Chapter 4

The Riemann-Hilbert problem and examples

In this chapter we will give some examples of moduli spaces of differential equations, and we describe the connection with the Riemann-Hilbert problem.

4.1 The classical Riemann-Hilbert problem

We start by briefly recalling the classical Riemann-Hilbert problem as described in [PS03] Chapter 6.

Let $(M, \nabla)$ be a regular singular connection over $\mathbb{C}(z)$ with singular locus equal to $S = \{s_1, \cdots, s_r\} \subset \mathbb{P}^1$. This means that $M$ is a finite dimensional $\mathbb{C}(z)$-vector space, and $\nabla : M \to \mathbb{C}(z)dz \otimes M$ is a regular singular connection (see [PS03] 6.4.2). One defines a monodromy map associated to $(M, \nabla)$ in the following manner. Write $V := \ker(\mathbb{C}((z - b)) \otimes_{\mathbb{C}(z)} M, \nabla)$ for the local solution space at a regular point $b \in \mathbb{P}^1 \setminus S$ and define $\pi_1 := \pi_1(\mathbb{P}^1 \setminus S, b)$. Let $\lambda \in \pi_1$ be a loop, then we can make an analytic continuation of the local solutions $V$ along $\lambda$. This analytic continuation defines a linear map on $V$, so we can associate an element of $\text{GL}(V)$ to $\lambda$. This process results in a map $\pi_1 \to \text{GL}(V)$, called the monodromy map. The question if for any given representation $\rho : \pi_1 \to \text{GL}(V)$ there is a regular singular connection with
Theorem 6.22 of [PS03]). For general \((M, \nabla)\) this is not always the case.

We will conclude by briefly describing how a connection \((\mathcal{M}, \nabla)\) on \(\mathbb{P}^1_{\mathbb{C}}\) with a prescribed monodromy representation can be constructed. We will use a generalization of this construction in the next section.

Write \(U := \mathbb{P}^1_{\mathbb{C}} \setminus S\), so \(\pi_1 := \pi_1(U, 0)\). We start by constructing a regular connection on \(U\) with the prescribed monodromy. For this consider the universal covering \(u : \tilde{U} \to U\) of \(U\). Define a connection \((\mathcal{N}, \nabla)\) on \(\tilde{U}\) by \(\mathcal{N} := \mathbb{C}^n \otimes \mathcal{O}_{\tilde{U}}\), and \(\nabla(v \otimes f) = v \otimes f'\) for all \(v \in \mathbb{C}^n\), \(f \in \mathcal{O}_{\tilde{U}}\). Furthermore we define a \(\pi_1\)-action on \(\mathcal{N}\) by \(\lambda(v \otimes f) = \rho(\lambda)(v) \otimes (f \circ \lambda^{-1})\) \(\forall \lambda \in \pi_1\). The vector bundle \(\mathcal{N}\) corresponds to the geometric vector bundle \(\mathbb{C}^n \times \tilde{U}\), and the corresponding \(\pi_1\)-action is given by \(\lambda(v, \tilde{u}) = (\rho(\lambda)(v), \lambda(\tilde{u}))\), \(v \in \mathbb{C}^n\), \(\tilde{u} \in \tilde{U}\). Indeed in this way we get for a section \(h \times id : \tilde{U} \to \mathbb{C}^n \times \tilde{U}\), \(h \in \mathcal{N}(U)\), that \((\lambda(h) \times id)(\tilde{u}) = \lambda(h \times id(\lambda^{-1}(\tilde{u})))\). It is clear that the quotient \(\pi_1(\mathbb{C}^n \times \tilde{U})\) defines a geometric vector bundle on \(U = \pi_1(\tilde{U})\), with corresponding vector bundle \(\mathcal{M}_U := \mathcal{N}/\pi_1\). The \(\pi_1\)-action on \(\mathcal{N}\) commutes with \(\nabla\). So we find an induced connection \(\nabla_U\) on \(\mathcal{M}_U\). Write \(\mathcal{L} := \ker(\nabla_{\mathcal{M}_U}, \mathcal{M}_U)\). The only thing left to show is that \(\mathcal{L}\) is the local system corresponding to \(\rho\). There is a one to one correspondence between local systems on \(U\) and (trivial) local systems on \(\tilde{U}\) with a \(\pi_1\)-action. Under this correspondence \(\mathcal{L}\) clearly corresponds to \(\mathbb{C}^n\) with the defined \(\pi_1\)-action which is given by \(\rho\). This proves that \((\mathcal{M}_U, \nabla_U)\) has monodromy given by \(\rho\).

We now want to extend this connection \((\mathcal{M}_U, \nabla_U)\) to a connection on \(\mathbb{P}^1_{\mathbb{C}}\). Let \(s \in S\), and consider the pointed disk \(U_s^* := 0 < |z - s| < \varepsilon\). We will construct a connection on \(U_s := |z - s| < \varepsilon\) that glues to the restriction of \((\mathcal{M}_U, \nabla_U)\) to \(U^*_s\). For this we consider the local solution space \(V_s\) at \(s + \frac{\varepsilon}{2}\). The circle around \(s\) through \(s + \frac{\varepsilon}{2}\) induces a monodromy map \(B \in \text{GL}(V_s)\). Choose \(A \in \text{End}(V)\) such that \(|z| A = B\), then we define the connection \(\nabla_s\) on the vector bundle \(\mathcal{M}_s := \mathcal{O}_{U_s} \otimes V_s\) by \(\nabla_s(f \otimes v) = df \otimes v + z^{-1} \otimes A(v)\). The restriction of \((\mathcal{M}_s, \nabla_s)\) to \(U^*_s\) clearly has local monodromy \(B\). By [PS03] 6.6-3 this restriction is isomorphic to the restriction of \((\mathcal{M}_U, \nabla_U)\) to \(U^*_s\). Therefore we can glue the connection \((\mathcal{M}_s, \nabla_s)\) to \((\mathcal{M}_U, \nabla_U)\). In this way we obtain the desired connection \((\mathcal{M}, \nabla)\) on \(\mathbb{P}^1_{\mathbb{C}}\) extending \((\mathcal{M}_U, \nabla_U)\).
4.2 The Riemann-Hilbert problem for families

We will now consider the Riemann-Hilbert problem for families of differential equations. Let $Y$ be an analytic manifold, and let $S := \{s_1, \ldots, s_r\}$ be a set of points in $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, \infty\}$. Suppose that $(\mathcal{M}, \nabla)$ is an analytic family of differential equations on $\mathbb{P}^1$, parametrized by $Y$ (the definition of an analytic family is a straightforward variation of Definition 3.13). We suppose that $S$ is the set of singular points of $\nabla$; more precisely, for every $y \in Y$ the set of singular points of $\nabla(y)$ is $S$.

We write $pr_1 : Y \times \mathbb{P}^1 \to Y$, $pr_2 : Y \times \mathbb{P}^1 \to \mathbb{P}^1$ for the two projection maps. Let $U := \mathbb{P}^1 \setminus S$, and $\pi_1 := \pi_1(U, 0)$. We will also write $pr_1$, $pr_2$ for the restrictions to $Y \times U$ of the two projection maps. The kernel $\mathcal{L} := \ker(\nabla)$ of $\nabla|_{Y \times U}$ is a locally free $pr_1^*(\mathcal{O}_Y)$-module of rank $n$, where $n = \dim(V)$. For any $a \in U$ the embedding $j_a : Y \cong Y \times \{a\} \hookrightarrow Y \times U$ defines a vector bundle $\mathcal{L}_a := j_a^*(\mathcal{L})$ on $Y$.

We will now define a monodromy map $\pi_1 \to \text{Aut}(\mathcal{L}_0)$. Let $\lambda : [0, 1] \to U$ be a path in $U$. Then $(id \times \lambda)^*(\mathcal{L})$ is a $pr^*(\mathcal{O}_Y)$-module on $Y \times [0, 1]$, where $pr : Y \times [0, 1] \to Y$ is the projection map. Since $(id \times \lambda)^*(\mathcal{L})$ is a locally free sheaf, we find a canonical isomorphism $\mathcal{L}_{\lambda(0)} \cong \mathcal{L}_{\lambda(1)}$. In particular, a closed path $\lambda$ with $\lambda(0) = 0 \in \mathbb{P}^1_{\mathbb{C}}$ yields an automorphism of $\mathcal{L}_0$ and this defines a homomorphism $\pi_1 \to \text{Aut}(\mathcal{L}_0)$.

**Definition 4.2** We call the map $\pi_1 \to \text{Aut}(\mathcal{L}_0)$ constructed above, the monodromy map associated to $(\mathcal{M}, \nabla)$.

We now present a converse construction, which we interpret as a solution to the Riemann-Hilbert problem for families.

**Theorem 4.3** Let $Y$ be an analytic irreducible reduced manifold with a vector bundle $\mathcal{L}$ on it. Suppose we are given a set $S := \{s_1, \ldots, s_r\} \subset \mathbb{P}^1_{\mathbb{C}}$ and a representation $\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus S) \to \text{Aut}(\mathcal{L})$ satisfying the following properties.

- Let $\lambda_i \in \pi_1$ be loops around the points $s_i$ with $\prod_{i=1}^r \lambda_i = 1$. For every $y \in Y$ we have that $\rho(\lambda_i)(y) \sim e^{2\pi i A_i}$ for some fixed $A_i \in M_n(\mathbb{C})$. Here
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\[ \rho(\lambda_i)(y) \text{ denotes the automorphism on } \mathcal{L}_y/(m_y\mathcal{L}_y) \cong \mathbb{C}^n \text{ induced by } \rho(\lambda_i). \]

- None of the differences of the eigenvalues of \( A_i \) is in \( \mathbb{Z} \setminus \{0\} \).

Then there exists an analytic connection \((\mathcal{M}, \nabla)\) on \( Y \times \mathbb{P}^1 \), with singular points in \( S \) and monodromy map given by \( \rho \).

**Proof:** We can cover \( Y \) by Stein-manifolds \( Y_i \) such that \( \mathcal{L}|_{Y_i} \) is free for all \( i \). We will now construct a solution to the Riemann-Hilbert problem for families over \( Y_i \). From the construction it can be seen that the connections on the \( Y_i \) glue to a connection on \( Y \) with the appropriate monodromy map, hence this also solves the Riemann-Hilbert problem for families over \( Y \). From now on we assume \( \mathcal{L} \) to be free and \( Y \) to be a Stein-manifold, and \( \rho \) is given as a homomorphism \( \rho : \pi_1 \to \text{GL}_n(\mathcal{O}(Y)) \).

Let \( \tilde{U} \) be the universal covering of \( U \). We can identify \( \pi_1 \) with \( \text{Aut}(\tilde{U}/U) \). Write \( \pi_1 : Y \times \tilde{U} \to Y \), \( \pi_2 : Y \times \tilde{U} \to \tilde{U} \) for the two projection maps. The vector bundle \( \mathcal{N} := \mathcal{O}^n_Y \) can be written as \( \pi_1^{-1}(\mathcal{O}^n_Y) \otimes \mathcal{O} \), where \( \otimes \) is an “analytic tensor product”, as defined in [GR71] p.179.

**Remark 4.4** We note that \( \pi_1^{-1}(\mathcal{O}_Y) \) are the analytic functions on \( Y \times \tilde{U} \) which are constant with respect to \( \tilde{U} \). The sheaf \( \pi_1^{-1}(\mathcal{O}_Y) \otimes \mathcal{O} \) (the usual tensor product over \( \mathbb{C} \)) consists of functions of the form \( \sum_{i=1}^n f_i \cdot g_i \), where the \( f_i \) are constant with respect to \( \tilde{U} \) and the \( g_i \) are constant with respect to \( Y \). Therefore the sheaf \( \pi_1^{-1}(\mathcal{O}_Y) \otimes \mathcal{O} \) consisting of \( n \)-tuples of such functions is much smaller then \( \mathcal{N} \).

We will now define a connection \((\mathcal{M}_U, \nabla_U)\) on \( Y \times U \) by a construction similar to the one in the previous section. Define a \( \pi_1 \)-action on \( \mathcal{N} \) given by the formula \( \lambda(v \otimes f) = \rho(\lambda)v \otimes (f \circ \lambda^{-1}) \) for \( v \in \pi_1^{-1}(\mathcal{O}_Y) \), \( f \in \pi_2^{-1}(\mathcal{O}_U) \), and \( \lambda \in \pi_1 \cong \text{Aut}(\tilde{U}/U) \). Let \( \mathcal{M}_U := \mathcal{N}^{\pi_1} \), then \( \mathcal{M}_U \) defines a vector bundle on \( Y \times U \). Let \( \nabla : \mathcal{N} \to \mathcal{N} \otimes \Omega^1_{Y \times \tilde{U}/Y} \) be given by \( \nabla(v \otimes f) = v \otimes df \). The connection \( \nabla \) commutes with the \( \pi_1 \)-action, and we get an induced connection \((\mathcal{M}_U, \nabla_U)\) on \( Y \times U \) with monodromy representation given by \( \rho \).

Now we will extend \((\mathcal{M}_U, \nabla_U)\) to a connection on \( Y \times \mathbb{P}^1 \). Let \( s \in S \),
and let \( O_s^ε := \{ z ∈ U \mid 0 < |z - s| < ε \} \) be a small neighborhood of \( a \). The inverse image of \( O_s^ε \) under the natural map \( u : \bar{U} → U \) consists of a number of connected components. Let \( O_s \) be one of them, then \( u : \bar{O}_s → O_s^ε \) is a universal covering ([F77] Section 31.4). Let \( λ ∈ π_1 \) be a loop around \( s \). The subgroup of \( π_1 \) mapping \( \bar{O}_s \) to itself is cyclic with generator \( λ \).

**Lemma 4.5** Let \( Y \) be an irreducible Stein manifold over \( \mathbb{C} \) and \( A ∈ M_n(\mathbb{C}) \) a matrix with the property that the differences of the eigenvalues of \( A \) are not in \( \mathbb{Z} \setminus \{0\} \). If \( M ∈ GL_n(\mathcal{O}(Y)) \) satisfies \( M(y) \sim e^{2πiλ} \forall y ∈ Y \), then there exists \( B ∈ M_n(\mathcal{O}(Y)) \) with \( M = e^{2πiB} \) and \( B(y) \sim A \forall y ∈ Y \).

**Proof.** Let \( K := \text{Frac}(\mathcal{O}(Y)) \), and let \( μ_1, \ldots, μ_p \) be the distinct eigenvalues of \( A \). Write \( ν_j = e^{2πiμ_j} \), then \( ν_1, \ldots, ν_p \) are the distinct eigenvalues of \( M \). We can make a decomposition \( M = M_s M_u \), with \( M_s \) semi-simple and \( M_u \) unipotent. One can write \( M_s \) and \( M_s^{-1} \) as polynomials in \( M \) with coefficients in \( \mathbb{C} \), so \( M_s, M_u ∈ GL_n(\mathcal{O}(Y)) \). Let \( V_i := \ker(M_s - ν_iI, K^n) \), then \( K^n = V_1 ⊕ ⋯ ⊕ V_p \). For \( w ∈ \mathcal{O}(Y)^n \) we can write \( w = w_1 + ⋯ + w_p \), with \( w_i ∈ V_i \). Now \( M_s(w) = ν_1^m w_1 + ⋯ + ν_p^m w_p ∈ \mathcal{O}(Y)^n, m ≥ 0 \). Using the fact that the Vandermonde matrix

\[
\begin{pmatrix}
1 & ν_1 & \cdots & ν_1^n \\
1 & ν_2 & \cdots & ν_2^n \\
& & \vdots & \\
1 & ν_p & \cdots & ν_p^n
\end{pmatrix}
\]

is invertible, we see that all \( w_i ∈ \mathcal{O}(Y)^n \), so we can write \( \mathcal{O}(Y)^n = ⊕W_i, W_i := \ker(M_s - ν_iI, \mathcal{O}(Y)^n) \).

Let \( B_s ∈ M_n(\mathcal{O}(Y)) \) be the linear map that acts as multiplication by \( μ_i \) on \( W_i \), and let \( B_n \) be defined as the finite sum \( \frac{1}{2πi} ∑_{j=1}^∞ \frac{(-1)^j+1}{j} (M_u - I)^j \). We will show that \( B := B_s + B_n \) satisfies the lemma. We have that \( e^{2πiB_s} = M_s \) and \( e^{2πiB_n} = M_u \) since \( B_s \) and \( B_n \) commute it is clear that \( M = e^{2πiB} \).

Furthermore \( e^{2πiB(y)} \sim e^{2πiA} \forall y ∈ Y \), and the eigenvalues of \( B(y) \) and \( A \) correspond. By construction we have \( B(y) \sim A \forall y ∈ Y \).

We find that we can write \( ρ(λ_i) = e^{2πiB_i} \), \( B_i ∈ M_n(\mathcal{O}(Y)) \), with \( B_i(y) \sim A_i \) for all \( y ∈ Y \). Let \( s = s_i \) be the singular point we fixed, then we write \( B \) for \( B_i \).

For notational convenience we replace the covering \( u : \bar{O}_s → O_s^ε \) by the covering \( exp : \mathbb{C} → \mathbb{C} \), \( z → e^{2πiz} \). The group \( \text{Aut}(\mathbb{C}/\mathbb{C}^*) \) is generated by \( t : z → z + 1 \). The restriction of \( \mathcal{N} \) to \( Y × \mathbb{C} \) is \( pr_1^{-1}(\mathcal{O}_Y^n) ⊗ pr_2^{-1}(\mathcal{O}_\mathbb{C}) \), and
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we want to calculate \( \mathcal{N}^{(t)} \), \( \nabla \) explicitly. So we have to calculate the action of \( t \in \text{Aut}(\mathbb{C}/\mathbb{C}^*) \) on \( \mathcal{N}|_{Y \times \mathbb{C}} \). Let \( v(y, z) \) be a section of \( \mathcal{N}|_{Y \times \mathbb{C}} \). Using the explicit description of the \( \pi_1 \)-action on \( \mathcal{N} \) given in the beginning of this construction we find \( t(v(y, z)) = e^{2\pi i B} v(y, z - 1) \). Write \( v(y, z) = e^{2\pi i B} w(y, z) \), then the condition \( t(v) = v \) is equivalent to \( w(y, z) = w(y, z - 1) \). So \( t(v) = v \iff w(y, z) = \tilde{w}(y, e^{2\pi i z}) \) for some section \( \tilde{w} \) of \( \mathcal{O}^n_{Y \times \mathbb{C}} \).

We find that \( (\mathcal{N}|_{Y \times \mathbb{C}})^{(t)} \cong \mathcal{O}^n_{Y \times \mathbb{C}} \) is a free vector bundle on \( Y \times \mathbb{C}^* \) with generators \( \{f_1, \ldots, f_n\}, \ f_i = e^{2\pi i B} e_i, \) where \( \{e_1, \ldots, e_n\} \) is the standard free basis for \( \mathcal{O}^n_{Y \times \mathbb{C}} \). Furthermore \( \nabla \) is given by \( \nabla(f_i) = 2\pi i B f_i dz \). We have that \( u := e^{2\pi i z} \) is a parameter on \( \mathbb{C}^* \), and we find \( \nabla(f_i) = B f_i du \). Using this formula, we can extend the connection \( (\mathcal{N}|_{Y \times \mathbb{C}})^{(t)} \), \( \nabla \) on \( \mathbb{C}^* \) to a connection on \( \mathbb{C} \supset \mathbb{C}^* \). In this way we can make an extension of \( (\mathcal{M}_U, \nabla_U) \) to a connection on \( Y \times \mathbb{P}^1 \).

In the following we want to construct a family of differential equations parametrized by a certain space of monodromy representations. Suppose we are given regular singular moduli problem in the sense of Chapter 2, with data \((V, \{s_1, \ldots, s_r\}, \{\frac{d}{dz} + C_j\})_{j=1}^r, C_j \in \text{GL}(V)\). Consider the set of corresponding monodromy representations \( \mathbf{M} := \{\rho \in \text{Repr}(\pi_1, V) | \rho(\lambda_j) \sim e^{2\pi i c_j} \} \).

We can identify \( \mathbf{M} \) with the set \( \{M_1, \ldots, M_r | \prod_{j=1}^r M_j = I \} \) by identifying \( \rho \) with \( \{\rho(\lambda_1), \ldots, \rho(\lambda_r)\} \).

**Lemma 4.6** The set \( \mathbf{M} \) is a Zariski constructible subset of \( \text{GL}(V)^r \) and, if all matrices \( C_i \) are diagonalizable, even Zariski closed. Furthermore the subset of \( \mathbf{M} \) consisting of irreducible representations is also Zariski constructible.

**Proof.** It clearly is sufficient to prove corresponding statements for the set \( \mathbf{M}' \) obtained by dropping the condition \( \prod_{j=1}^r M_j = I \). In proving the first statement, we may suppose \( r = 1 \). For a diagonal matrix \( C \in \text{GL}(V) \) with characteristic polynomial \( P_C = \prod (T - \mu_i)^{m_i} \), the set \( \{A \in \text{GL}(V) | PC = \prod (T - \mu_i)^{m_i} \} \) is given by \( \{B \in \text{GL}(V) | P_B = P_C, \text{rank}(B - \mu_i I) = n - m_i \ \forall \ i \} \). The latter condition is equivalent to the condition that the determinant of all \( l \times l \)-submatrices of \( B - \mu I \), with \( l > n - m_i \), is zero. This clearly defines a closed set. For an arbitrary matrix \( C \in \text{GL}(V) \), we have that \( B \) is similar to \( C \) if and only if \( P_B = P_C \), and \( \text{rank}((B - \mu I)^m) = \text{rank}((C - \mu I)^m), m = 1, \ldots, n, \) for every eigenvalue \( \mu \). This defines a constructible set. To be precise,
rank(\(A\)) \leq m \text{ defines a closed subset of } GL(V), \text{ so rank}(A) = m \text{ defines a constructible subset.}

We will now prove the second statement. Note that the set of matrices in GL(V) that leave a line \(C \cdot v, v \in V\) invariant is given by \(\{M \mid Mv \cap v = 0\}\), where \(\cap\) denotes the exterior product. So the set of tuples \((M_1, \cdots, M_n)\) that leave a line invariant is obtained by first taking the kernel of the map \(V \setminus \{0\} \times GL(V)^n \rightarrow \mathbb{C}^n, (v, M_1, \cdots, M_n) \mapsto (M_1 v \cap v, \cdots, M_n v \cap v)\) and then taking the projection of this kernel onto \(GL(V)^n\). This clearly defines a constructible set. The matrices that leave a subspace of dimension \(l < n\) invariant, are the matrices that leave a decomposable line in \(\wedge^l V\) invariant. This also defines a constructible set. Since the complement of a constructible set is constructible, this proves the lemma. \(\square\)

The family \(M\) of representations gives rise to a family of differential equations parametrized by \(M\), according to Theorem 4.3. In more detail, given \(\lambda \in \pi_1\), a representation \(m \in M\) yields an element \(m(\lambda) \in GL(V)\). This defines a morphism \(\rho(\lambda) : M \rightarrow GL(V)\) which we regard as an element \(\rho(\lambda) \in GL(O(M) \otimes V)\). We obtain a representation \(\rho : \pi_1 \rightarrow GL(O(M) \otimes V)\). By Theorem 4.3 the representation \(\rho\) gives rise to a family of differential equations \((\mathcal{M}, \nabla, V, d_0 + \frac{C_5}{t})\) parametrized by \(M\). For \(m \in M\), the monodromy representation of \((\mathcal{M}(m), \nabla(m))\) is clearly congruent to \(m\). By the classical Riemann-Hilbert theorem, and Lemma 4.1 the connection \((\mathcal{M}(m), \nabla(m))\) is unique up to isomorphism.

We conclude this section by a lemma on the local invertibility of the exponential map. It states that under more general conditions than in Lemma 4.5 one can still locally construct a logarithm.

**Lemma 4.7** The map \(E : M_\delta(\mathbb{C}) \rightarrow GL_\delta(\mathbb{C}), A \mapsto e^{2\pi i A}\) is locally invertible in \(A\) if and only if \(\lambda_i - \lambda_j \notin \mathbb{Z} \setminus \{0\}\) for all couples of eigenvalues \(\lambda_i, \lambda_j\) of \(A\).

**Proof.** We start by proving that if there are two eigenvalues \(\lambda_1, \lambda_2\) of \(A\) with \(\lambda_1 - \lambda_2 \in \mathbb{Z} \setminus \{0\}\), then \(E\) is not locally invertible. Write \(A = SJJS^{-1}\), with \(J\) in Jordan normal form. We will show that there exists a matrix \(B \neq 0\), with \(E(J + \varepsilon B) = E(J), \varepsilon^2 = 0\) (where we use the extension of \(E\) to a map on \(M_\delta(\mathbb{C}[\varepsilon])\)). If \(E(J + \varepsilon B) = E(J)\) then also \(E(A + \varepsilon SBS^{-1}) = E(A)\) holds. We can suppose that \(J\) has only two eigenvalues \(\lambda, \lambda + m, m \in \mathbb{Z} \setminus \{0\}\) and
only two Jordan blocks of size $j$ and $d - j$ respectively. Subtracting $\lambda \cdot I_d$ from $J$, we may assume that $J$ has eigenvalues $0, m$. Define $B$ by $B_{j+1, j} = 1$, and zeros everywhere else. Then $JB = mB$, $BJ = 0$. It follows that

$$E(J + \varepsilon B) = E(J) = \left( \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p} \varepsilon \right) = \left( \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} m^{n-1} B \varepsilon \right) = \frac{1}{m} (e^{2\pi i m} - 1) B \varepsilon = 0.$$  

To prove the converse, again write $A = SJS^{-1}$. We will first consider the case where $J$ is a diagonal matrix. For a matrix $B$ with only one nonzero entry $B_{ij} = 1$, we have that $E(J + \varepsilon B) = E(J)$ also has (at most) one nonzero entry at the same place. The fact that the remaining coefficient is nonzero follows from an explicit calculation in the case $J = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \notin \mathbb{Z}$. We conclude that the derivative of $E$ at $A$ is bijective. For the general case, write $J = D + N$, where $D$ is diagonal, $N$ is nilpotent, and $ND = DN$. We will use the matrix norm $\|A\| = \max\{|A_{ij}|, 1 \leq i, j \leq d\}$, which has the property $\|AB\| \leq d \|A\| \|B\|$. The idea of the proof is as follows. Local invertibility at $J$ is equivalent to local invertibility at a conjugate $SJS^{-1}$. We can pick $S$ such that $\|SNS^{-1}\|$ becomes arbitrary small. An estimate then shows that local invertibility at $SDS^{-1}$ implies local invertibility at $SJS^{-1}$.

We have

$$E(J + \varepsilon B) = E(J) = \left( \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p} \varepsilon \right) = \varepsilon \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} (J^p - D^p) B J^{n-1-p} + D^p B (J^{n-1-p} - D^{n-1-p}) + \varepsilon \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} D^p B D^{n-1-p}.$$  

Write $a := \| \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} D^p B D^{n-1-p} \|$. Then $a > 0$ by the argument above. We can make the following estimate:

$$\| \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p} \| \geq$$
\[ a - \left\| \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} (J^p - D^p) B J^{n-1-p} + D^p B (J^{n-1-p} - D^{n-1-p}) \right\|. \]

Write \( \delta := \|N\|, s := \|D\|, t := s + \delta \). We will now use the estimate
\[ \|J^p - D^p\| = \|N \sum_{k=1}^{p} \left( \frac{p}{k!} \right) D^{p-k} N^{k-1}\| \leq d^p \delta \sum_{k=1}^{p} \left( \frac{p}{k!} \right) s^{p-k} \delta^{k-1} \leq d^p p t^{p-1} \delta. \]

Writing \( b := \|B\| \), we find that
\[ a - \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} J^p B J^{n-1-p} \geq \]
\[ a - \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} d^{n-1} (p t^{n-1} b t^{n-1-p} \delta + (n-1-p) t^p b t^{n-1-p-2} \delta) = \]
\[ a - \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \sum_{p=0}^{n-1} d^{n-1} (n-1) t^{n-2} b \delta = a - \left( \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} n(n-1)(dt)^{n-2} \right) d b \delta = \]
\[ a - (2\pi i)^2 \sum_{n \geq 0} \frac{(2\pi i)^n}{n!} (dt)^n \delta d b \delta = a - (2\pi i)^2 e^{2\pi i b \delta} d b \delta. \]

So for any matrix \( B \), we can pick a basis (and therefore a small \( \delta \)), such that \( \|E(J + \varepsilon B) - E(J)\| > 0 \), which shows that \( E \) is locally invertible at \( J \), and therefore at \( A \).

For a vector bundle \( M \) obtained by Theorem 4.3, there can be points \( y \in Y \) such that the induced vector bundle \( M(y) \) on \( \mathbb{P}^1_C \) is not free. This situation already appears in the Lamé example as we will see later on. Before we get to the Lamé example, we will first study connections on non-free vector bundles in detail.

### 4.3 Connections on non-free vector bundles

We will now give a precise description of connections on non-free vector bundles, and construct a fine moduli space for such connections.
Let $\mathcal{M} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$, $a_1 \geq \cdots \geq a_n$ be a vector bundle, and $D = \sum_{i=1}^r k_i[s_i]$ a divisor of degree $k := \sum_{i=1}^r k_i$, with all $s_i \neq \infty$. We can write $\mathcal{O}(a_i)(U_0) = \mathbb{C}[z]e_i$, $\mathcal{O}(a_i)(U_{\infty}) = \mathbb{C}[z^{-1}]f_i$, with $f_i = z^{a_i}e_i$. A connection $\nabla : \mathcal{M} \to \Omega(D) \otimes \mathcal{M}$ is given by two connections on the free vector bundles $\mathcal{M}(U_0), \mathcal{M}(U_{\infty})$, say $\nabla_0 : \mathcal{M}(U_0) \to \Omega(D)(U_0) \otimes \mathcal{M}(U_0)$ and $\nabla_{\infty} : \mathcal{M}(U_{\infty}) \to \Omega(D)(U_{\infty}) \otimes \mathcal{M}(U_{\infty})$ that glue on $U_0 \cap U_{\infty}$. We have that $\Omega(D)(U_0) = \mathbb{C}[z] \frac{dz}{\prod_{i=1}^r z-s_i}$ (where as always $t_i = z-s_i$), so $\nabla_0$ is given by a $C[z]$-linear map $A$ on $\mathbb{C}[z]\{e_1, \ldots, e_n\}$, taking $\nabla_0(e_i) = A(e_i) \frac{dz}{\prod_{i=1}^r z-s_i}$. We will also write $A$ for the matrix of $A$ on the basis $\{e_1, \ldots, e_n\}$. In the same way the connection $\nabla_{\infty}$ is defined by $\nabla_{\infty}(f_i) = B(f_i) \frac{dz}{\prod_{i=1}^r z-s_i}$, with $B$ given by a matrix $B \in M_n(\mathbb{C}[z^{-1}])$. For the connections $\nabla_0$ and $\nabla_{\infty}$ to glue, we must have $\nabla_0(z^{a_i}e_i) = \nabla_{\infty}(f_i)$. This translates into $\prod_{i=1}^r t_i^{k_i} a_i z^{a_i-1} + z^n A_{ii} = z^n B_{ii}$ for $i = 1, \ldots, n$ and $z^{a_i}A_{ij} = z^{a_i}B_{ij}$ for $i, j = 1 \cdots n$, $i \neq j$.

From this we obtain the following properties for $A$:

- $\deg(A_{ij}) \leq k + a_i - a_j - 2$ for $i \neq j$,
- $\deg(A_{ii}) = k - 1$,
- $A_{ii}$ has as highest order coefficient $-a_i$.

Conversely, a matrix $A \in M_n(\mathbb{C}[z])$ satisfying these properties defines a connection on $\mathcal{M}$.

In the following we will use the group of automorphisms of $\mathcal{M}$, so we give an explicit description of it. An automorphism $\psi$ of $\mathcal{M}$ is given by a $C[z]$-linear automorphism of $\mathcal{M}(U_0)$ and a $C[z^{-1}]$-linear automorphism of $\mathcal{M}(U_{\infty})$ that glue. So $\psi(U_0)$ is given on the basis $\{e_1, \ldots, e_n\}$ by a matrix $A \in \text{GL}_n(\mathbb{C}[z])$. Furthermore $\psi(U_{\infty})$ is given on the basis $\{f_1, \ldots, f_n\}$ by a matrix $B \in \text{GL}_n(\mathbb{C}[z^{-1}])$ with $B = Z^{-1}AZ$, where $Z$ is the diagonal matrix with $Z_{ii} = z^{a_i}$. Let $a_{n_1}, \ldots, a_{n_p}$ be the subsequence of $a_1, \ldots, a_n$ consisting of $a_1$ and the $a_i$ with $a_i - a_{i-1} < 0$. Then we can write $A$ in block form

$$
A = \begin{pmatrix}
A_{11} & \cdots & A_{1p} \\
0 & \ddots & \vdots \\
\vdots & \ddots & A_{pp}
\end{pmatrix}.
$$
Here the $A_{ij} \in \text{GL}_{n_i-1} \circ \text{GL}(C)$ (where we take $n_{p+1} = n+1$) and the coefficients of $A_{ij}$, $i > j$ are polynomials of degree $\leq a_{n_i} - a_{n_j}$. Conversely any such matrix $A$ defines an automorphism of $\mathcal{M}$.

### 4.3.1 Moduli spaces of non-free connections

We will define a moduli space of connections on a vector bundle of some fixed type associated to a data set $(V, \{s_1, \cdots , s_r\}, \{\frac{d}{dz} + B_i\}_{i=1}^{r})$ as in Chapter 2. We fix an ordered basis for $V$, say $\{e_1, \cdots , e_n\}$. Define a vector bundle $\mathcal{M}$ of type $(a_1, \cdots , a_n)$, $a_1 \geq \cdots \geq a_n$ by $\mathcal{M}(U_0) = \mathbb{C}[z] \otimes V$, $\mathcal{M}(U_\infty) = \mathbb{C}[z^{-1}] \otimes (\oplus \mathbb{C}z^i e_i)$. We fix a type $(a_1, \cdots , a_n)$ with $a_1 - a_n \leq r - 1$, and we will only consider connections on the corresponding vector bundle $\mathcal{M}$. Note that in case $\mathcal{M}$ has rank 2, and there exists an irreducible connection on $\mathcal{M}$, then by [PS03] Proposition 6.21 we get $a_2 - a_1 \leq r - 2$.

We start by defining a functor $\mathcal{F}^+$ in a similar way to the definition of $\mathcal{F}$ in Chapter 2, but now we do not divide out equivalence.

**Definition 4.8** The functor $\mathcal{F}^+ : \{\mathbb{C}\text{-algebras}\} \rightarrow \{\text{sets}\}$ is defined as follows. For any $\mathbb{C}$-algebra $R$, the set $\mathcal{F}^+(R)$ consists of the tuples $(A, \{\phi_i\}_{i=1}^{r})$, where:

- $A \in M_n(R[z])$ satisfies $\deg(A_{ij}) \leq k + a_i - a_j - 2$ for $i \neq j$ and $\deg(A_{ii}) = k - 1$. Furthermore $A_{ii}$ has as highest order coefficient $-a_i$.

- the $\phi_i = \sum_{j=0}^{\infty} \phi_i(j)(t_i)^j$, $\phi_i(j) \in M_n(R)$ are automorphisms of $R[[t_i]]^n$.

- $\phi_i(\frac{d}{dz} + \frac{A}{t_i^{k+1} t_i^j}) \phi_i^{-1} = \frac{d}{dz} + B_i$, $i = 1, \cdots , s$, where we see $-\frac{A}{t_i^{k+1} t_i^j}$ and $\phi_i$ as elements of $\text{End}(R[[t_i]][t_i^{-1}]^n)$. This condition can be restated as $\phi_i' = \phi_i - \frac{A}{t_i^{k+1} t_i^j} - B_i \phi_i$.

This functor $\mathcal{F}^+$ is represented by a $\mathbb{C}$-algebra of finite type $U$, as can be shown in a way similar to the proof of Theorem 2.9. We can also consider $\mathcal{F}^+$ as a contravariant functor on schemes of finite type over $\mathbb{C}$. In this setting $\mathcal{F}^+$ is represented by $\mathcal{M} := \text{Spec}(U)$.

We say that two tuples $(A_1, \{\phi_1^i\}), (A_2, \{\phi_2^i\}) \in \mathcal{F}^+(R)$ are equivalent if there
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exists an automorphism \( \psi \) of \( \mathcal{M} \otimes R \) such that \( \frac{d}{dz} + \frac{A_2}{\prod_{i=1}^{r-1} t_i} = \psi^{-1} \left( \frac{d}{dz} + \frac{A_1}{\prod_{i=1}^{r-1} t_i} \right) \psi \)
and \( \phi_i^2 = \phi_i \circ \psi, \ i = 1, \ldots, r \) where we consider \( \psi \) as an element of \( \text{GL}_n(R[[z]]) \)
and \( \text{GL}_n(R[[t]]) \) respectively. We define a functor \( \mathcal{F} \) by \( \mathcal{F}(R) = \mathcal{F}^+(R) / \sim \).

**Theorem 4.9** There is a coarse moduli scheme for the functor \( \mathcal{F} \) defined above, which is in fact a quasi-projective variety.

**Proof.** Consider the group \( G := \text{Aut}(\mathcal{M}) \). This group acts on \( \mathcal{M}(\mathbb{C}) \) and we want to make a quotient. We can make an embedding \( G \subset \text{GL}_n(\mathbb{C}[z]) \).
From the description of \( G \) above we see that the degree of the coefficients of elements of \( G \) is bounded by \( \max_{i=1 \ldots n-1} (a_i-a_{i+1}) \). By our assumption on \( \mathcal{M} \) this bound is less or equal to \( r-2 \). Therefore the map \( \psi : G \rightarrow \text{GL}_n(\mathbb{C})^r \) given by \( A(z) \mapsto (A(s_1), \ldots, A(s_r)) \) is injective. In this way we can consider \( G \) as a linear algebraic subgroup of \( \text{GL}_n(\mathbb{C})^r \).
By [Br69] Theorem 6.8, the quotient \( \text{GL}_n(\mathbb{C})^r / G \) exists and is given by \((Q, \pi), \pi : \text{GL}_n(\mathbb{C})^r \rightarrow Q, \) with \( Q \) a quasi-projective variety. Let \( \phi : \mathcal{M} \rightarrow \text{GL}_n(\mathbb{C})^r, \ (A, \{ \phi_i \}) \mapsto \ (_, \{ \phi_i(0), \ldots, \phi_i(0) \}) \) be the projection map. We want to use the following proposition to prove that a geometric quotient \( \mathcal{M} / G \) exists and is quasi-projective.

**Proposition 4.10** (Proposition 7.1 of [MF94])
Let \( G \) be a group scheme, flat and of finite type over \( S \). Let \( X \) and \( Y \) be schemes of finite type over \( S \), let \( \sigma \) and \( \tau \) be actions of \( G \) on \( X \) and \( Y \), and let \( \phi : X \rightarrow Y \) be a \( G \)-linear morphism. Assume that \( Y \) is a principal fibre bundle over an \( S \)-scheme \( Q \), with group \( G \), and with projection \( \pi : Y \rightarrow Q \).
Assume that there exists an \( L \in \text{Pic}^G(X) \) which is relatively ample for \( \phi \), and that \( Q \) is quasi-projective over \( S \). Then there is a scheme \( P \), quasi-projective over \( S \), and an \( S \)-morphism \( \omega : X \rightarrow P \) such that \( X \) becomes a principal fibre bundle over \( P \) with group \( G \), and projection \( \omega \).

This needs some explanation. A principal fibre bundle is defined as follows: let \( \sigma : G \times_S X \rightarrow X \) be an action, with a geometric quotient \((Q, \pi)\), then \( X \) is a principal fibre bundle over \( Q \) with group \( G \) if

- \( \pi \) is a flat morphism of finite type,
- the map \( (\sigma, pr_2) : G \times_S X \rightarrow X \times_Q X \subset X \times_S X \) is an isomorphism.
By Proposition 0.9 of [MFK94] for a free action of an algebraic group $G$ on an algebraic scheme $X$ with geometric quotient $(Q, \pi)$, the scheme $X$ always is a principal fibre bundle over $Q$ with group $G$.

We further remark that $Pic^G(X)$ is the group of $G$-linearized line bundles on $X$. For details see [MFK94].

We want to apply this proposition with $S = \text{Spec}(\mathbb{C})$, $X = \mathbb{M}$, $Y = \text{GL}_n^r$, and $G, \phi, Q, \pi$ as above. There are a number of conditions to be checked.

(1) $\phi$ is $G$-linear.

(2) $\text{GL}_n^r$ is a principal fibre bundle over $Q$ with group $\text{Aut}(M)$.

(3) There exists an $L$ as in the proposition.

Condition (1) is clearly fulfilled. For the line bundle $L$ in (3) we can take the trivial line bundle since $\mathbb{M}$ is affine. By Proposition 0.9 of [MFK94] for a free action of an algebraic group $G$ on an algebraic scheme $Y$ with geometric quotient $(Q, \pi)$, the scheme $Y$ always is a principal fibre bundle over $Q$ with group $G$. So to prove (3) it suffices to show that the action of $G$ on $\text{GL}_n^r$ is free, and that $Q$ is a geometric quotient. The action being free means that $(\sigma, p^{(2)}) : G \times \text{GL}_n^r \to \text{GL}_n^r \times_Q \text{GL}_n^r$ is a closed immersion, which is the case. The fact that $(Q, \pi)$ is a geometric quotient follows from the definition of a quotient in [Br69].

We will now proof that $P$ is a coarse moduli scheme for $\mathcal{F}$ by an argument as in the proof of Proposition 5.4 of [MFK94]. There is a natural isomorphism $\phi^* : \mathcal{F}^* \to \text{Hom}(\ast, P)$, which induces a natural isomorphism $\phi : \mathcal{F} \to \text{Hom}(\ast, P)$. For $(P, \phi)$ to be a coarse moduli space, the following conditions have to be verified.

- for every algebraically closed field $k$, the map
  
  $\phi(\text{Spec } k) : \mathcal{F}(\text{Spec } k) \to \text{Hom}(\text{Spec } k, P)$

  is an isomorphism.

- given a scheme $N$ and a morphism $\psi : \mathcal{F} \to \text{Hom}(\ast, N)$, there is a unique morphism $\chi : \text{Hom}(\ast, P) \to \text{Hom}(\ast, N)$, such that $\psi = \chi \circ \phi$. 

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The first condition is verified since \((P, \omega)\) is a geometric quotient. To prove that the second condition is verified, consider the element \(\tilde{id} \in \mathcal{F}(\mathbb{M})\) induced by \(id \in \mathcal{F}^*(\mathbb{M}) \cong \text{Hom}(\mathbb{M}, \mathbb{M})\). To a morphism \(\psi : \mathcal{F} \to \text{Hom}(\ast, N)\), we associate the morphism \(f := \psi\tilde{id} : \mathbb{M} \to N\). This induces a bijection of the set of morphisms from \(\mathcal{F}\) to representable functors and the set of morphisms \(f : \mathbb{M} \to N\) with \(N\) a scheme. It follows that the second condition is verified, and therefore \((P, \phi)\) is a coarse moduli space. \(\square\)

4.4 The Lamé equation

We will now consider the moduli problem with data

\[
\left\{ s_1, \cdots, s_4 \right\}, \left\{ \frac{d}{dt_i} + \frac{1}{t_i} \begin{pmatrix} \frac{1}{i} & 0 \\ 0 & -\frac{1}{i} \end{pmatrix} \right\}_{i=1}^4.
\]

The corresponding set of “monodromy representations” \(\mathbb{M}\) defined above is given by \(\mathbb{M} = \{ (M_1, \cdots, M_4) | M_i \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \prod_{i=1}^4 M_i = 1 \}. \) Write

\[
M_i := \begin{pmatrix} p_i^{s_i} & q_i^{s_i} \\ r_i & -p_i^{s_i} \end{pmatrix}, \quad i = 1, 2, 3,
\]

then the coordinate ring of \(\mathbb{M}\) is given by

\[
R := \mathbb{C}[p_1, q_1, r_1, p_2, q_2, r_2, p_3, q_3, r_3]/I,
\]

\[
I = \langle p_1^2 + q_1 r_1 + 1, p_2^2 + q_2 r_2 + 1, p_3^2 + q_3 r_3 + 1, f \rangle
\]

\[
f := -pq_3r_3 + pq_2r_2 + pq_1r_1 - pr_3q_3 - pr_2q_2 - pr_1q_1 + q_1q_2q_3.
\]

The following properties of \(\mathbb{M}\) are known (but also easily verified).

- \(\mathbb{M}\) is a five dimensional variety.
- The group \(\text{PGL}_2(\mathbb{C})\) acts on \(\mathbb{M}\) (by conjugation) and on its coordinate ring \(R\). The ring \(R^{\text{PGL}_2} := \mathbb{C}[t_1, t_2, t_3]/\langle t_1^2 + t_2^2 + t_3^2 + t_1 t_2 t_3 - 4 \rangle\) is the ring of invariants, where \(t_1 := \text{Tr}(M_2 M_3), \ t_2 := \text{Tr}(M_1 M_3), \ t_3 := \text{Tr}(M_1 M_2)\). This follows immediately from [Bo03] Section 2.
- The variety \(\mathbb{M}^{\text{PGL}_2} := \text{Spec}(R^{\text{PGL}_2})\) has 4 singular points, namely \((t_1, t_2, t_3) \in \{ (-2, 2, 2), (2, -2, 2), (2, 2, -2), (-2, -2, -2) \}\). Each one of these points corresponds to multiple \(\text{PGL}_2(\mathbb{C})\)-orbits. After deleting the 4 singular points and their preimages in \(\mathbb{M}\) we obtain a good quotient under \(\text{PGL}_2(\mathbb{C})\). In particular \(\mathbb{M}\) is reduced and irreducible.
• The preimage of the 4 singular points of $\mathbb{M}^{\text{PGl}_2}$ in $\mathbb{M}$ precisely consists of all reducible representations in $\mathbb{M}$.

• The complement $\mathbb{M}_{\text{ir}}$ is a smooth connected variety.

By Theorem 4.3 we obtain a family of differential equations parametrized by $\mathbb{M}$, say $(\mathcal{M}, \nabla, \mathbb{C}^2, \text{local data})$. For every irreducible representation $m \in \mathbb{M}$, the following lemma shows that $\mathcal{M}(m)$ is either free, or of type $(1, -1)$.

**Lemma 4.11** Let $(\mathcal{M}, \nabla)$ be an irreducible connection of rank 2 on $\mathbb{C}^2$ with four singular points such that the sum of the local exponents at each singular point is 0. Then the vector bundle $\mathcal{M}$ is of the type $\mathcal{O}(a) \oplus \mathcal{O}(-a), a \in \{0, 1\}$.

**Proof.** Because the sum of the local exponents of $(\mathcal{M}, \nabla)$ is zero at each singular point, the induced connection $\Lambda^2 \nabla$ on $\Lambda^2 \mathcal{M}$ is everywhere regular. Since $\mathbb{C}^2$ is simply connected, $\Lambda^2 \mathcal{M}$ is the trivial line bundle, and $\Lambda^2 \nabla$ is the trivial connection. So $\mathcal{M}$ is of the type $\mathcal{O}(a) \oplus \mathcal{O}(-a), a \geq 0$. By [PS03] Proposition 6.21, the defect of $\mathcal{M}$ is $\leq 2$. This proves the lemma. \[\square\]

We will now show that the set $\mathbb{M}^{(1, -1)} := \{m \in \mathbb{M}_{\text{ir}} | \mathcal{M}(m) \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)\}$ is nonempty.

**Proposition 4.12** $\mathbb{M}^{(1, -1)}$ is a non-trivial divisor in $\mathbb{M}_{\text{ir}}$.

**Proof.** By Remark 3.12 (4) we have that $\mathbb{M}^{(1, -1)}$ is a divisor. So we only need to show that $\mathbb{M}^{(1, -1)}$ is non-empty. As we saw in Section 4.2 the connection $(\mathcal{M}(m), \nabla(m))$ is uniquely determined for every $m \in \mathbb{M}$. Therefore we only have to construct a connection on $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ with the correct local behavior. By the description of connections on non-free vector bundles above, we get that such a connection is given by a matrix $A$ of the form

$$A = \begin{pmatrix}
a_0 + a_1 z + a_2 z^2 - z^3 & b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 \\
c_0 & d_0 + d_1 z + d_2 z^2 + z^3
\end{pmatrix}.$$ 

We want that the connection is locally at the points $s_i$ formally isomorphic to $d \frac{dz}{t^i} + \frac{1}{i^i} \begin{pmatrix}
\frac{1}{t} & 0 \\
0 & -\frac{1}{t}
\end{pmatrix}$, so the Laurent series expansion of $\frac{A}{|t_i|^{1/2}}$ at a point $s_i$ has to be of the form $\frac{A_i}{t^i} + \text{h.o.t.}$, with $A_i$ similar to $\begin{pmatrix}
\frac{a}{t^i} & 0 \\
0 & -\frac{1}{t^i}
\end{pmatrix}$. The
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\( A_i \) are of the form \( \left( \begin{array}{c} \frac{p_i}{r_i} \\ -\frac{q_i}{r_i} \end{array} \right) \), \( \frac{A}{\prod_{\nu} r_\nu} = \sum_{i=1}^4 \frac{A_i}{r_i} + \left( \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right) \). We find that \( p_1 + p_2 + p_3 + p_4 + 1 = 0 \), and \( r_2, r_3, r_4 \) are multiples of \( r_1 \). So we get a 5-dimensional family of tuples \((A_1, \cdots, A_4, b_4)\), and hence a 5-dimensional family \( X \) of connections on \( \mathcal{O}(1) \oplus \mathcal{O}(-1) \).

The automorphism group \( G \) of \( \mathcal{O}(1) \oplus \mathcal{O}(-1) \) is \( \{ \left( \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right) : a, e \neq 0 \} \). A connection given by a matrix \( A \) is equivalent to the one given by the matrix \( \tilde{A} = \Phi^{-1} \Phi' + \Phi^{-1} A \Phi \), with \( \Phi \in G \). We can construct a one dimensional subfamily of \( X \) consisting of matrices of the form \( \left( \begin{array}{ccc} -z^3 & 0 \\ 0 & z^3 \end{array} \right) \) parameterized by \( b_4 \) with no equivalent elements. In case \( s_i = i, i = 1, \cdots, 4 \) this family is

\[
\left\{ \left( \begin{array}{ccc} -z^3 & \frac{625}{4} - 3000z + 1800z^2 - 3500z^3 + b_4 (24 - 50z + 35z^2 - 10z^3 + z^4) \\ 0 & z^3 \end{array} \right) \right\}, \quad b_4 \in \mathbb{C}
\]

\( \square \)

We remark that the above does not imply that \( M^{1, -1} \) is 1-dimensional. Indeed, let \( \mathcal{M} \) be the vector bundle on \( \mathbb{P}^3_\mathcal{M} \) given by the Riemann-Hilbert construction. The type of \( \mathcal{M}(m) \), \( m \in \mathcal{M} \) is \((0, 0)\) or \((1, -1)\), and since \( \mathcal{M} \) is irreducible we find by Remarks 3.12 that \( M^{1, -1} \) is an analytic divisor on \( \mathcal{M} \) (since \( M^{1, -1} \) is nonempty). We find that \( M^{1, -1} \) is 4-dimensional, and so there are isomorphic connections in \( M^{1, -1} \).

We can make similar calculations (which are in fact simpler) for the case of a free vector bundle. We get a 5-dimensional space of connections, with an action of the group \( SL_2 \). There exists a categorical quotient, which maps all reducible connections to four points. This 2-dimensional quotient is actually a geometric quotient on the space of irreducible connections.
monodromy representation equivalent to $\rho$ is known as the weak Riemann-Hilbert problem. This question has a positive answer, which is precisely formulated in [PS03], Theorem 6.15.

The strong Riemann-Hilbert problem asks whether for a given representation $\rho : \pi_1 \rightarrow \text{GL}(V)$ there is a Fuchsian connection over $\mathbb{C}(z)$ with monodromy representation equivalent to $\rho$. A Fuchsian connection is a connection which for the differentiation $\frac{d}{dz}$ can be written in the form $\frac{d}{dz} + \sum_{i=1}^{n} \frac{A_i}{z-a_i}$, with $A_i \in \mathbb{M}_n(\mathbb{C})$. In general, for a given $\rho$ there is no such Fuchsian connection. However under some conditions on $\rho$ the strong Riemann-Hilbert problem has a positive answer, see sections 6.4 and 6.5 of [PS03] for details.

The strong Riemann-Hilbert problem can be restated in terms of connections on vector bundles. For a connection $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1_{\mathbb{C}}$ (where $\mathcal{M}$ is not necessarily free), we get an induced connection $(\mathcal{M}_q, \nabla_q)$ over $\mathbb{C}(z)$ by localization at the generic fibre. Therefore we can associate a monodromy map to $(\mathcal{M}, \nabla)$. It is easily seen that the strong Riemann-Hilbert problem precisely asks whether there is a connection on a free vector bundle with some given monodromy map.

For a representation $\rho : \pi_1 \rightarrow \text{GL}(V)$, by [PS03] Theorem 6.15, we find an associated connection $(\mathcal{M}, \nabla)$ over $\mathbb{C}(z)$. The following lemma states how we can associate a connection over $\mathbb{P}^1_{\mathbb{C}}$ to $(\mathcal{M}, \nabla)$.

**Lemma 4.1 (Lemma 6.18 of [PS03])** Let $(\mathcal{M}, \nabla)$ be a regular singular connection over $\mathbb{C}(z)$ with singular locus $S$. For every $s \in S$ we choose a local parameter $t_s$. For every $s \in S$ let $\Lambda_s \subset \widehat{\Lambda}_s := \mathbb{C}((t_s)) \otimes M$ be a lattice that satisfies $\nabla(\Lambda_s) \subset \frac{d}{dt_s} \otimes \Lambda_s$ (the existence of such a lattice is equivalent to $(\mathcal{M}, \nabla)$ being regular singular at $s$). Then there is a unique regular singular connection $(\tilde{\mathcal{M}}, \tilde{\nabla})$ on $\mathbb{P}^1_{\mathbb{C}}$ with singular locus in $S$ such that:

1. For every open $U \subset \mathbb{P}^1_{\mathbb{C}}$, one has $\mathcal{M}(U) \subset M$.
2. The generic fibre of $(\tilde{\mathcal{M}}, \tilde{\nabla})$ is $(\mathcal{M}, \nabla)$.
3. $\tilde{\Lambda}_s = \Lambda_s$ for all $s \in S$.

In the case where $(\mathcal{M}, \nabla)$ is irreducible, one can make a choice for the lattices $\Lambda_s$ in such a way that the corresponding vector bundle $\mathcal{M}$ is free (see