\mathcal{L}$ is the local system corresponding to $\rho$. There is a one to one correspondence between local systems on $U$ and (trivial) local systems on $\tilde{U}$ with a $\pi_1$-action. Under this correspondence $\mathcal{L}$ clearly corresponds to $\mathbb{C}^n$ with the defined $\pi_1$-action which is given by $\rho$. This proves that $(\mathcal{M}_U, \nabla_U)$ has monodromy given by $\rho$.

We now want to extend this connection $(\mathcal{M}_U, \nabla_U)$ to a connection on $\mathbb{P}^1_C$. Let $s \in S$, and consider the pointed disk $U_s^* := 0 < |z-s| < \varepsilon$. We will construct a connection on $U_s^* := |z-s| < \varepsilon$ that glues to the restriction of $(\mathcal{M}_U, \nabla_U)$ to $U_s^*$. For this we consider the local solution space $V_s$ at $s + \frac{\varepsilon}{2}$. The circle around $s$ through $s + \frac{\varepsilon}{2}$ induces a monodromy map $B \in \text{GL}(V_s)$. Choose $A \in \text{End}(V)$ such that $e^{2\pi i A} = B$, then we define the connection $\nabla_s$ on the vector bundle $\mathcal{M}_s := \mathcal{O}_{U_s} \otimes V_s$ by $\nabla_s (f \otimes v) = df \otimes v + z^{-1} \otimes A(v)$. The restriction of $(\mathcal{M}_s, \nabla_s)$ to $U_s^*$ clearly has local monodromy $B$. By [PS03] 6.6.3 this restriction is isomorphic to the restriction of $(\mathcal{M}_U, \nabla_U)$ to $U_s^*$. Therefore we can glue the connection $(\mathcal{M}_s, \nabla_s)$ to $(\mathcal{M}_U, \nabla_U)$. In this way we obtain the desired connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1_C$ extending $(\mathcal{M}_U, \nabla_U)$.
4.2 The Riemann-Hilbert problem for families

We will now consider the Riemann-Hilbert problem for families of differential equations. Let $Y$ be an analytic manifold, and let $S := \{s_1, \ldots, s_r\}$ be a set of points in $\mathbb{P}^1 \setminus \{0, \infty\}$. Suppose that $(\mathcal{M}, \nabla)$ is an analytic family of differential equations on $\mathbb{P}^1$, parametrized by $Y$ (the definition of an analytic family is a straightforward variation of Definition 3.13). We suppose that $S$ is the set of singular points of $\nabla$; more precisely, for every $y \in Y$ the set of singular points of $\nabla(y)$ is $S$.

We write $pr_1 : Y \times \mathbb{P}^1 \to Y$, $pr_2 : Y \times \mathbb{P}^1 \to \mathbb{P}^1$ for the two projection maps. Let $U := \mathbb{P}^1 \setminus S$, and $\pi_i := \pi_1(U, 0)$. We will also write $pr_1$, $pr_2$ for the restrictions to $Y \times U$ of the two projection maps. The kernel $\mathcal{L} := \ker(\nabla)$ of $\nabla|_{Y \times U}$ is a locally free $pr_1^*(\mathcal{O}_Y)$-module of rank $n$, where $n = \dim(\mathcal{V})$. For any $a \in U$ the embedding $j_a : Y \cong Y \times \{a\} \hookrightarrow Y \times U$ defines a vector bundle $\mathcal{L}_a := j_a^*(\mathcal{L})$ on $Y$.

We will now define a monodromy map $\pi_1 \to \text{Aut}(\mathcal{L}_0)$. Let $\lambda : [0, 1] \to U$ be a path in $U$. Then $(id \times \lambda)^*(\mathcal{L})$ is a $pr_1^*(\mathcal{O}_Y)$-module on $Y \times [0, 1]$, where $pr : Y \times [0, 1] \to Y$ is the projection map. Since $(id \times \lambda)^*(\mathcal{L})$ is a locally free sheaf, we find a canonical isomorphism $\mathcal{L}_{\lambda(0)} \sim \mathcal{L}_{\lambda(1)}$. In particular, a closed path $\lambda$ with $\lambda(0) = 0 \in \mathbb{P}^1 \cong Y$ yields an automorphism of $\mathcal{L}_0$ and this defines a homomorphism $\pi_1 \to \text{Aut}(\mathcal{L}_0)$.

**Definition 4.2** We call the map $\pi_1 \to \text{Aut}(\mathcal{L}_0)$ constructed above, the monodromy map associated to $(\mathcal{M}, \nabla)$.

We now present a converse construction, which we interpret as a solution to the Riemann-Hilbert problem for families.

**Theorem 4.3** Let $Y$ be an analytic irreducible reduced manifold with a vector bundle $\mathcal{L}$ on it. Suppose we are given a set $S := \{s_1, \ldots, s_r\} \subset \mathbb{P}^1$ and a representation $\rho : \pi_1(\mathbb{P}^1 \setminus S) \to \text{Aut}(\mathcal{L})$ satisfying the following properties.

- Let $\lambda_i \in \pi_1$ be loops around the points $s_i$ with $\prod_{i=1}^r \lambda_i = 1$. For every $y \in Y$ we have that $\rho(\lambda_i)(y) \sim e^{2\pi i A_i}$ for some fixed $A_i \in M_n(\mathbb{C})$. Here
4.2. THE RIEMANN-HILBERT PROBLEM

\[ \rho(\lambda_i)(y) \] denotes the automorphism on \( \mathcal{L}_y/(m_y \mathcal{L}_y) \cong \mathbb{C}^n \) induced by \( \rho(\lambda_i) \).

- None of the differences of the eigenvalues of \( A_i \) is in \( \mathbb{Z} \setminus \{0\} \).

Then there exists an analytic connection \((\mathcal{M}, \nabla)\) on \( Y \times \mathbb{P}^1 \), with singular points in \( S \) and monodromy map given by \( \rho \).

**Proof:** We can cover \( Y \) by Stein-manifolds \( Y_i \) such that \( \mathcal{L}|_{Y_i} \) is free for all \( i \). We will now construct a solution to the Riemann-Hilbert problem for families over \( Y_i \). From the construction it can be seen that the connections on the \( Y_i \) glue to a connection on \( Y \) with the appropriate monodromy map, hence this also solves the Riemann-Hilbert problem for families over \( Y \). From now on we assume \( \mathcal{L} \) to be free and \( Y \) to be a Stein-manifold, and \( \rho \) is given as a homomorphism \( \rho : \pi_1 \to \text{GL}_n(\mathcal{O}(Y)) \).

Let \( \tilde{U} \) be the universal covering of \( U \). We can identify \( \pi_1 \) with \( \text{Aut}(\tilde{U}/U) \). Write \( pr_1 : Y \times \tilde{U} \to Y \), \( pr_2 : Y \times \tilde{U} \to \tilde{U} \) for the two projection maps. The vector bundle \( \mathcal{N} := \mathcal{O}_{Y \times \tilde{U}}^n \) can be written as \( pr_1^{-1}(\mathcal{O}_Y^n) \otimes pr_2^{-1}(\mathcal{O}_{\tilde{U}}) \), where \( \otimes \) is an “analytic tensor product”, as defined in [GR71] p.179.

**Remark 4.4** We note that \( pr_1^{-1}(\mathcal{O}_Y) \) are the analytic functions on \( Y \times \tilde{U} \) which are constant with respect to \( \tilde{U} \). The sheaf \( pr_1^{-1}(\mathcal{O}_Y^n) \otimes_{\mathbb{C}} pr_2^{-1}(\mathcal{O}_{\tilde{U}}) \) (the usual tensor product over \( \mathbb{C} \)) consists of functions of the form \( \sum_{i=1}^n f_i \cdot g_i \), where the \( f_i \) are constant with respect to \( \tilde{U} \) and the \( g_i \) are constant with respect to \( Y \). Therefore the sheaf \( pr_1^{-1}(\mathcal{O}_Y^n) \otimes_{\mathbb{C}} pr_2^{-1}(\mathcal{O}_{\tilde{U}}) \) consisting of \( n \)-tuples of such functions is much smaller then \( \mathcal{N} \).

We will now define a connection \((\mathcal{M}_U, \nabla_U)\) on \( Y \times U \) by a construction similar to the one in the previous section. Define a \( \pi_1 \)-action on \( \mathcal{N} \) given by the formula \( \lambda(v \otimes f) = \rho(\lambda)v \otimes (f \circ \lambda^{-1}) \) for \( v \in pr_1^{-1}(\mathcal{O}_Y^n) \), \( f \in pr_2^{-1}(\mathcal{O}_{\tilde{U}}) \), and \( \lambda \in \pi_1 \cong \text{Aut}(\tilde{U}/U) \). Let \( \mathcal{M}_U := \mathcal{N}^{\pi_1} \), then \( \mathcal{M}_U \) defines a vector bundle on \( Y \times U \). Let \( \nabla : \mathcal{N} \to \mathcal{N} \otimes \Omega^1_{Y \times \tilde{U}/Y} \) be given by \( \nabla(v \otimes f) = v \otimes df \). The connection \( \nabla \) commutes with the \( \pi_1 \)-action, and we get an induced connection \((\mathcal{M}_U, \nabla_U)\) on \( Y \times U \) with monodromy representation given by \( \rho \).

Now we will extend \((\mathcal{M}_U, \nabla_U)\) to a connection on \( Y \times \mathbb{P}^1 \). Let \( s \in S \),
and let $O^*_a := \{ z \in U | 0 < |z - s| < \varepsilon \}$ be a small neighborhood of $a$. The inverse image of $O^*_a$ under the natural map $u : \tilde{U} \to U$ consists of a number of connected components. Let $O_s$ be one of them, then $u : O_s \to O^*_a$ is a universal covering ([F77] Section 31.4). Let $\lambda \in \pi_1$ be a loop around $s$. The subgroup of $\pi_1$ mapping $O_s$ to itself is cyclic with generator $\lambda$.

**Lemma 4.5** Let $Y$ be an irreducible Stein manifold over $\mathbb{C}$ and $A \in M_n(\mathbb{C})$ a matrix with the property that the differences of the eigenvalues of $A$ are not in $\mathbb{Z} \setminus \{0\}$. If $M \in \text{GL}_n(\mathcal{O}(Y))$ satisfies $M(y) \sim e^{2\pi i A} \forall y \in Y$, then there exists $B \in M_n(\mathcal{O}(Y))$ with $M = e^{2\pi i B}$ and $B(y) \sim A \forall y \in Y$.

**Proof.** Let $K := \text{Frac}(\mathcal{O}(Y))$, and let $\mu_k, \cdots, \mu_p$ be the distinct eigenvalues of $A$. Write $\nu_j := e^{2\pi i \mu_j}$, then $\nu_1, \cdots, \nu_p$ are the distinct eigenvalues of $M$. We can make a decomposition $M = M_s M_u$, with $M_s$ semi-simple and $M_u$ unipotent. One can write $M_s$ and $M^{-1}_s$ as polynomials in $M$ with coefficients in $\mathbb{C}$, so $M_s, M_u \in \text{GL}_n(\mathcal{O}(Y))$. Let $V_i := \ker(M_s - \nu_i I, K^n)$, then $K^n = V_1 \oplus \cdots \oplus V_p$. For $w \in \mathcal{O}(Y)^n$ we can write $w = w_1 + \cdots + w_p$, with $w_i \in V_i$. Now $M_s^n (w) = \nu_1^m w_1 + \cdots + \nu_p^m w_p \in \mathcal{O}(Y)^n, m \geq 0$. Using the fact that the Vandermonde matrix $\begin{pmatrix} 1 & \nu_1 & \cdots & \nu_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \nu_p & \cdots & \nu_p^n \end{pmatrix}$ is invertible, we see that all $w_i \in \mathcal{O}(Y)^n$, so we can write $\mathcal{O}(Y)^n = \oplus W_i$, $W_i := \ker(M_s - \nu_i I, \mathcal{O}(Y)^n)$.

Let $B_s \in M_n(\mathcal{O}(Y))$ be the linear map that acts as multiplication by $\mu_i$ on $W_i$, and let $B_n$ be defined as the finite sum $\frac{1}{2\pi i} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} (M_u - I)^j$. We will show that $B := B_s + B_n$ satisfies the lemma. We have that $e^{2\pi i B_s} = M_s$ and $e^{2\pi i B_n} = M_u$. Since $B_s$ and $B_n$ commute it is clear that $M = e^{2\pi i B}$. Furthermore $e^{2\pi i B(y)} \sim e^{2\pi i A} \forall y \in Y$, and the eigenvalues of $B(y)$ and $A$ correspond. By construction we have $B(y) \sim A \forall y \in Y$. 

We find that we can write $\rho(\lambda_j) = e^{2\pi i B_i}, B_i \in M_n(\mathcal{O}(Y))$, with $B_i(y) \sim A_i$ for all $y \in Y$. Let $s = s_i$ be the singular point we fixed, then we write $B$ for $B_i$.

For notational convenience we replace the covering $u : \tilde{O}_s \to O_s^*$ by the covering $exp : \mathbb{C} \to \mathbb{C}^*, z \mapsto e^{2\pi i z}$. The group $\text{Aut}(\mathbb{C}/\mathbb{C}^*)$ is generated by $t : z \mapsto z + 1$. The restriction of $\mathcal{N}$ to $Y \times \mathbb{C}$ is $p_1^{-1}(\mathcal{O}_Y) \otimes p_2^{-1}(\mathcal{O}_\mathbb{C})$, and
we want to calculate \( (N_e, \nabla) \) explicitly. So we have to calculate the action of \( t \in \text{Aut}(\mathbb{C}/\mathbb{C}^*) \) on \( N|_{\mathbb{C} \times \mathbb{C}} \). Let \( v(y, z) \) be a section of \( N|_{\mathbb{C} \times \mathbb{C}} \). Using the explicit description of the \( \pi_1 \)-action on \( \mathcal{N} \) given in the beginning of this construction we find \( t(v(y, z)) = e^{2\pi i B} v(y, z - 1) \). Write \( v(y, z) = e^{2\pi i B z} w(y, z) \), then the condition \( t(v) = v \) is equivalent to \( w(y, z) = w(y, z - 1) \). So \( t(v) = v \iff w(y, z) = \tilde{w}(y, e^{2\pi i z}) \) for some section \( \tilde{w} \) of \( O_{Y \times \mathbb{C}} \).

We find that \( (N|_{\mathbb{C} \times \mathbb{C}})^{(t)} \cong O_{Y \times \mathbb{C}} \) is a free vector bundle on \( Y \times \mathbb{C}^* \) with generators \( \{f_1, \ldots, f_n\} \), \( f_i = e^{2\pi i B z} e_i \), where \( \{e_1, \ldots, e_n\} \) is the standard free basis for \( O_{Y \times \mathbb{C}} \). Furthermore \( \nabla \) is given by \( \nabla(f_i) = 2\pi i B f_i d z \). We have that \( u := e^{2\pi i z} \) is a parameter on \( \mathbb{C}^* \), and we find \( \nabla(f_i) = B f_i \frac{d}{dz} \). Using this formula, we can extend the connection \( ((N|_{\mathbb{C} \times \mathbb{C}})^{(t)}, \nabla) \) on \( \mathbb{C}^* \) to a connection on \( \mathbb{C} \supset \mathbb{C}^* \). In this way we can make an extension of \( (M_U, \nabla_U) \) to a connection on \( Y \times \mathbb{P}^1 \). ☐

In the following we want to construct a family of differential equations parametrized by a certain space of monodromy representations. Suppose we are given regular singular moduli problem in the sense of Chapter 2, with data \((V, \{s_1, \ldots, s_r\}, \{\frac{d}{dz} + C_i\})_{j=1}^r\), \( C_j \in \text{GL}(V) \). Consider the set of corresponding monodromy representations \( M := \{\rho \in \text{Repr}(\pi_1, V) | \rho(\lambda_j) \sim e^{-2\pi i C_j} \} \).

We can identify \( M \) with the set \( \{(M_1, \ldots, M_r) | M_j \sim e^{-2\pi i C_j}, \prod_{j=1}^r M_j = I \} \) by identifying \( \rho \) with \( \{\rho(\lambda_1), \ldots, \rho(\lambda_r)\} \).

**Lemma 4.6** The set \( M \) is a Zariski constructible subset of \( \text{GL}(V)^r \) and, if all matrices \( C_i \) are diagonalizable, even Zariski closed. Furthermore the subset of \( M \) consisting of irreducible representations is also Zariski constructible.

**Proof.** It clearly is sufficient to prove corresponding statements for the set \( M' \) obtained by dropping the condition \( \prod_{i=1}^r M_i = I \). In proving the first statement, we may suppose \( r = 1 \). For a diagonal matrix \( C \in \text{GL}(V) \) with characteristic polynomial \( P_C = \prod (T - \mu_i)^{m_i} \), the set \( \{A C A^{-1} | A \in \text{GL}(V) \} \) is given by \( \{B \in \text{GL}(V) | P_B = P_C, \text{rank}(B - \mu_i I) = n - m_i \ \forall \ i \} \). The latter condition is equivalent to the condition that the determinant of all \( l \times l \) submatrices of \( B - \mu I \), with \( l > n - m_i \), is zero. This clearly defines a closed set. For an arbitrary matrix \( C \in \text{GL}(V) \), we have that \( B \) is similar to \( C \) if and only if \( P_B = P_C \), and \( \text{rank}((B - \mu I)^m) = \text{rank}((C - \mu I)^m), m = 1, \ldots, n \), for every eigenvalue \( \mu \). This defines a constructible set. To be precise,
rank\( (A) \leq m \) defines a closed subset of \( GL(V) \), so rank\( (A) = m \) defines a constructible subset.

We will now prove the second statement. Note that the set of matrices in \( GL(V) \) that leave a line \( \mathbb{C} \cdot v, \ v \in V \) invariant is given by \( \{ M | Mv \wedge v = 0 \} \), where \( \wedge \) denotes the exterior product. So the set of tuples \( (M_1, \cdots, M_n) \) that leave a line invariant is obtained by first taking the kernel of the map \( V \setminus \{ 0 \} \times GL(V)^n \to \mathbb{C}^n, \ (v, M_1, \cdots, M_n) \mapsto (M_1 v \wedge v, \cdots, M_n v \wedge v) \) and then taking the projection of this kernel onto \( GL(V)^n \). This clearly defines a constructible set. The matrices that leave a subspace of dimension \( l < n \) invariant, are the matrices that leave a decomposable line in \( \Lambda^l V \) invariant. This also defines a constructible set. Since the complement of a constructible set is constructible, this proves the lemma. \( \square \)

The family \( \mathcal{M} \) of representations gives rise to a family of differential equations parametrized by \( \mathcal{M} \), according to Theorem 4.3. In more detail, given \( \lambda \in \pi_1 \), a representation \( m \in \mathcal{M} \) yields an element \( m(\lambda) \in GL(V) \). This defines a morphism \( \rho(\lambda) : \mathcal{M} \to GL(V) \) which we regard as an element \( \rho(\lambda) \in GL(\mathcal{O}(\mathcal{M}) \otimes V) \). We obtain a representation \( \rho : \pi_1 \to GL(\mathcal{O}(\mathcal{M}) \otimes V) \).

By Theorem 4.3 the representation \( \rho \) gives rise to a family of differential equations \( (\mathcal{M}, \nabla, V, \frac{d}{dt} + \frac{C_\lambda}{t}) \) parametrized by \( \mathcal{M} \). For \( m \in \mathcal{M} \), the monodromy representation of \( (\mathcal{M}(m), \nabla(m)) \) is clearly congruent to \( m \). By the classical Riemann-Hilbert theorem, and Lemma 4.1 the connection \( (\mathcal{M}(m), \nabla(m)) \) is unique up to isomorphism.

We conclude this section by a lemma on the local invertibility of the exponential map. It states that under more general conditions than in Lemma 4.5 one can still locally construct a logarithm.

**Lemma 4.7** The map \( E : M_d(\mathbb{C}) \to GL_d(\mathbb{C}), A \mapsto e^{2\pi i A} \) is locally invertible in \( A \) if and only if \( \lambda_i - \lambda_j \notin \mathbb{Z} \setminus \{ 0 \} \) for all couples of eigenvalues \( \lambda_i, \lambda_j \) of \( A \).

**Proof.** We start by proving that if there are two eigenvalues \( \lambda_1, \lambda_2 \) of \( A \) with \( \lambda_1 - \lambda_2 \in \mathbb{Z} \setminus \{ 0 \} \), then \( E \) is not locally invertible. Write \( A = SJS^{-1} \), with \( J \) in Jordan normal form. We will show that there exists a matrix \( B \neq 0 \), with \( E(J + \varepsilon B) = E(J), \varepsilon^2 = 0 \) (where we use the extension of \( E \) to a map on \( M_d(\mathbb{C}[\varepsilon]) \)). If \( E(J + \varepsilon B) = E(J) \) then also \( E(A + \varepsilon SBS^{-1}) = E(A) \) holds. We can suppose that \( J \) has only two eigenvalues \( \lambda, \lambda + m, m \in \mathbb{Z} \setminus \{ 0 \} \) and
only two Jordan blocks of size $j$ and $d - j$ respectively. Subtracting $\lambda \cdot I_d$
from $J$, we may assume that $J$ has eigenvalues $0, m$. Define $B$ by $B_{j+1,j} = 1$, 
and zeros everywhere else. Then $JB = mB$, $BJ = 0$. It follows that

$$E(J + \varepsilon B) - E(J) = \left( \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p} \varepsilon \right) = \left( \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} m^{n-1} B \varepsilon \right) =$$

$$\frac{1}{m} (e^{2\pi i m} - 1) B \varepsilon = 0.$$ 

To prove the converse, again write $A = SJS^{-1}$. We will first consider the
case where $J$ is a diagonal matrix. For a matrix $B$ with only one nonzero 
entry $B_{ij} = 1$, we have that $E(J + \varepsilon B) - E(J)$ also has (at most) one nonzero 
entry at the same place. The fact that the remaining coefficient is nonzero 
follows from an explicit calculation in the case $J = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \notin \mathbb{Z}$. We 
conclude that the derivative of $E$ at $A$ is bijective. For the general case, write 
$J = D + N$, where $D$ is diagonal, $N$ is nilpotent, and $ND = DN$. We will 
use the matrix norm $\|A\| = \max\{|A_{ij}|, 1 \leq i, j \leq d\}$, which has the property 
$\|AB\| \leq d \|A\| \|B\|$. The idea of the proof is as follows. Local invertibility at 
$J$ is equivalent to local invertibility at a conjugate $SJS^{-1}$. We can pick $S$ 
such that $\|SNS^{-1}\|$ becomes arbitrary small. An estimate then shows that 
local invertibility at $SJS^{-1}$ implies local invertibility at $SJS^{-1}$.

We have

$$E(J + \varepsilon B) - E(J) = \left( \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p} \varepsilon \right) =$$

$$\varepsilon \left( \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} (J^p - D^p) B J^{n-1-p} + D^p B (J^{n-1-p} - D^{n-1-p}) + \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} D^p B D^{n-1-p} \right).$$

Write $a := \left\| \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} D^p B D^{n-1-p} \right\|$. Then $a > 0$ by the argument 
above. We can make the following estimate:

$$\left\| \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p} \right\| \geq$$
\[ a - \left\| \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} (J^p - D^p) B J^{n-1-p} + D^p B (J^{n-1-p} - D^{n-1-p}) \right\| . \]

Write \( \delta := \|N\|, s := \|D\|, t := s + \delta \). We will now use the estimate

\[ \|J^p - D^p\| = \|N \sum_{k=1}^{p} \left( \frac{p}{k} \right) D^{p-k} N^{k-1}\| \leq d^p \delta \sum_{k=1}^{p} \left( \frac{p}{k} \right) s^{p-k} \delta^{k-1} \leq d^p p t^{p-1} \delta. \]

Writing \( b := \|B\| \), we find that

\[ \left\| \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p} \right\| \geq \]

\[ a - \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} d^{n-1} (p b^{n-1} b t^{n-1-p} \delta + (n-1-p) t^{p} b t^{n-1-p-2} \delta) = \]

\[ a - \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} d^{n-1} (n-1) t^{n-2} b \delta = a - \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} n(n-1)(dt)^{n-2} d b \delta = \]

\[ a - (2\pi i)^2 \sum_{n \geq 0} \frac{(2\pi i)^n}{n!} (dt)^n d b \delta = a - (2\pi i)^2 e^{2\pi i dt} d b \varepsilon. \]

So for any matrix \( B \), we can pick a basis (and therefore a small \( \delta \)), such that \( \|E(J + \varepsilon B) - E(J)\| > 0 \), which shows that \( E \) is locally invertible at \( J \), and therefore at \( A \). \( \square \)

For a vector bundle \( \mathcal{M} \) obtained by Theorem 4.3, there can be points \( y \in Y \) such that the induced vector bundle \( \mathcal{M}(y) \) on \( \mathbb{P}^1 \) is not free. This situation already appears in the Lamé example as we will see later on. Before we get to the Lamé example, we will first study connections on non-free vector bundles in detail.

### 4.3 Connections on non-free vector bundles

We will now give a precise description of connections on non-free vector bundles, and construct a fine moduli space for such connections.
Let $\mathcal{M} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$, $a_1 \geq \cdots \geq a_n$ be a vector bundle, and $D = \sum_{i=1}^r k_i[s_i]$ a divisor of degree $k := \sum_{i=1}^r k_i$, with all $s_i \neq \infty$. We can write $\mathcal{O}(a_i)(U_0) = \mathbb{C}[z]e_i$, $\mathcal{O}(a_i)(U_\infty) = \mathbb{C}[z^{-1}]f_i$, with $f_i = z^{a_i}e_i$. A connection $\nabla : \mathcal{M} \to \Omega(D) \otimes \mathcal{M}$ is given by two connections on the free vector bundles $\mathcal{M}(U_0), \mathcal{M}(U_\infty)$, say $\nabla_0 : \mathcal{M}(U_0) \to \Omega(D)(U_0) \otimes \mathcal{M}(U_0)$ and $\nabla_\infty : \mathcal{M}(U_\infty) \to \Omega(D)(U_\infty) \otimes \mathcal{M}(U_\infty)$ that glue on $U_0 \cap U_\infty$. We have that $\Omega(D)(U_0) = \mathbb{C}[z] \frac{dz}{\prod_{i=1}^r t_i^{k_i}}$ (where as always $t_i = z-s_i$), so $\nabla_0$ is given by a $C[z]$-linear map $A$ on $\mathbb{C}[z] \langle e_1, \cdots, e_n \rangle$, taking $\nabla_0(e_i) = A(e_i) = A(e_i) \frac{dz}{\prod_{i=1}^r t_i^{k_i}}$. We will also write $A$ for the matrix of $A$ on the basis $\{e_1, \cdots, e_n\}$. In the same way the connection $\nabla_\infty$ is defined by $\nabla_\infty(f_i) = B(f_i) \frac{dz}{\prod_{i=1}^r t_i^{k_i}}$, with $B$ given by a matrix $B \in M_n(z^{k-2}\mathbb{C}[z^{-1}])$. For the connections $\nabla_0$ and $\nabla_\infty$ to glue, we must have $\nabla_0(z^{a_i}e_i) = \nabla_\infty(f_i)$. This translates into $\prod_{i=1}^r t_i^{k_i} a_i z^{a_i-1} + z^n A_{ii} = z^n B_{ii}$ for $i = 1, \cdots, n$ and $z^{a_j}A_{ij} = z^{a_i}B_{ij}$ for $i, j = 1 \cdots n$, $i \neq j$. From this we obtain the following properties for $A$:

- $\deg(A_{ij}) \leq k + a_i - a_j - 2$ for $i \neq j$,
- $\deg(A_{ii}) = k - 1$,
- $A_{ii}$ has as highest order coefficient $-a_i$.

Conversely, a matrix $A \in M_n(\mathbb{C}[z])$ satisfying these properties defines a connection on $\mathcal{M}$.

In the following we will use the group of automorphisms of $\mathcal{M}$, so we give an explicit description of it. An automorphism $\psi$ of $\mathcal{M}$ is given by a $C[z]$-linear automorphism of $\mathcal{M}(U_0)$ and a $C[z^{-1}]$-linear automorphism of $\mathcal{M}(U_\infty)$ that glue. So $\psi(U_0)$ is given on the basis $\{e_1, \cdots, e_n\}$ by a matrix $A \in GL_n(\mathbb{C}[z])$. Furthermore $\psi(U_\infty)$ is given on the basis $\{f_1, \cdots, f_n\}$ by a matrix $B \in GL_n(\mathbb{C}[z^{-1}])$ with $B = Z^{-1}AZ$, where $Z$ is the diagonal matrix with $Z_{ii} = z^{a_i}$. Let $a_{n_1}, \cdots, a_{n_p}$ be the subsequence of $a_1, \cdots, a_n$ consisting of $a_1$ and the $a_i$ with $a_i - a_{i-1} < 0$. Then we can write $A$ in block form:

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1p} \\ 0 & \ddots & \vdots \\ \vdots & \ddots & A_{pp} \end{pmatrix}.$$
Here the $A_{ij} \in \text{GL}_{n+1-n_0}(C)$ (where we take $n_{p+1} = n+1$) and the coefficients of $A_{ij}$, $i > j$ are polynomials of degree $\leq a_{n_i} - a_{n_j}$. Conversely any such matrix $A$ defines an automorphism of $\mathcal{M}$.

### 4.3.1 Moduli spaces of non-free connections

We will define a moduli space of connections on a vector bundle of some fixed type associated to a data set $(V, \{s_1, \cdots, s_r\}, \{\frac{d}{dz} + B_i\}_{i=1}^s)$ as in Chapter 2. We fix an ordered basis for $V$, say $\{e_1, \cdots, e_n\}$. Define a vector bundle $\mathcal{M}$ of type $(a_1, \cdots, a_n)$, $a_1 \geq \cdots \geq a_n$ by $\mathcal{M}(U_0) = \mathbb{C}[z] \otimes V$, $\mathcal{M}(U_\infty) = \mathbb{C}[z^{-1}] \otimes (\oplus \mathbb{C}^d e_i)$. We fix a type $(a_1, \cdots, a_n)$ with $a_1 - a_n \leq r - 1$, and we will only consider connections on the corresponding vector bundle $\mathcal{M}$.

Note that in case $\mathcal{M}$ has rank 2, and there exists an irreducible connection on $\mathcal{M}$, then by [PS03] Proposition 6.21 we get $a_2 - a_1 \leq r - 2$.

We start by defining a functor $\mathcal{F}^+$ in a similar way to the definition of $\mathcal{F}$ in Chapter 2, but now we do not divide out equivalence.

**Definition 4.8** The functor $\mathcal{F}^+: \{\mathbb{C}\text{-algebras}\} \to \{\text{sets}\}$ is defined as follows. For any $\mathbb{C}$-algebra $R$, the set $\mathcal{F}^+(R)$ consists of the tuples $(A, \{\phi_i\}_{i=1}^s)$, where:

- $A \in M_n(R[z])$ satisfies $\deg(A_{ij}) \leq k + a_i - a_j - 2$ for $i \neq j$ and $\deg(A_{ii}) = k - 1$. Furthermore $A_{ii}$ has as highest order coefficient $-a_i$.
- the $\phi_i = \sum_{j=0}^\infty \phi_i(j)(t_i)^j$, $\phi_i(j) \in M_n(R)$ are automorphisms of $R[[t_i]]^n$.
- $\phi_i(\frac{d}{dz} + A_{i-1}t_i^{-1})\phi_i^{-1} = \frac{d}{dz} + B_i$, $i = 1, \cdots, s$, where we see $-\frac{A_{i-1}t_i^{-1}}{t_i}$ and $\phi_i$ as elements of $\text{End}(R[[t_i]][t_i^{-1}]^n)$. This condition can be restated as $\frac{d}{dz} + A_{i-1}t_i^{-1} - B_i\phi_i$.

This functor $\mathcal{F}^+$ is represented by a $\mathbb{C}$-algebra of finite type $U$, as can be shown in a way similar to the proof of Theorem 2.9. We can also consider $\mathcal{F}^+$ as a contravariant functor on schemes of finite type over $\mathbb{C}$. In this setting $\mathcal{F}^+$ is represented by $\mathcal{M} := \text{Spec}(U)$.

We say that two tuples $(A_1, \{\phi_1^i\}), (A_2, \{\phi_2^i\}) \in \mathcal{F}^+(R)$ are equivalent if there
exists an automorphism $\psi$ of $\mathcal{M} \otimes R$ such that
\[ \frac{A_2}{dz} + \frac{A_1}{\Pi_{i=1}^{r-1} \tau_i} = \psi^{-1} \left( \frac{d}{dz} + \frac{A_1}{\Pi_{i=1}^{r-1} \tau_i} \right) \psi \]
and $\phi_i^2 = \phi_i^1 \circ \psi$, $i = 1, \ldots, r$ where we consider $\psi$ as an element of $GL_n(\mathbb{C}[z])$ and $GL_n(R[[z]])$ respectively. We define a functor $\mathcal{F}$ by $\mathcal{F}(R) = \mathcal{F}^+(R)/\sim$.

Theorem 4.9 There is a coarse moduli scheme for the functor $\mathcal{F}$ defined above, which is in fact a quasi projective variety.

Proof. Consider the group $G := Aut(\mathcal{M})$. This group acts on $\mathbb{M}(\mathbb{C})$ and we want to make a quotient. We can make an embedding $G \subset GL_n(\mathbb{C}[z])$. From the description of $G$ above we see that the degree of the coefficients of elements of $G$ is bounded by $\max_{i=1, \ldots, n-1} (a_i - a_i + 1)$. By our assumption on $\mathcal{M}$ this bound is less or equal to $r - 2$. Therefore the map $\psi : G \to GL_n(\mathbb{C})^r$ given by $A(z) \mapsto (A(s_1), \ldots, A(s_r))$ is injective. In this way we can consider $G$ as a linear algebraic subgroup of $GL_n(\mathbb{C})^r$. By [Br69] Theorem 6.8, the quotient $GL_n(\mathbb{C})^r/G$ exists and is given by $(Q, \pi)$, $\pi : GL_n(\mathbb{C})^r \to Q$, with $Q$ a quasi-projective variety. Let $\phi : \mathbb{M} \to GL_n(\mathbb{C})^r$, $(A, \{\phi_i\}) \mapsto (\phi_i(0), \ldots, \phi_r(0))$ be the projection map. We want to use the following proposition to prove that a geometric quotient $\mathbb{M}/G$ exists and is quasi-projective.

Proposition 4.10 (Proposition 7.1 of [MFK94])
Let $G$ be a group scheme, flat and of finite type over $S$. Let $X$ and $Y$ be schemes of finite type over $S$, let $\sigma$ and $\tau$ be actions of $G$ on $X$ and $Y$, and let $\phi : X \to Y$ be a $G$-linear morphism. Assume that $Y$ is a principal fibre bundle over an $S$-scheme $Q$, with group $G$, and with projection $\pi : Y \to Q$. Assume that there exists an $L \in \text{Pic}^G(X)$ which is relatively ample for $\phi$, and that $Q$ is quasi-projective over $S$. Then there is a scheme $P$, quasi-projective over $S$, and an $S$-morphism $\omega : X \to P$ such that $X$ becomes a principal fibre bundle over $P$ with group $G$, and projection $\omega$.

This needs some explanation. A principal fibre bundle is defined as follows: let $\sigma : G \times_X X \to X$ be an action, with a geometric quotient $(Q, \pi)$, then $X$ is a principal fibre bundle over $Q$ with group $G$ if

- $\pi$ is a flat morphism of finite type,
- the map $(\sigma, pr_2) : G \times_X X \to X \times_X X \subset X \times_X X$ is an isomorphism.
By Proposition 0.9 of [MFK94] for a free action of an algebraic group $G$ on an algebraic scheme $X$ with geometric quotient $(Q, \pi)$, the scheme $X$ always is a principal fibre bundle over $Q$ with group $G$.

We further remark that $Pic^G(X)$ is the group of $G$-linearized line bundles on $X$. For details see [MFK94].

We want to apply this proposition with $S = \text{Spec} (\mathbb{C})$, $X = \mathcal{M}$, $Y = \text{GL}_n^r$, and $G, \phi, Q, \pi$ as above. There are a number of conditions to be checked.

1. $\phi$ is $G$-linear.

2. $\text{GL}_n^r$ is a principal fibre bundle over $Q$ with group $\text{Aut}(M)$.

3. There exists an $L$ as in the proposition.

Condition (1) is clearly fulfilled. For the line bundle $L$ in (3) we can take the trivial line bundle since $\mathcal{M}$ is affine. By Proposition 0.9 of [MFK94] for a free action of an algebraic group $G$ on an algebraic scheme $Y$ with geometric quotient $(Q, \pi)$, the scheme $Y$ always is a principal fibre bundle over $Q$ with group $G$. So to prove (3) it suffices to show that the action of $G$ on $\text{GL}_n^r$ is free, and that $Q$ is a geometric quotient. The action being free means that $(\sigma, \mu_{\mathcal{M}}) : G \times \text{GL}_n^r \to \text{GL}_n^r \times_Q \text{GL}_n^r$ is a closed immersion, which is the case. The fact that $(Q, \pi)$ is a geometric quotient follows from the definition of a quotient in [Br69].

We will now proof that $P$ is a coarse moduli scheme for $\mathcal{F}$ by an argument as in the proof of Proposition 5.4 of [MFK94]. There is a natural isomorphism $\phi^+: \mathcal{F}^+ \to \text{Hom}(\ast, P)$, which induces a natural isomorphism $\phi : \mathcal{F} \to \text{Hom}(\ast, P)$. For $(P, \phi)$ to be a coarse moduli space, the following conditions have to be verified.

- for every algebraically closed field $k$, the map
  $$\phi(\text{Spec} k) : \mathcal{F}(\text{Spec} k) \to \text{Hom}(\text{Spec} k, P)$$
  is an isomorphism.

- given a scheme $N$ and a morphism $\psi : \mathcal{F} \to \text{Hom}(\ast, N)$, there is a unique morphism $\chi : \text{Hom}(\ast, P) \to \text{Hom}(\ast, N)$, such that $\psi = \chi \circ \phi$. 
4.2. The Riemann-Hilbert Problem

The first condition is verified since \((P, \omega)\) is a geometric quotient. To proof that the second condition is verified, consider the element \(i\hat{d} \in \mathcal{F}(\mathcal{M})\) induced by \(i\hat{d} \in \mathcal{F}^+(\mathcal{M}) \cong \text{Hom}(\mathcal{M}, \mathcal{M})\). To a morphism \(\psi : \mathcal{F} \to \text{Hom}(\ast, N)\), we associate the morphism \(f := \psi_{\mathcal{M}}(i\hat{d}) : \mathcal{M} \to N\). This induces a bijection of the set of morphisms from \(\mathcal{F}\) to representable functors and the set of morphisms \(f : \mathcal{M} \to N\) with \(N\) a scheme. It follows that the second condition is verified, and therefore \((P, \phi)\) is a coarse moduli space. \(\square\)

4.4 The Lamé equation

We will now consider the moduli problem with data

\[
\{( \{s_1, \ldots, s_4\}, \{ \frac{d}{dt_i} + \frac{1}{t_i} \left( \begin{array}{cc} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{array} \right) \}_{i=1}^4 \}.
\]

The corresponding set of "monodromy representations" \(\mathcal{M}\) defined above is given by \(\mathcal{M} = \{ (M_1, \ldots, M_4) | M_i \sim \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \prod_{i=1}^4 M_i = 1 \}. \) Write

\[
M_i := \left( \begin{array}{cc} p_i & q_i \\ r_i & -p_i \end{array} \right), ~ i = 1, 2, 3,
\]

then he coordinate ring of \(\mathcal{M}\) is given by

\[
R := \mathbb{C}[p_1, q_1, r_1, p_2, q_2, r_2, p_3, q_3, r_3]/I,
\]

\[
I = \langle p_1^2 + q_1r_1 + 1, p_2^2 + q_2r_2 + 1, p_3^2 + q_3r_3 + 1, f \rangle
\]

\[
f := -p_3q_2r_1 + p_3q_1r_2 + p_2q_3r_1 + p_2q_1r_2 - p_1q_3r_2 - p_1q_1r_3 + p_1q_2r_3.
\]

The following properties of \(\mathcal{M}\) are known (but also easily verified).

- \(\mathcal{M}\) is a five dimensional variety.

- The group \(\text{PGL}_2(\mathbb{C})\) acts on \(\mathcal{M}\) (by conjugation) and on its coordinate ring \(R\). The ring \(R^{\text{PGL}_2} = \mathbb{C}[t_1, t_2, t_3]/\langle t_1^2 + t_2^2 + t_3^2 + t_1t_2t_3 - 4 \rangle\) is the ring of invariants, where \(t_1 := \text{Tr}(M_2M_3), t_2 := \text{Tr}(M_1M_3), t_3 := \text{Tr}(M_1M_2)\). This follows immediately from [Bo03] Section 2.

- The variety \(M^{\text{PGL}_2} := \text{Spec}(R^{\text{PGL}_2})\) has 4 singular points, namely \((t_1, t_2, t_3) \in \{ (-2, 2, 2), (2, -2, 2), (2, 2, -2), (-2, -2, -2) \}\). Each one of these points corresponds to multiple \(\text{PGL}_2(\mathbb{C})\)-orbits. After deleting the 4 singular points and their preimages in \(\mathcal{M}\) we obtain a good quotient under \(\text{PGL}_2(\mathbb{C})\). In particular \(\mathcal{M}\) is reduced and irreducible.
• The preimage of the 4 singular points of $\mathbb{M}^{PGL_2}$ in $\mathbb{M}$ precisely consists of all reducible representations in $\mathbb{M}$.

• The complement $\mathbb{M}_\text{ir}$ is a smooth connected variety.

By Theorem 4.3 we obtain a family of differential equations parametrized by $\mathbb{M}$, say $(\mathcal{M}, \nabla, \mathbb{C}^2$, local data). For every irreducible representation $m \in \mathbb{M}$, the following lemma shows that $\mathcal{M}(m)$ is either free, or of type $(1, -1)$.

**Lemma 4.11** Let $(\mathcal{M}, \nabla)$ be an irreducible connection of rank 2 on $\mathbb{P}^1$, with four singular points such that the sum of the local exponents at each singular point is 0. Then the vector bundle $\mathcal{M}$ is of the type $\mathcal{O}(a) \oplus \mathcal{O}(-a)$, $a \in \{0, 1\}$.

**Proof.** Because the sum of the local exponents of $(\mathcal{M}, \nabla)$ is zero at each singular point, the induced connection $\bigwedge^2 \nabla$ on $\bigwedge^2 \mathcal{M}$ is everywhere regular. Since $\mathbb{P}^1$ is simply connected, $\bigwedge^2 \mathcal{M}$ is the trivial line bundle, and $\bigwedge^2 \nabla$ is the trivial connection. So $\mathcal{M}$ is of the type $\mathcal{O}(a) \oplus \mathcal{O}(-a)$, $a \geq 0$. By [PS03] Proposition 6.21, the defect of $\mathcal{M}$ is $\leq 2$. This proves the lemma. \(\square\)

We will now show that the set $\mathbb{M}^{(1,-1)} := \{ m \in \mathbb{M}_\text{ir}[\mathcal{M}(m) \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)\}$ is nonempty.

**Proposition 4.12** $\mathbb{M}^{(1,-1)}$ is a non-trivial divisor in $\mathbb{M}_\text{ir}$.

**Proof.** By Remark 3.12 (4) we have that $\mathbb{M}^{(1,-1)}$ is a divisor. So we only need to show that $\mathbb{M}^{(1,-1)}$ is non-empty. As we saw in Section 4.2 the connection $(\mathcal{M}(m), \nabla(m))$ is uniquely determined for every $m \in \mathbb{M}$. Therefore we only have to construct a connection on $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ with the correct local behavior. By the description of connections on non-free vector bundles above, we get that such a connection is given by a matrix $A$ of the form

$$A = \begin{pmatrix} a_0 + a_1 z + a_2 z^2 - z^3 & b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 \\ c_0 & d_0 + d_1 z + d_2 z^2 + z^3 \end{pmatrix}.$$  

We want that the connection is locally at the points $s_i$ formally isomorphic to $d \frac{dz}{dz} + \frac{1}{t_i} \begin{pmatrix} \frac{1}{t_i} & 0 \\ 0 & -\frac{1}{t_i} \end{pmatrix}$, so the Laurent series expansion of $\frac{A}{t_i^2}$ at a point $s_i$ has to be of the form $\frac{A_i}{t_i} + \text{h.o.t.}$, with $A_i$ similar to $\begin{pmatrix} \frac{1}{t_i} & 0 \\ 0 & -\frac{1}{t_i} \end{pmatrix}$. The
4.2. THE RIEMANN-HILBERT PROBLEM

\( A_i \) are of the form \( \left( \frac{p_i}{r_i}, \frac{q}{p_i} \right), p_i^2 + q_i^2 = \frac{1}{16} \) for all \( i \), and we can write

\[
\sum_{i=1}^4 \frac{A_{ij}}{t_i^j} = \sum_{i=1}^4 \frac{A_{ij}}{t_i^j} + \left( \begin{array}{cc}
0 & b_4 \\
0 & 0
\end{array} \right).
\]

We find that \( p_1 + p_2 + p_3 + p_4 + 1 = 0 \), and \( r_1, r_2, r_3, r_4 \) are multiples of \( r_1 \). So we get a 5-dimensional family of tuples \((A_1, \ldots, A_4, b_4)\), and hence a 5-dimensional family \( X \) of connections on \( \mathcal{O}(1) \oplus \mathcal{O}(-1) \).

The automorphism group \( G \) of \( \mathcal{O}(1) \oplus \mathcal{O}(-1) \) is \( \left\{ \left( \begin{array}{cc}
a & b + cz + dz^2 \\
o & e \end{array} \right) \right\}, a, e \neq 0 \} \).

A connection given by a matrix \( \Delta \) is equivalent to the one given by the matrix \( \tilde{\Delta} = \Phi^{-1}\Phi' + \Phi^{-1}A\Phi \), with \( \Phi \in G \). We can construct a one dimensional subfamily of \( X \) consisting of matrices of the form \( \left( \begin{array}{cc}
\frac{-z^3}{s^3} & z \\
1 & 1
\end{array} \right) \) parameterized by \( b_4 \) with no equivalent elements. In case \( s_i = i, i = 1, \cdots, 4 \) this family is

\[
\{ \left( \begin{array}{cc}
\frac{-z^3}{s^3} & \frac{6295}{2} - 3015z + 1800z^2 - 350z^3 + b_4 (24 - 50z + 35z^2 - 10z^3 + z^4) \\
1 & 1
\end{array} \right), b_4 \in \mathbb{C} \}.
\]

\( \Box \)

We remark that the above does not imply that \( \mathcal{M}^{(1,-1)} \) is 1-dimensional. Indeed, let \( \mathcal{M} \) be the vector bundle on \( \overline{\mathcal{M}} \) given by the Riemann-Hilbert construction. The type of \( \mathcal{M}(m), m \in \mathcal{M} \) is \((0,0)\) or \((1,-1)\), and since \( \mathcal{M} \) is irreducible we find by Remarks 3.12 that \( \mathcal{M}^{(1,-1)} \) is an analytic divisor on \( \mathcal{M} \) (since \( \mathcal{M}^{(1,-1)} \) is nonempty). We find that \( \mathcal{M}^{(1,-1)} \) is 4-dimensional, and so there are isomorphic connections in \( \mathcal{M}^{(1,-1)} \).

We can make similar calculations (which are in fact simpler) for the case of a free vector bundle. We get a 5-dimensional space of connections, with an action of the group \( SL_2 \). There exists a categorical quotient, which maps all reducible connections to four points. This 2-dimensional quotient is actually a geometric quotient on the space of irreducible connections.
monodromy representation equivalent to \( \rho \) is known as the weak Riemann-Hilbert problem. This question has a positive answer, which is precisely formulated in [PS03], Theorem 6.15.

The strong Riemann-Hilbert problem asks whether for a given representation \( \rho : \pi_1 \to \text{GL}(V) \) there is a Fuchsian connection over \( \mathbb{C}(z) \) with monodromy representation equivalent to \( \rho \). A Fuchsian connection is a connection which for the differentiation \( \frac{d}{dz} \) can be written in the form \( \frac{d}{dz} + \sum_{i=1}^{r} \frac{A_i}{z-z_i} \), with \( A_i \in M_n(\mathbb{C}) \). In general, for a given \( \rho \) there is no such Fuchsian connection. However under some conditions on \( \rho \) the strong Riemann-Hilbert problem has a positive answer, see sections 6.4 and 6.5 of [PS03] for details.

The strong Riemann-Hilbert problem can be restated in terms of connections on vector bundles. For a connection \((\mathcal{M}, \nabla)\) on \( \mathbb{P}^1_{\mathbb{C}} \) (where \( \mathcal{M} \) is not necessarily free), we get an induced connection \((\mathcal{M}_q, \nabla_q)\) over \( \mathbb{C}(z) \) by localization at the generic fibre. Therefore we can associate a monodromy map to \((\mathcal{M}, \nabla)\). It is easily seen that the strong Riemann-Hilbert problem precisely asks whether there is a connection on a free vector bundle with some given monodromy map.

For a representation \( \rho : \pi_1 \to \text{GL}(V) \), by [PS03] Theorem 6.15, we find an associated connection \((\mathcal{M}, \nabla)\) over \( \mathbb{C}(z) \). The following lemma states how we can associate a connection over \( \mathbb{P}^1_{\mathbb{C}} \) to \((\mathcal{M}, \nabla)\).

**Lemma 4.1 (Lemma 6.18 of [PS03])** Let \((\mathcal{M}, \nabla)\) be a regular singular connection over \( \mathbb{C}(z) \) with singular locus \( S \). For every \( s \in S \) we choose a local parameter \( t_s \). For every \( s \in S \) let \( \Lambda_s \subset \mathcal{M}_s := \mathcal{O}(t_s) \otimes \mathcal{M} \) be a lattice that satisfies \( \nabla(\Lambda_s) \subset \frac{dt_s}{t_s} \otimes \Lambda_s \) (the existence of such a lattice is equivalent to \((\mathcal{M}, \nabla)\) being regular singular at \( s \)). Then there is a unique regular singular connection \((\mathcal{M}, \nabla)\) on \( \mathbb{P}^1_{\mathbb{C}} \) with singular locus in \( S \) such that:

1. For every open \( U \subset \mathbb{P}^1_{\mathbb{C}} \), one has \( \mathcal{M}(U) \subset M \).
2. The generic fibre of \((\mathcal{M}, \nabla)\) is \((\mathcal{M}, \nabla)\).
3. \( \mathcal{M}_s = \Lambda_s \) for all \( s \in S \).

In the case where \((\mathcal{M}, \nabla)\) is irreducible, one can make a choice for the lattices \( \Lambda_s \) in such a way that the corresponding vector bundle \( \mathcal{M} \) is free (see