Algorithms and moduli spaces for differential equations

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Chapter 3  

Singer’s Theorem for families of differential equations

This chapter is joint work with Marius van der Put.

3.1 Introduction

In this chapter we will define families of differential equations on the projective line \( \mathbb{P}^1_C \), parametrized by a scheme of finite type \( X \). As before we suppose \( C \) to be algebraically closed and of characteristic zero. These families are of a more general nature than the moduli spaces, defined in Chapter 2. Theorem 2.17 is extended to a family of differential equations of dimension \( n \), parametrized by some \( X \). Thus the condition “\( \text{Gal}(x) \subset G \)” for closed points \( x \) of \( X \) (i.e., \( x \in X(C) \)) defines a closed subset of \( X \). The aim is to show that the set of closed points \( x \in X \) for which the differential Galois group \( \text{Gal}(x) \) of the corresponding equation is equal to \( G \) is a constructible subset of \( X \) i.e., of the form \( \bigcup_{i=1}^n (O_i \cap F_i) \) for open sets \( O_i \) and closed sets \( F_i \). This statement (and the earlier one) has to be made more precise by providing a suitable definition of “family of differential equations” and a meaning for the expression \( \text{Gal}(x) \subset G \). Moreover, a condition on the group \( G \) is essential.

In his paper [S93], M.F. Singer defines a set of differential operators, by giving some local data. He proves that under a certain condition on \( G \),
the subset of differential equation with Galois group equal to $G$ is constructible. This condition on $G$ will be called the Singer condition. We will consider the same problem, in our context of families of differential equations parametrized by a scheme $X$. We will construct for any group $G$ that does not satisfy the “Singer condition” an example of a moduli family $\mathcal{M}$ such that $\{x \in \mathcal{M} | \text{Gal}(x) = G\}$ is not constructible. Finally, from these constructions one deduces an alternative description of the Singer condition.

3.2 The Singer condition

Let $G$ be a linear algebraic group over $C$. First we will recall the Singer condition on $G$, as given in [S93]. A character $\chi$ of $G$ is a morphism of algebraic groups $\chi : G \to \mathbb{G}_m$, where $\mathbb{G}_m$ stands for the multiplicative group $C^\times$. The set $X(G)$ of all characters is a finitely generated abelian group. Let $\ker X(G)$ denote the intersection of the kernels of all $\chi \in X(G)$. This intersection is a characteristic (closed) subgroup of $G$. As usual, $G^o$ denotes the connected component of the identity of $G$. The group $\ker X(G^o)$ is a normal, closed subgroup of $G^o$ and $G$. Let $\chi_1, \ldots, \chi_r$ generate $X(G^o)$. Then $\ker X(G^o)$ is equal to the intersection of the kernels of $\chi_1, \ldots, \chi_r$. In other words $\ker X(G^o)$ is the kernel of the morphism $G^o \to \mathbb{G}_m^r$, given by $g \mapsto (\chi_1(g), \ldots, \chi_r(g))$. The image is a connected subgroup of $\mathbb{G}_m^r$ and therefore a torus $T$. Hence $G^o / \ker X(G^o)$ is isomorphic to $T$. Moreover, by definition, $T$ is the largest torus factor group of $G^o$. One considers the exact sequence:

$$1 \to G^0 / \ker X(G^0) \to G / \ker X(G^0) \to G / G^0 \to 1.$$ 

Since $G^0 / \ker X(G^0)$ is abelian, this sequence induces an action of $G / G^0$ on $G^0 / \ker X(G^0)$ by conjugation.

**Definition 3.1** A linear algebraic group $G$ satisfies the Singer Condition if the action of $G / G^0$ on $G^0 / \ker X(G^0)$ is trivial.

The Singer condition can be stated somewhat simpler, using $U(G) \subset G$, the subgroup generated by all unipotent elements in $G$.

**Lemma 3.2** $U(G) = U(G^o)$ is equal to $\ker X(G^o)$ and the Singer condition is equivalent to “$G^o / U(G)$ lies in the center of $G / U(G)$”.
3.2. THE SINGER CONDITION

Proof. Fix an embedding $G \subset \text{GL}(V)$, where $V$ is a finite dimensional vector space over $C$. First we prove that $U(G)$ is a closed connected normal subgroup of $G$. Let $I + B$, $B \neq 0$ be a unipotent element of $G$. Then $I + B = e^D$, for some nilpotent element $D = \sum (-1)^{i-1}i^{-1}B^i \in \text{End}(V)$. The Zariski closure $\langle (I + B)^n \mid n \in \mathbb{Z} \rangle$ of the group generated by $I + B$ lies in $G$ and is equal to the group $\{e^{tD} \mid t \in C\}$, which is isomorphic to the additive group $\mathbb{G}_a$ over $C$. Hence $U(G)$ is generated by this connected subgroups of $G$ and by Proposition 2.2.6 of [Sp98] the group $U(G)$ is closed and connected. Further $U(G)$ is a normal subgroup and even a characteristic subgroup, since the set of unipotent elements of $G$ is stable under any automorphism of $G$. The connectedness of $U(G)$ implies $U(G) = U(G^o)$.

Now we will show that $G^o/U(G^o)$ is a torus. Since the unipotent radical $R_u(G^o)$ lies in $U(G^o)$, we may divide $G^o$ by $R_u(G^o)$ and assume $G^o$ to be reductive. Then by [Sp98] Corollary 8.1.6, we have $G^o = R(G^o) \cdot (G^o, G^o)$, where $R(G^o)$ is the radical of $G^o$, and where $(G^o, G^o)$ is the commutator subgroup of $G^o$. The latter group is a semi simple subgroup, according to the same corollary. By [Sp98] Theorem 8.1.5, we get that $(G^o, G^o)$ is generated by unipotent elements, so $(G^o, G^o) \subset U(G^o)$. Since $R(G^o)$ is a torus, its image $G^o/U(G^o)$ is a torus, too. This proves $U(G^o) \supset \ker X(G^o)$. The other inclusion follows from the observation that every unipotent element lies in the kernel of every character. Finally, the triviality of the action of $G/G^o$ on $G^o/U(G^o)$ is clearly equivalent to $G^o/U(G^o)$ lies in the center of $G/U(G^o)$.

Assumptions 3.3 Let $G \subset \text{GL}(V)$ be a linear algebraic group. For the moment we assume the following items (see Definition 3.13, Remark 3.14, Proposition 3.16 and Corollary 3.17).

- The definition of a family of differential equations, parametrized by $X$.
- The meaning of $\text{Gal}(x) \subset G$ for $x \in X(C)$.
- $\{x \in X(C) \mid \text{Gal}(x) \subset G\}$ is closed.
- $\{x \in X(C) \mid \text{Gal}(x) \subset hGh^{-1} \text{ for some } h \in \text{GL}(V)\}$ is constructible.

Lemma 3.4 Let $G, X$ be as in the above assumptions. If $G$ has finitely many proper closed subgroups $H_1, \ldots, H_r$, such that every proper closed subgroup is
contained in a conjugate of one of the $H_i$, then $\{x \in X(C) \mid \text{Gal}(x) = G\}$ is constructible.

The proof is easy.

**Remarks 3.5**

(1) If $G$ satisfies the group-condition of the lemma, then $G$ satisfies the Singer condition, too. This follows from the fact that $G/U(G)$ is a finite group. Indeed, if $T := G^o/U(G) \neq \{1\}$, then one can produce many proper normal subgroups of $G/U(G)$. For example, for any integer $m > 1$ the subgroup $T[m]$, consisting of the $m$-torsion elements of $T$. By lifting this subgroups, we obtain a contradiction.

(2) Consider $G := \text{SL}_2(C)$. The classification of the proper closed subgroups $H$ of $G$ states that $H$ is either contained in a Borel subgroup or in a conjugate of the infinite dihedral group $D^\infty_{\text{SL}_2}$ or is conjugated to one of the special finite groups: the tetrahedral group, the octahedral group, the icosahedral group. Thus $G$ satisfies the conditions of the lemma and moreover, $G/U(G) = \{1\}$.

(3) The infinite dihedral group $G = D^\infty_{\text{SL}_2}$ has the properties: $G^o = G_m$, $U(G^o) = 1$ and $G/G^o$ acts non-trivially on $G^o$. Thus $G$ does not satisfy the Singer condition. For this group one can produce moduli spaces $\mathcal{M}$ such that $\{x \in \mathcal{M}(C) \mid \text{Gal}(x) = G\}$ is not constructible (see example 3.8).

(4) For the following two examples, namely moduli spaces and the groups $\mathbb{G}_a^3$ and $\mathbb{G}_m^3$, the Singer condition is valid, but $G$ does not satisfy the condition of the lemma. We will show explicitly that these groups define constructible subsets.

**Example 3.6** A moduli space with differential Galois groups in $\mathbb{G}_a^3$.

$V$ is a 4-dimensional vector space over $C$ with basis $e_1, \ldots, e_4$. $N \in \text{End}(V)$ is given by $N(e_i) = 0$ for $i = 1, 2, 3$ and $N(e_4) = e_1$. The data for the moduli problem are.

- Three distinct singular points $s_1, s_2, s_3 \in C^\times$. The point $\infty$ is allowed to have a, non prescribed, regular singularity.
- For each singular point $s_i$, the differential operator $\frac{d}{d(z-s_i)} + \frac{N}{z-s_i}$. 
3.2. THE SINGER CONDITION

Some calculations lead to an identification $\text{GL}(4, C) \times \text{GL}(4, C) \rightarrow \mathcal{M}$, where $\mathcal{M}$ is the moduli space of the problem. Let $m := (\phi_2, \phi_3)$ denote a closed point of the first space, then the corresponding universal differential operator is

$$
\frac{d}{dz} + \frac{N}{z - s_1} + \frac{\phi_2^{-1}N\phi_2}{z - s_2} + \frac{\phi_3^{-1}N\phi_3}{z - s_3}.
$$

Let $G := \mathbb{G}_a^3$ the subgroup of $\text{GL}(V)$ consisting of the maps of the form $I + B$, $Be_i = 0$ and $Be_i \in Ce_i$ for $i = 2, 3, 4$. The condition $\text{Gal}(m) \subset \mathbb{G}_a^3$ can be seen to be equivalent to $\phi_2(e_1), \phi_3(e_1) \in C(e_1)$. This describes the set $\{m \in \mathcal{M} \mid \text{Gal}(m) \subset G\}$ completely. The above differential operator evaluated at a point of $\{m \in \mathcal{M} \mid \text{Gal}(m) \subset G\}$ has the form

$$
\frac{d}{dz} + \begin{pmatrix}
0 & h_1 & h_2 & h_3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

where $(h_1, h_2, h_3) = \frac{1}{z - s_1}(0, 0, 1) + \frac{1}{z - s_2}(f_1, f_2, f_3) + \frac{1}{z - s_3}(g_1, g_2, g_3)$. Moreover, $f_1, f_2, f_3$ are polynomials of degree $\leq 2$ in the entries of $\phi_2$ and $g_1, g_2, g_3$ are polynomials of degree $\leq 2$ in the entries of $\phi_3$.

$G$ has many (non-conjugated) maximal proper closed subgroups and there is no obvious reason why $\{m \in \mathcal{M} \mid \text{Gal}(m) = G\}$ should be constructible. We continue the calculation. The differential Galois group $\text{Gal}(m)$, with $m$ such that $\text{Gal}(m) \subset G$, is in fact the differential Galois group for the three inhomogeneous equations $y'_i = h_i$, $i = 1, 2, 3$ over $C(z)$. Thus $\text{Gal}(m)$ is a proper subgroup of $G$ if and only if there is a non trivial linear combination $c_1h_1 + c_2h_2 + c_3h_3$ with $c_1, c_2, c_3 \in C$ such that $y' = c_1h_1 + c_2h_2 + c_3h_3$ has a solution in $C(z)$. Now $y$ exists if and only if $c_1h_1 + c_2h_2 + c_3h_3$ has residue 0 at the points $s_1$, $s_2$, $s_3$. The existence of such a linear combination translates into a linear dependence and the explicit equation $f_1(s_2)g_2(s_3) - f_2(s_2)g_1(s_3) = 0$. This defines a closed subset of $\{m \in \mathcal{M} \mid \text{Gal}(m) \subset G\}$ and therefore $\{m \in \mathcal{M} \mid \text{Gal}(m) = G\}$ is constructible.

**Example 3.7** A moduli space with differential Galois groups in $\mathbb{G}_a^n$.

The data for the moduli problem are.

- A vector space $V$ of dimension $n$ over $C$, with basis $e_1, \ldots, e_n$. 
Singular points \( s_1, \ldots, s_r \in \mathbb{C}^4 \). We allow \( \infty \) to have a non-prescribed regular singularity.

Local differential operators
\[
\frac{d}{dz - s_1} + \frac{C_i}{z - s_i},
\]
where \( e_1, \ldots, e_n \) are eigenvectors for all \( C_i \in \text{End}(V) \).

The moduli space \( \mathbb{M} \) can be identified with \( \text{GL}(V)^{r-1} \). At a closed point \( m = (\phi_2, \ldots, \phi_r) \in \text{GL}(V)^{r-1} \) the universal differential operator reads
\[
\frac{d}{dz} + \sum_{i=1}^s \frac{\phi_i^{-1} C_i \phi_i}{z - s_i},
\]
where \( \phi_i = I \). The group \( G_m \cong G \subset \text{GL}(V) \) consists of the maps for which each \( e_i \) is an eigenvector. Above the closed subset \( \{ m \in \mathbb{M} \mid \text{Gal}(m) \subset G \} \) the differential operator has the form
\[
L := \frac{d}{dz} + \sum_{i=1}^s \frac{A_i}{z - s_i},
\]
with \( A_1 = C_1 \) and each \( A_i \) is a diagonal matrix w.r.t. the basis \( e_1, \ldots, e_n \) and having the same eigenvalues as \( C_i \). The space \( \{ m \in \mathbb{M} \mid \text{Gal}(m) \subset G \} \) has a positive dimension if there is at least one \( C_i \) with \( i > 1 \) having an eigenvalue with multiplicity \( > 1 \). However the number of differential operators \( L \) is finite! Thus only a finite number of algebraic subgroups of \( G \cong \mathbb{G}_m \) occur as differential Galois group \( \text{Gal}(m) \). One concludes that for every algebraic subgroup \( H \subset G \), the set \( \{ m \in \mathbb{M} \mid \text{Gal}(m) = H \} \) is constructible.

The above example is the general pattern for “families” with differential Galois groups contained in some commutative algebraic group \( G \). Again, there are only finitely many distinct differential operators \( L \) possible above the moduli family. Hence there are only finitely many possibilities for the differential Galois groups. This implies that for every algebraic subgroup \( H \subset G \) the set of the points with differential Galois group equal to \( H \) is constructible.

**Example 3.8** A moduli space with differential Galois groups in \( D_{\infty}^{SL_2} \).
Let \( V = Ce_1 + Ce_2 \). By \( D_{\infty}^{SL_2} \) we will denote the subgroup of \( \text{SL}(V) \) consisting of the maps which permute the lines \( Ce_1, Ce_2 \). The data for the moduli problem are.
3.3. FAMILIES OF DIFFERENTIAL EQUATIONS

- Singular points $s_1, \ldots, s_4 \in C^a$, and $\infty$ is supposed to be regular.

- For each point $s_i$ the differential operator \( \frac{d}{d(z-s_i)} + \frac{1}{z-s_i} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \) with respect to the basis $e_1, e_2$.

The moduli space $\mathcal{M}$ for this problem can be made explicit. The universal differential equation has 4 regular singular points with local exponents $1/4$ and $-1/4$. This is essentially the Lamé equation. It has a closed subset $\{ m \in \mathcal{M} \mid \text{Gal}(m) \subset D_{\infty}^{SL_2} \}$. Let $D_n^{SL_2} \subset D_{\infty}^{SL_2}$ denote the dihedral subgroup (of order $4n$). It turns out that $D_n^{SL_2}$ and $D_{2n}^{SL_2}$ for $n \geq 1$ occur as differential Galois groups Gal$(m)$ for closed points. The conclusion is that $\{ m \in \mathcal{M} \mid \text{Gal}(m) = D_{\infty}^{SL_2} \}$ is not constructible! One way to explain this is to consider the case where $C$ is the field of the complex numbers. Since the $s_i$ are regular singular points, the differential Galois group is the algebraic closure of the monodromy group. This monodromy group is generated by four elements $A_1, \ldots, A_4 \in SL_2(C)$ having product 1 and such that each $A_i^2 = -I$.

Above the moduli space $\mathcal{M}$ essentially all groups with these generators and relations do occur. Therefore each $D_n^{SL_2}$ occurs as differential Galois group. A detailed study of this moduli space will be given in Chapter 4.  

3.3 Families of differential equations

We will now come to the definition of families of differential equations on $\mathbb{P}^3$, parametrized by a scheme $X$. We will first recall some facts on local differential modules.

3.3.1 Formal connections and semi-simple modules

The usual differentiation on the field of formal Laurent series $C((u))$ is given by the formula $\sum a_n u^n \mapsto \frac{d}{du}(\sum a_n u^n) := \sum a_n n u^{n-1}$. For notational convenience we will use (in this section) the differentiation $f \mapsto \delta(f) := u \frac{du}{f}$. A differential module $M$ over $C((u))$ is a finite dimensional vector space over $C((u))$ provided with an additive map $\delta = \delta_M : M \to M$ satisfying $\delta(fm) = f \delta(m) + \delta(f)m$. Put $\mathcal{Q} := \bigcup_{m \geq 1} u^{-1/m}C[u^{-1/m}]$. The Galois group of the algebraic closure of $C((u))$ acts on $\mathcal{Q}$. Take $q \in \mathcal{Q}$ and let $m \geq 1$
be minimal such that \( q \in u^{-1/m} C[u^{-1/m}] \). The differential module \( E(q) \) over \( C((u^{1/m})) \) is defined by \( E(q) = C((u^{1/m}))e \) and \( \delta(e) = qe \). This module can also be viewed as a differential module of dimension \( m \) over \( C((u)) \). As such, it depends only on the Galois orbit \( \omega q \) of \( q \) in \( \Omega \). We write \( E(\omega q) \) for \( E(q) \) considered as a differential module over \( C((u)) \). We note that \( E(\omega q) \) is an irreducible differential module. The classification of differential modules over \( C((u)) \) can be formulated as follows:

*Every differential module \( M \) over \( C((u)) \) can be written uniquely as \( \oplus_{i=1}^r E(\omega_i q) \otimes M_i \), where the \( \omega_1, \ldots, \omega_r \) are distinct Galois orbits in \( \Omega \) and where the \( M_i \) are regular singular differential modules.*

We recall that a differential module \( N \) is regular singular if there exists a basis \( b_1, \ldots, b_r \) of \( N \) over \( C((u)) \), with the property that the free \( C[[u]] \)-module \( \Lambda := C[[u]]b_1 + \cdots + C[[u]]b_r \) is invariant under \( \delta \). One associates to a regular singular \( N \) a semi-simple regular singular differential module \( N_{ss} \) by the following construction. (compare [Levelt, Jordan decomposition for a class of singular differential operators. Arkiv för matematik, 13 (1): 1-27, may 1975]). The operator \( \delta \) leaves \( u^m \Lambda \) invariant for each \( m \geq 0 \). Thus \( \delta \) induces a \( C \)-linear endomorphism \( \delta_m \) on \( \Lambda/u^m \Lambda \). The additive Jordan decomposition of \( \delta_m \) is written as \( \delta_m = \delta_{m,ss} + \delta_{m,nilp} \). Here \( ss \) denotes the semi-simple part and \( nilp \) denotes the nilpotent part. It is easily seen that the families of endomorphisms \( \{ \delta_{m,ss} \} \) and \( \{ \delta_{m,nilp} \} \) form projective systems. Now we write \( \delta_{ss} \) and \( \delta_{nilp} \) for the induced maps on \( \Lambda \). One verifies that \( \delta_{nilp} \) is \( C[[u]] \)-linear and that \( \delta_{ss}(fm) = f\delta_{ss}(m) + \delta(f)m \) for \( f \in C[[u]] \) and \( m \in \Lambda \). Both operators are extended to \( N \). The vector space \( N \) provided with \( \delta_{ss} \) is denoted by \( N_{ss} \). It is a differential module over \( C((u)) \) and it is semi-simple in the sense that every submodule of \( N_{ss} \) has a complement.

In terms of matrix differential equations this construction has an easy translation. One knows that \( N \) contains a basis such that the corresponding matrix differential equation has the form \( u \frac{d}{du} + A \), where \( A \) is a constant matrix (i.e., has entries in \( C \)). Then \( N_{ss} \) corresponds (on the same basis) with the matrix differential equation \( u \frac{d}{du} + A_{ss} \), where \( A = A_{ss} + A_{nilp} \) is the usual Jordan decomposition of \( A \). We note that the “classical” solution space for the matrix differential equation \( u \frac{d}{du} + A \) contains logarithmic terms if \( A_{nilp} \neq 0 \).
Let $M$ be a differential module over $C((u))$, with canonical decomposition $\oplus_{i=1}^s E(\alpha_i) \otimes M_i$. Then we define $M_{ss} := \oplus_{i=1}^s E(\alpha_i) \otimes M_{i,ss}$. Thus $M_{ss}$ is equal to $M$ as vector space over $C((u))$. One has $\delta_M = \delta_{M_{ss}} + \nu$ where $\nu$ is a nilpotent endomorphism of $M$ commuting with $\delta_{M_{ss}}$ and $\delta_M$. In particular, every submodule of $M$ is also a submodule of $M_{ss}$. Moreover, the differential module $M_{ss}$ is semi-simple.

A formal connection is a connection $\nabla : N \to C[[u]]u^{-k}du \otimes N$, where $N$ is a free $C[[u]]$-module of finite rank. One associates to $N$ the differential module $M = C((u)) \otimes N$ (with $\delta_M$ induced by $\nabla_{u^{-k}}$). The formal connection $N_{ss}$ is now defined as the connection on $N$ induced by the $\delta_{M_{ss}}$ on $M_{ss}$. We will call $N_{ss}$ and $M_{ss}$ the semi-simplifications of $N$ and $M$. Suppose that $R \subset N$ is a $C[[u]]$-submodule such that $N/R$ is free and $\nabla R \subset C[[u]]u^{-k}du \otimes R$. Then also $\nabla_{ss} R \subset C[[u]]u^{-k}du \otimes R$.

### 3.3.2 Defining families

The statement that we want to prove concerns the closed points of $X$ and therefore we may suppose that $X$ is reduced. For the same reason we may suppose (at every stage of the proof) that $X$ is irreducible and affine. Assume that $X = \text{Spec}(R)$ with $R$ reduced and finitely generated over $C$. In order to avoid technical complications we will consider families for which the singular points (apparent or not) lie in a fixed subset $\{s_1, \ldots, s_s\}$ of $\mathbb{P}^1$. For convenience we suppose that $0, \infty \notin \{s_1, \ldots, s_s\}$.

A first attempt to define a family parametrized by $X = \text{Spec}(R)$, is to consider a matrix differential equation $\frac{d}{dz} + A$, where $A$ is an $R[z, \frac{1}{(z-s_1) \cdots (z-s_s)}] \otimes_C V$. More explicitly, $A$ has the form $\sum_{j=1}^s \sum_{i=1}^k \frac{A(i,j)}{(z-s_i)^j}$, where each $A(i,j)$ is an $R$-linear endomorphism of $R \otimes V$. For every closed point $x$ of $X$, i.e., $x \in X(C)$, one writes $A(x)$ for the $C[z, \frac{1}{(z-s_1) \cdots (z-s_s)}] \otimes_V C[x]$-linear endomorphism of $C[z, \frac{1}{(z-s_1) \cdots (z-s_s)}] \otimes_C V$, obtained by applying $x : R \to C$ to $A$. In this way, $\frac{d}{dz} + A$ is a family of differential equations on the projective line over $C$. The equation $\frac{d}{dz} + A$ is regular at $z = 0$. One considers $R[[z]] \otimes_C V$ and the canonical map

$$\text{Mod}_z : R[[z]] \otimes_C V \to R[[z]] \otimes_C V/(z) \cong R \otimes_C V.$$
Lemma 3.9 Consider the kernels:

\[ S = \ker \left( \frac{d}{dz} + A, R[[z]] \otimes_C V \right) \text{ and } S(x) = \ker \left( \frac{d}{dz} + A(x), C[[z]] \otimes_C V \right). \]

The maps \( \text{Mod}_z : S \to R \otimes_C V \) and \( \text{Mod}_z : S(x) \to V \) are bijections. Moreover, the image of \( S \) under the map \( R[[z]] \otimes_C V \to C[[z]] \otimes_C V \), induced by \( x : R \to C \), is equal to \( S(x) \).

Proof. One considers an endomorphism \( F = F_0 + zF_1 + \cdots \) of \( R[[z]] \otimes_C V \) (i.e., each \( F_i \) is an endomorphism of \( R \otimes_C V \)) with \( F_0 = 1 \). One requires that \( F \) is a “fundamental matrix”, which means that \( F^\prime + AF = 0 \). Put \( A = A_0 + A_1z + \cdots \). This leads to equations

\[(n + 1)F_{n+1} + A_0F_n + A_1F_{n-1} + \cdots + A_nF_0 = 0 \text{ for all } n \geq 0.\]

Clearly \( F \) exists and is unique. This implies that \( \text{Mod}_z : S \to R \otimes_C V \) is a bijection. Let \( F(x) \), for a closed point \( x \), be obtained from \( F \) by the map \( x : R \to C \); then \( F(x) \) is a fundamental matrix for \( \frac{d}{dz} + A(x) \). The other two statements of the lemma follow from this. \( \square \)

\( \frac{d}{dz} + A(x) \) is viewed as a differential equation over the ring \( C[z, \frac{1}{(z-s_1)\cdots(z-s_r)}] \). Let \( PV R(x) \) denote the subring of \( C[[z]] \) generated over \( C[z, \frac{1}{(z-s_1)\cdots(z-s_r)}] \) by all the entries of \( F(x) \) and the inverse of the determinant of \( F(x) \). Then \( PV R(x) \) is a Picard-Vessiot ring for \( \frac{d}{dz} + A(x) \). Let \( \text{Gal}(x) \) denote the group of the differential automorphisms of \( PV R(x) \) over \( C[z, \frac{1}{(z-s_1)\cdots(z-s_r)}] \). By construction \( S(x) = \ker \left( \frac{d}{dz} + A(x), PV R(x) \otimes_C V \right) \) and \( \text{Gal}(x) \) acts faithfully on \( S(x) \). Using the isomorphism \( \text{Mod}_z : S(x) \to V \), one finds a faithful action of \( \text{Gal}(x) \) on \( V \). We conclude that the above constructions provide a canonical way to embed every \( \text{Gal}(x) \) into \( \text{GL}(V) \).

The next lemma will not be used in the proof of the main result. However, its contents and the ideas behind it are closely related to our main theme. In what follows we will prove a converse of this lemma.

Lemma 3.10 (Specialization of the differential Galois group)

We use the above notation. Suppose that \( R \) is a domain with field of fractions \( K \). We can consider \( \frac{d}{dz} + A \) as a differential equation over \( K[z, \frac{1}{(z-s_1)\cdots(z-s_r)}] \). Let \( \bar{K} \) denote an algebraic closure of \( K \). Then the following holds:
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(a) the differential Galois group $G_{\overline{K}}$ over the field of constants $\overline{K}$ descends to an algebraic subgroup $G$ of $GL(K \otimes V)$,

(b) the schematic closure $G_R$ of $G$ as algebraic subgroup of $GL(R \otimes V)$ has the property: for every closed point $x$, with corresponding maximal ideal $m_x$, there is an inclusion $\text{Gal}(x) \subset (G_R \otimes R/m_x)$.

We note that this lemma and its proof are rather close to a result of O. Gabber (see [Kt90], Theorem 2.4.1 on page 39).

Proof.

(a) The solution space $\ker\left(\frac{d}{dx} + A, K[[z]] \otimes_C V\right)$ is equal to $K \otimes_R S$. Let $PVR$ denote the subring of $K[[z]]$, generated over $K[z, \frac{1}{z-z_1}, \ldots, \frac{1}{z-z_n}]$ by the entries of $F$ and the inverse of the determinant of $F$. Then $\overline{K} \otimes_K PVR$ is a Picard-Vessiot ring and we write $G_{\overline{K}}$ for its differential Galois group. The latter is characterized as the group of the $\overline{K}[[z, \frac{1}{z-z_1}, \ldots, \frac{1}{z-z_n}]]$-linear differential automorphisms of $\overline{K} \otimes PVR$. The group $G_{\overline{K}}$ acts faithfully on $\overline{K} \otimes_C V$. Choose a basis of $V$ over $C$. The affine ring of $GL(\overline{K} \otimes_C V)$ can be written as $\overline{K}[[X_{i,j}]]_{i,j=1}^n, \frac{1}{\det}]$, where $\det$ denotes the determinant of the matrix $(X_{i,j})$. The ideal $J$ defining $G_{\overline{K}}$ is the kernel of the $\overline{K}$-homomorphism $\phi: \overline{K}[[X_{i,j}]]_{i,j=1}^n, \frac{1}{\det}] \to \overline{K} \otimes PVR$, given by $\phi(X_{i,j})$ is equal to $F_{i,j}$ (the $(i,j)$-entry of the matrix $F$). Since $\phi$ “descends” to $K$, the ideal $J$ descends to an ideal $I$ of $K[[X_{i,j}]]_{i,j=1}^n, \frac{1}{\det}]$. The latter defines an algebraic subgroup $G$ of $GL(K \otimes_C V)$ satisfying $G \otimes_K \overline{K} = G_{\overline{K}}$.

(b) The schematic closure $G_R$ of $G$ is the group scheme over $R$ given by the ideal $I_R := I \cap R[[X_{i,j}]]_{i,j=1}^n, \frac{1}{\det}]$. The inclusion $\text{Gal}(x) \subset G_R \otimes R/m_x$ follows from a combination of Chevalley’s theorem and some properties of matrix differential operators (or connections). The expression $\frac{d}{dx} + A$ is seen as a regular differential operator on $\text{Spec}(R) \times (\mathbb{P}^1_C \setminus \{s_1, \ldots, s_n\})$. Let $V^*$ denote the tensor product $V^* \otimes \cdots \otimes V^* \otimes V \otimes \cdots \otimes V$ (of $a$ copies of the dual $V^*$ of $V$ and $b$ copies of $V$). There is a K-subspace $W$ of some finite direct sum $K \otimes_C \otimes_i V^*_b$ such that $G$ is the stabilizer of $W$. The differential operator $\frac{d}{dx} + B$ on $R[[z, \frac{1}{z-z_1}, \ldots, \frac{1}{z-z_n}]] \otimes_C V$ induces a differential operator $\frac{d}{dx} + B$ on $R[[z, \frac{1}{z-z_1}, \ldots, \frac{1}{z-z_n}]] \otimes_C \otimes_i V^*_b$. By differential Galois theory, $K[[z, \frac{1}{z-z_1}, \ldots, \frac{1}{z-z_n}]] \otimes_R W$ is invariant under $\frac{d}{dx} + B$. Put $\hat{W} := W \cap (R \otimes_C \otimes_i V^*_b)$. Then $\hat{W}$ is invariant under $G_R$ and moreover
$R[z, \frac{1}{(z-s_1)\cdot\ldots\cdot(z-s_r)}] \otimes_R \hat{W}$ is invariant under $\frac{d}{dz} + B$. The regularity of this differential operator implies that $\hat{W}$ is a projective $R$-module (see loc.cit. for more details). Let $x \in X(C)$. The group $R/m_x \otimes G_R$ is defined by the invariance of the subspace $R/m_x \otimes_R \hat{W}$ of $R/m_x \otimes C \oplus_i V_{b_i} = \oplus_i V_{b_i}$. Furthermore, the space $C[z, \frac{1}{(z-s_1)\cdot\ldots\cdot(z-s_r)}] \otimes_C (R/m_x \otimes_R \hat{W})$ is invariant under $\frac{d}{dz} + B(x)$. By differential Galois theory, the group $Gal(x)$ leaves $R/m_x \otimes_R \hat{W}$ invariant. Hence $Gal(x) \subseteq (R/m_x \otimes G_R)$.

In our present setup the constructibility result that we want to prove is not valid. This is illustrated by the rather obvious example: $R = C[t]$ and the differential operator $\frac{d}{dz} + \frac{1}{z^{q+1}}$. If the value of $t$ is rational number of the form $\frac{p}{q}$ with $q \geq 1$ and $(p, q) = 1$, then the differential Galois group is a cyclic group of order $q$. For other values of $t$ in $C$, the differential Galois group is the multiplicative group $\mathbb{G}_m$. However, the group $\mathbb{G}_m$ satisfies the "Singer condition".

In order to avoid this and other examples of this sort we will suppose that there are only finitely many possibilities for the formal local structure of $\frac{d}{dz} + A(x)$ at any of the singular points $s_1, \ldots, s_r$. Again this is not sufficient for our goal, namely the statement that the set of closed points $x$ with $Gal(x) = G$ is constructible. The new problem is that the formal isomorphism between $\frac{d}{dz} + A(x)$ at $s_j$ and one of the prescribed formal connections can have a pole at $s_j$ of arbitrarily high order. A remedy for this is the introduction of connections on the projective line over $C$. In order to work out this idea the following (probably known) result on vector bundles on $\mathbb{P}^1_X := X \times \mathbb{P}^1_C$ is needed. We introduce some notation. Let $pr_1 : X \times \mathbb{P}^1_C \to X$ and $pr_2 : X \times \mathbb{P}^1_C \to \mathbb{P}^1_C$ denote the two projections. For vector bundles $\mathcal{A}$ and $\mathcal{B}$ on $X$ and $\mathbb{P}^1_C$, we write $\mathcal{A} \otimes \mathcal{B}$ for the vector bundle $pr_1^*\mathcal{A} \otimes pr_2^*\mathcal{B}$. The line bundle of degree $d$ on $\mathbb{P}^1_C$ is denoted by $\mathcal{O}(d)$. For $\mathcal{O}_X \otimes \mathcal{O}(d) = pr_2^*\mathcal{O}(d)$ we also write $\mathcal{O}_X(d)$. We recall that any vector bundle of rank $n$ on $\mathbb{P}^1_C$ has the form $\mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_n)$ with unique $a_1 \geq a_2 \geq \cdots \geq a_n$. We call the sequence $a_1 \geq \cdots \geq a_n$ the type of the vector bundle.

**Proposition 3.11** Let $X$ be a scheme of finite type over $C$ and let $\mathcal{M}$ be a vector bundle on $\mathbb{P}^1_X$ of rank $n$. Let $x \in X$ be a closed point. Suppose that the induced vector bundle $\mathcal{M}(x)$ on $\mathbb{P}_C$ is free. Then there exists an open neighbourhood $U$ of $x$ such that the restriction of $\mathcal{M}$ to $\mathbb{P}^1_U$ is free.
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Proof. We remark that $\mathcal{M}(x)$ denotes the vector bundle on $\mathbb{P}^1_C$ obtained by evaluating $\mathcal{M}$ at $x$. More precisely, write $j_x : \text{Spec}(C) \to X$ for the morphism corresponding to $x$ and write $g_x = j_x \times id : \mathbb{P}^1_C = \text{Spec}(C) \times \mathbb{P}^1_C \to X \times \mathbb{P}^1_C$. Then $\mathcal{M}(x)$ is defined as $g_x^* \mathcal{M}$.

One may suppose that $X$ is affine. Let $D_0$ and $D_\infty$ denote the divisors $X \times \{0\}$ and $X \times \{\infty\}$. Define the sheaf $\mathcal{N} = \mathcal{O}(-D_\infty) \otimes \mathcal{M}$ and consider the covering of $\mathbb{P}^1_X$ by the affine sets $U_0 = \mathbb{P}^1_X - D_\infty$ and $U_\infty = \mathbb{P}^1_X - D_0$. Put $U_{0,\infty} = U_0 \cap U_\infty$. The following sequence

$$0 \to H^0(\mathcal{N}) \to \mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty) \to \mathcal{N}(U_{0,\infty}) \to H^1(\mathcal{N}) \to 0$$

is exact. The two $O(X)$-modules $H^0(\mathcal{N})$ and $H^1(\mathcal{N})$ are finitely generated. Indeed, since the natural projection $pr : \mathbb{P}^1_X \to X$ is proper one has that $pr_* \mathcal{N}$ and $R^1 pr_* \mathcal{N}$ are coherent. Moreover, $H^0(\mathcal{N}) = H^0(X, pr_* \mathcal{N})$ and $H^1(\mathcal{N}) = H^0(X, R^1 pr_* \mathcal{N})$ (by Leray’s spectral sequence). Let $m_x$ denote the maximal ideal of $O(X)$ corresponding to the closed point $x$. The assumption that $\mathcal{M}(x)$ is free implies that $H^0(\mathcal{N}(x)) = H^1(\mathcal{N}(x)) = 0$. This implies that the map $\alpha \otimes_{O(X)} O(X)/m_x$ is a bijection. Hence $x$ does not lie in the support of the $O(X)$-module $H^1(\mathcal{N})$. After shrinking $X$, we may assume that $H^1(\mathcal{N}) = 0$ and that $\alpha$ is surjective. The $O(U_{0,\infty})$-module $\mathcal{N}(U_{0,\infty})$ is projective. Therefore $\mathcal{N}(U_{0,\infty})$ is also a projective module over the ring $O(X)$. Hence the exact sequence of $O(X)$-modules

$$0 \to H^0(\mathcal{N}) \to \mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty) \to \mathcal{N}(U_{0,\infty}) \to 0$$

splits. The bijectivity of the map $\alpha \otimes_{O(X)} O(X)/m_x$ implies that the module $H^0(\mathcal{N}) \otimes_{O(X)} O(X)/m_x = 0$. After shrinking $X$, we may suppose that $H^0(\mathcal{N}) = 0$. Define the sheaf $\mathcal{Q}$ by the exactness of

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{Q} \to 0.$$
denotes the maximal ideal of the local ring $\mathcal{O}_{x_{K}}$. The point $w$ lies on a divisor $D = X \times \{ p \}$ for some closed point $p$ of $\mathbb{P}_{C}^{1}$. Put $w = (q, p)$. Define the sheaf $S$ by the exact sequence

$$0 \to \mathcal{O}(-D) \otimes \mathcal{M} \to \mathcal{M} \to S \to 0.$$  

We note that $\mathcal{O}(-D)$ is isomorphic to $\mathcal{O}(-D_{\infty})$. As before one concludes that $S$ is a vector bundle on $X \cong X \times \{ p \}$ and that $H^{0}(\mathcal{M}) \to H^{0}(X \times \{ p \}, S)$ is surjective. Since $X \times \{ p \}$ is affine, the map $H^{0}(X \times \{ p \}, S) \to S_{q}/m_{q}S$ (where $m_{q}$ denotes the maximal ideal corresponding to the point $w = (q, p)$) is surjective. Finally, $\mathcal{M}_{w}/m_{w}\mathcal{M}_{w} \to S_{q}/m_{q}S_{q}$ is an isomorphism. $\Box$

**Remarks 3.12 More on vector bundles on $\mathbb{P}^{1}_{X}$.**

1. We start with an example showing that the type of a vector bundle on $\mathbb{P}^{1}_{X}$ is not locally constant, i.e., the type of $\mathcal{M}(x)$ is not locally constant in $X$. Take $X = \text{Spec}(C[t])$ and consider a vector bundle $\mathcal{M}$ of rank 2 on $\mathbb{P}^{1}_{X}$. Let $z$ denote the usual global parameter on $\mathbb{P}^{1}_{C}$. Write again $D_{0} = X \times \{ 0 \}$, $D_{\infty} = X \times \{ \infty \}$, $U_{0} = \mathbb{P}^{1}_{X} - D_{\infty}$ and $U_{\infty} = \mathbb{P}^{1}_{X} - D_{0}$. The restriction of $\mathcal{M}$ to the two affine sets $U_{0}, U_{\infty}$ is free (since every projective module over a polynomial ring is free). Hence $\mathcal{M}$ is given by a matrix $A \in \text{GL}(2, C[t][z, z^{-1}])$. This matrix defines a unique double coset $\text{GL}(2, C[t][z]) \cdot A \cdot \text{GL}(2, C[t][z^{-1}])$. On the other hand each double coset, as above, defines a vector bundle of rank 2 on $\mathbb{P}^{1}_{X}$. We consider now the vector bundle associated to

$$A = \begin{pmatrix} z & 0 \\ t & z^{-1} \end{pmatrix}.$$  

For $t = 0$, this defines the vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ on $\mathbb{P}^{1}_{C}$. For $t \neq 0$, this defines the free vector bundle on $\mathbb{P}^{1}_{C}$. Indeed,

$$A = \begin{pmatrix} 1 & t^{-1}z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -t^{-1} \\ t & z^{-1} \end{pmatrix}.$$  

2. Let $\mathcal{M}$ be a vector bundle on $\mathbb{P}^{1}_{X}$ of rank $n$. Then the set of closed points $x \in X(C)$, such that $\mathcal{M}(x)$ has type $a_{1} \geq a_{2} \geq \cdots \geq a_{n}$, is a constructible subset. We sketch the proof of this result. It suffices to consider the case where $X$ is affine and connected. For a point $x \in X(C)$, the type $a_{1} \geq \cdots \geq a_{n}$ of the vector bundle $\mathcal{M}(x)$ is determined by the dimensions $h^{i}(k, x)$, $i = 0, 1$ of the cohomology groups $H^{i}(\mathbb{P}^{1}_{C}, \mathcal{M}(x) \otimes \mathcal{O}(k))$, for all
$k \in \mathbb{Z}$. The degree $D$ of $\mathcal{M}(x)$ is independent of $x \in X(C)$. By Riemann-Roch, $h^0(k, x) - h^1(k, x) = D + n$ for all $k$. There exists an integer $N$, depending on $\mathcal{M}$, such that for $k \geq N$ one has $h^1(k, x) = 0$ and for $k \leq -N$ one has $h^0(k, x) = 0$. Hence the type of $\mathcal{M}(x)$ is determined by the values of $h^1(k, x)$ for $-N < k < N$. Therefore we have to investigate the dependence of $h^1(k, x)$ on $x$. For convenience we consider $h^1(0, x)$. The proof of Proposition 3.11 asserts that $H^1(\mathbb{P}_C^1, \mathcal{M}(x)) = O(X)/m_x \otimes H^1(\mathcal{M})$, where $m_x$ denotes the maximal ideal corresponding to $x$. This implies that for any integer $d \geq 0$ the set $\{x \in X(C) \mid h^1(0, x) \leq d\}$ is open. From this observation the above statement follows.

(3) The defect of a vector bundle on $\mathbb{P}_C^1$, of type $a_1 \geq \cdots \geq a_n$, is defined as $a_1 - a_n$. The reasoning in (2) above implies that for any integer $d \geq 0$ the set $\{x \in X(C) \mid$ the defect of $\mathcal{M}(x)$ is $\leq d\}$ is open. This generalizes the statement of Proposition 3.11.

(4) Suppose that $X$ is a reduced, irreducible scheme of finite type over $C$. Let $\mathcal{M}$ be a vector bundle on $\mathbb{P}_X^1$. Suppose that there exists a closed point $x_0 \in X$ such that $\mathcal{M}(x_0)$ is free. Then the set of closed points $x$ such that $\mathcal{M}(x)$ is not free is either empty or equal to a divisor on $X$.

Sketch of the proof. We may suppose that $X = \text{Spec}(R)$ with $R$ a finitely generated $C$-algebra having no zero-divisors. We will use the notation of the proof of Proposition 3.11. The statement that we want to prove is equivalent to: the $R$-module $H^1(\mathcal{N})$ is either 0, or its support is a divisor on $X$.

Consider again the exact sequence

$$0 \rightarrow H^0(\mathcal{N}) \rightarrow \mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty) \rightarrow \mathcal{N}(U_{0, \infty}) \rightarrow H^1(\mathcal{N}) \rightarrow 0$$

The assumption that $\mathcal{M}(x_0)$ is free implies that $\alpha$ becomes an isomorphism after localizing $R$ at a suitable non-zero element. Thus $H^0(\mathcal{N}) = 0$ (since $R$ has no zero-divisors) and $H^1(\mathcal{N})$ is a finitely generated torsion module over $R$. The modules $\mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty)$ and $\mathcal{N}(U_{0, \infty})$ are projective $R$-modules of infinite rank. The above exact sequence is therefore a resolution of $H^1(\mathcal{N})$ by projective modules of infinite rank. Consider an exact sequence

$$0 \rightarrow V_1 \xrightarrow{f} V_0 \rightarrow H^1(\mathcal{N}) \rightarrow 0$$

with $V_0$ a finitely generated free $R$-module. Then $V_1$ is a projective $R$-module (of finite rank). After replacing $\text{Spec}(R)$ by the elements of an open affine
covering, we may suppose that $V_i$ is a free $R$-module, too. Furthermore, $V_i$ and $V_0$ have the same rank. The support of $H^1(N)$ is equal to the closed subset defined by $\text{Det}(f) = 0$. This finishes the proof.

The above result is also valid in the complex analytic case. A proof is given by B. Malgrange in [M83], Section 4.

(5) In trying to classify the vector bundles on $X \times \mathbb{P}_C^l$, one encounters the question whether a vector bundle $\mathcal{M}$ of rank $n$ on $X \times \mathbb{P}_C^l$ has, at least locally with respect to $X$, the property that the restrictions of $\mathcal{M}$ to the affine sets $\text{Spec}(R) \times (\mathbb{P}_C^l - \{\infty\})$ and $\text{Spec}(R) \times (\mathbb{P}_C^l - \{0\})$ are free. If the answer is positive, then $\mathcal{M}$ is (locally with respect to $X$) defined by a double coset $\text{GL}(n, R[z]) \cdot A \cdot \text{GL}(n, R[z^{-1}])$ with $A \in \text{GL}(n, R[z, z^{-1}])$. This seems a useful way to present $\mathcal{M}$. The above question is directly related to the following question posed by H. Bass and D. Quillen:

Let $R$ be a regular noetherian ring. Does every finitely generated projective module over $R[z]$ come from a finitely generated projective module over $R$?

There are partial answers to this question (see [L78]). It seems that the general problem remains unsolved.

\textbf{Definition 3.13}
A family of differential equations on $\mathbb{P}^l$, parametrized by $X$
Distinct points $\{s_1, \ldots, s_r\} \subset \mathbb{P}_C^l \setminus \{0, \infty\}$ are given. Moreover, a finite set $I$ of semi-simple formal connections $\nabla_i : C[[u]]^n \to C[[u]]u^{-k} du \otimes C[[u]]^n$ (with $i \in I$) is given. This collection will be called the \textit{local data}. The next items are $X, \mathcal{M}, \nabla, V$ where:

(i) $X$ is a reduced scheme of finite type over $C$.

(ii) $\mathcal{M}$ is a vector bundle on $\mathbb{P}_X^l$ of the form $\mathcal{O}_X \otimes \mathcal{N}$, where $\mathcal{N}$ is a vector bundle on $\mathbb{P}_C^l$.

(iii) $V$ is a vector space of dimension $n$ over $C$, and there is given an isomorphism $\mathcal{N}_0/(z) \xrightarrow{\sim} V$.

(iv) A connection $\nabla : \mathcal{M} \to \Omega(k[s_1] + \cdots + k[s_r]) \otimes \mathcal{M}$. 
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For every point \( x \in X(C) \), we find a vector bundle \( \mathcal{M}(x) \) on \( \mathbb{P}_C^1 \), where \( \mathcal{M}(x) = j_x^*(\mathcal{M}) \), \( j_x : \{ x \} \times \mathbb{P}_C^1 \rightarrow \mathbb{P}_C^1 \). The above data induce a connection \( \nabla(x) : \mathcal{M}(x) \rightarrow \Omega(k[s_1] + \cdots + k[s_r]) \otimes \mathcal{M}(x) \). For every point \( x \in X(C) \) and every \( j \), we write \( u \) for the local parameter \( z - s_j \). We require that the semi-
simplification of the connection \( \nabla(x)_j : \mathcal{M}(x)_{s_j} \rightarrow C[[u]]u^{-k}du \otimes \mathcal{M}(x)_{s_j} \) is isomorphic to \( \nabla_i \) for some \( i \in I \). More precisely, there exists a \( C[[u]] \)-linear isomorphism \( \mathcal{M}(x)_{s_j, ss} \rightarrow C[[u]]^n \) that is compatible with the connections.

\[\nabla : \mathcal{M} \rightarrow \Omega_{\mathbb{P}_X} \left( \sum k[X \times \{ s_i \}] \right) \otimes \mathcal{M},\]

where the \( [X \times \{ s_i \}] \) are divisors on \( \mathbb{P}_X \). Moreover, the integer \( k \) occurring here can be replaced by any integer \( \ell \geq k \) without changing the family.

(2) A moduli space \( \mathcal{M}_a \), as defined in Chapter 2, is a special case of a family. Such a moduli problem yields a universal family, parametrized by \( \mathcal{M}_a \). The corresponding family is given by \( X = \mathcal{M}_a \), \( \mathcal{M} = \mathcal{O}_{\mathbb{P}_X} \otimes V \), and \( \nabla \), such that the universal family is \( (\mathcal{M}, \nabla, \{ \phi_i \}) \).

(3) Let a family, parametrized by, say, \( X = \text{Spec}(R) \) be given. For every \( x \in X \), we have a full solution space \( W(x) \) of \( \nabla(x) \) in \( \mathcal{M}(x)_0 \). We want to make an identification of \( W(x) \) with \( V \). By (iii) of the definition, we have an isomorphism \( \mathcal{N}_0/(z) \rightarrow V \). This isomorphism can be lifted to an isomorphism \( \mathcal{N}_0 \rightarrow C[z]_{(z)} \otimes V \). The latter is unique up to a \( C[z]_{(z)} \)-linear automorphism of \( C[z]_{(z)} \otimes V \) that is the identity modulo the ideal \( (z) \). The isomorphism \( \mathcal{N}_0 \rightarrow C[z]_{(z)} \otimes V \) can be extended to an isomorphism \( \mathcal{N}_0 \rightarrow C[[z]]_{(z)} \otimes V \), which is unique up to an element \( h \in \text{GL}(C[z]_{(z)} \otimes V) \) that is the identity modulo \( (z) \). We have that \( \mathcal{M}(x) \) is canonically isomorphic to \( \mathcal{N}_0 \), so the above map gives a canonical way to identify \( \ker(\nabla, \mathcal{M}(x)_0) \) with \( V \) (via mod \( z \)). So the differential Galois group \( \text{Gal}(x) \) is canonically embedded into \( \text{GL}(V) \).

(4) Let a family, parametrized by, say, \( X = \text{Spec}(R) \), be given. We will make some changes to this family. The isomorphism \( V \rightarrow \mathcal{N}_0/(z) \) can be
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lifted to a map $V \to \mathcal{N}_0$. Now $V$ can be considered as a subspace of the generic fiber $\mathcal{N}_\xi$, using the canonical map $\mathcal{N}_0 \to \mathcal{N}_\xi$. Now $\mathcal{N}$ is replaced by $\mathcal{N}_1 := \mathcal{N}(\ell[b_1] + \cdots + \ell[b_r])$ for suitable $\ell > 0$ and points $b_1, \ldots, b_r \neq 0$, such that $V \subset H^0(\mathcal{N}_1)$. Then we consider the free vector bundle $\mathcal{F} := \mathcal{O}_{\mathbb{P}_\xi} \otimes V$, subbundle of $\mathcal{N}_1$, and the free vector bundle $\mathcal{F}_X = \mathcal{O}_X \otimes \mathcal{F}$.

In general, $\nabla(\mathcal{F}_X) \subset \Omega(\sum_{i=1}^{t} k[s_i]) \otimes \mathcal{F}_X$ does no hold. At the cost of introducing some points $\{s_{i+1}, \ldots, s_t\}$ as new (apparent) singularities and adding finitely many new items to the local data, one obtains a new family, parametrized by $X$, with

$$\nabla : \mathcal{F}_X \to \Omega(\sum_{i=1}^{t} k[s_i]) \otimes \mathcal{F}_X \text{ (for a suitable, large enough } k > 0).$$

One of the new singular points $s_j$ might be the point $\infty$. An automorphism of $\mathbb{P}_\xi$, which fixes 0, takes care of that. The original family is closely related to this new family. In particular, $\text{Gal}(x) \subset \text{GL}(V)$ remains unchanged for every $x \in X$. So for the constructibility result that we want to prove, we may replace the original family by the new one. In what follows we may therefore (at any stage of the proof) assume that the vector bundle $\mathcal{M}$ on $\mathbb{P}_X$ is equal to $\mathcal{O}_X \otimes \mathcal{N}$ with $\mathcal{N}$ a free vector bundle on $\mathbb{P}_\xi$. Moreover $V$ is identified with $H^0(\mathcal{N})$. In other terms $\mathcal{M} = \mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}_\xi} \otimes V)$.

(5) For an algebraic subgroup $H$ of $\text{GL}(V)$ we write $X(\subset H)$ (resp. $X(H \subset)$) for the set of closed points $x \in X$ such that $\text{Gal}(x) \subset H$ (resp. $H \subset \text{Gal}(x)$). For two algebraic subgroups $H_1 \subset H_2$ we write $X(H_1 \subset \subset H_2)$ for $X(H_1 \subset) \cap X(\subset H_2)$. Furthermore, $X(\subset = H) := X(\subset \subset H)$. The main result of this chapter is the following.

Theorem 3.15 Suppose that the linear algebraic subgroup $G \subset \text{GL}(V)$ satisfies the “Singer condition”. Let a family of differential equations on $\mathbb{P}_1$, parametrized by $X$ be given. Then $X(\subset = G)$ is a constructible subset of $X$.

In the proof we follow some of the steps of the proof given in [S93]. However, we like to point out some important differences. In our setup, the differential Galois group $\text{Gal}(x)$ is given as a subgroup of $\text{GL}(V)$, whereas in [S93] this group is only determined up to conjugacy in $\text{GL}(V)$. The bounds $B$ and real algebraic subspaces $\mathcal{L}(n, m, B)$ of $\mathcal{L}(n, m)$ are not present in our proof.
The prescribed local connections and the type of the vector bundle \( \mathcal{M} \) provide the necessary bounds on the degrees of \( \nabla \)-invariant line bundles. The "constructions of linear algebra", needed in the proof, are rather involved for differential operators (especially when one has to produce another "cyclic vector"). Here the constructions are the natural ones known for differential modules. Our proof can be adapted to the case where the singular points are not fixed. However we prefer to avoid the technical complications introduced by "moving singularities". Finally, Singer's result applies to certain sets of differential equations. It seems possible to make a translation between those sets of differential equations, and our families of differential equations on \( \mathbb{P}^1 \), but now with moving singularities.

3.4 Proof of Singer's theorem for families

Throughout this section we will mainly consider families of differential equations \( (\mathcal{M}, \nabla, V, \{\nabla_i\}_{i \in I}) \), with \( \mathcal{M} = \mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1} \otimes V) \). We will write such a family as \( (\nabla, V, \{\nabla_i\}) \).

3.4.1 The set \( X(\subset G) \) is closed

We denote by \( V_b^a \) the tensor product of \( a \) copies of the dual \( V^* \) of \( V \) and of \( b \) copies of \( V \). One considers a subspace \( W \) of dimension \( d \) of a finite sum \( \oplus_i V_{b_i}^{a_i} \). Then \( G := \{ g \in \text{GL}(V) \mid gW = W \} \) is an algebraic subgroup of \( \text{GL}(V) \). According to Chevalley's theorem, every algebraic subgroup of \( \text{GL}(V) \) has this form. Put \( Z := \bigwedge^d (\oplus_i V_{b_i}^{a_i}) \) and \( L := \bigwedge^d W \). Then \( G \) is equal to \( \{ g \in \text{GL}(V) \mid gL = L \} \), too. The subgroups of \( \text{GL}(V) \), conjugated to \( G \), are the stabilizers of the lines \( hL \subset Z \) with \( h \in \text{GL}(V) \). This family of lines in \( Z \) is a constructible subset of \( \mathbb{P}(Z) \). Write \( L = C_{x_0} \). Then the set \( \{ h_{x_0} \mid h \in \text{GL}(V) \} \) is also constructible. Indeed, the action of \( \text{GL}(V) \) on \( Z \) and \( \mathbb{P}(Z) \) is algebraic.

**Proposition 3.16** Let be given a family of differential equations on the projective line, parametrized by \( X \). Then \( X(\subset G) \) is closed.

**Proof.** We have to extend the proofs of Chapter 2 to the present more general situation. We suppose that \( X \) is reduced, irreducible and affine. Let
$G$ be given as above as the stabilizer of a (special) line $L$ in a construction $Z$. Each step in the construction of $Z$ can be supplemented by a new family of differential equations parametrized by the same $X$. Indeed, for the dual $V^*$ one constructs from the given family, a new family obtained by taking everywhere duals. This works well since the free vector bundle $O_{\mathbb{P}^1} \otimes V$ has an obvious dual $O_{\mathbb{P}^1} \otimes V^*$. For a tensor product, like $V^*_g$, one can form the tensor product of the corresponding vector bundles (including their connections and the local data). Direct sums and exterior powers are treated in the obvious way. Thus we find a family, parametrized by $X$ and corresponding to $Z$, consisting of a free vector bundle $N$, identified with $O_X \otimes (O_{\mathbb{P}^1} \otimes Z)$, a connection $\nabla$ on $N$ and a new finite set of prescribed formal connections over $C[[u]]$. Then, according to Lemma 2.16, the set $X(\subset G)$ consists of the closed points $x$ such that there is a line bundle $\mathcal{L}$, contained in $N(x)$ and satisfying:

(i) $\mathcal{L}$ is invariant under $\nabla(x)$,

(ii) $N(x)/\mathcal{L}$ is again a vector bundle,

(iii) $\mathcal{L}_0/\mathcal{L}_0$ is equal to $L$.

We follow closely the proof of Theorem 2.17. Write $L = C\mathcal{L}_0$ and let $-d \leq 0$ denote the degree of a putative $\mathcal{L}$. Then one finds an equation for the generator $v_0 + v_1 z + \cdots + v_d z^d$ of $\mathcal{L}(d \cdot [\infty])$ (see the proof of Theorem 2.17). This equation has the form

$$\frac{d}{dz} + \sum_{i=1}^{s} \sum_{j=1}^{k} \frac{B_{i,j}(x)}{(z - s_i)^j} - T(\sum_{i=0}^{d} v_i z^i) = 0,$$

where the $B_{i,j}$ are endomorphisms of $O(X) \otimes Z$; $B_{i,j}(x)$ is the evaluation of $B_{i,j}$ at $x$, and $T := \sum \frac{g_{i,j}}{(z - s_i)^j}$ with $g_{i,j} \in C$. We note that $T$ does not depend on $x \in X$. There are finitely many possibilities for $T$, according to Lemma 2.18 and Definition 3.13. Each possibility yields at most one value for $d$. Now we consider a fixed choice for the term $T$. The equation

$$\frac{d}{dz} + \sum_{i=1}^{s} \sum_{j=1}^{k} \frac{B_{i,j}}{(z - s_i)^j} - T(\sum_{i \geq 0} v_i z^i) = 0,$$
with the prescribed \( v_0 \in Z \) and \( v_i \in O(X) \otimes Z \) for \( i \geq 1 \) has a unique solution (which is denoted by the same symbols). One can see \( v_i \), for \( i \geq 1 \), as a morphism from \( X \) to \( Z \). This determines a closed subset, say \( X(T) \) of \( X \), defined by \( v_i(x) = 0 \) for \( i > d \). In other words, \( X(T) \) is the intersection \( \cap_{i > d} v_i^{-1}(0) \). Finally, \( X(\subseteq G) \) is the union of the finitely many closed sets \( X(T) \). \hfill \Box

**Corollary 3.17** Let a family \( (\nabla, V, \{ \nabla_i \}_{i \in I}) \) of differential equations, parametrized by \( X \), be given.

1. Consider a vector space \( Z \) of the form \( \bigwedge^d (\oplus_i V_{b_i}^e) \) and a constructible subset \( S \) of \( Z \setminus \{0\} \). The set of the closed points \( x \in X(C) \) such that \( \text{Gal}(x) \subset \text{GL}(V) \) fixes a line \( C_s \subset Z \) with \( s \in S \) (for the induced action of \( \text{Gal}(x) \) on \( Z \)), is constructible.

2. Let \( G \) be an algebraic subgroup of \( \text{GL}(V) \). The set of the closed points \( x \in X(C) \), such that \( \text{Gal}(x) \) lies in a conjugate of \( G \), is constructible.

**Proof.**

(1) As in the proof of Proposition 3.16, one supposes that \( M \) is equal to \( O_X \otimes (O_{Z_C} \otimes V) \). There is an induced family \( (\nabla, Z, \text{ local data}) \). As in that proof, a fixed choice for the term \( T \) is made. The element \( v_0 \) is not fixed but lies in a given constructible subset \( S \) of \( Z \setminus \{0\} \). The elements \( v_i \) with \( i \geq 1 \) are now viewed as morphisms \( S \times X \rightarrow Z \). The set \( \bigcap_{i > d} v_i^{-1}(0) \) is a closed subset of \( S \times X \). Its image \( X(T, S) \), under the projection \( S \times X \rightarrow X \), is constructible. The union of the finitely many \( X(T, S) \) is the set of the closed points \( x \) such that \( \text{Gal}(x) \subset \text{GL}(V) \) fixes, for its action on \( Z \), a line \( L \) of the form \( L = Cs \) with \( s \in S \).

(2) Take \( Z \) as in (1) and a line \( L \subset Z \) such that \( G = \{ g \in \text{GL}(V) \mid gL = L \} \). Write \( L = Cv_0 \). Then (1), applied to the constructible set \( S = \{ hv_0 \mid h \in \text{GL}(V) \} \), yields (2). \hfill \Box

### 3.4.2 Galois invariant subspaces and subbundles

Let a family of differential equations \( (\nabla, V, \{ \nabla_i \}) \), parametrized by a reduced, irreducible, affine \( X \) be given. Let \( W \) be a subspace of \( V \) such that \( W \) is
invariant under all $\text{Gal}(x)$. Our aim is to prove that there is a subbundle of $\mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1} \otimes V)$, invariant under $\nabla$, corresponding to $W$. We start by discussing the special case where $W = Ce$ (with $e \neq 0$). We can give $\nabla_{\frac{d}{dz}}$ in the explicit form
\[
\frac{d}{dz} + A = \frac{d}{dz} + \sum_{i,j} \frac{A_{i,j}}{(z - s_j)^v},
\]
where the $A_{i,j}$ are $O(X)$-linear endomorphisms of $O(X) \otimes V$ and where $\sum_{j} A_{i,j} = 0$. We return to the proof of Theorem 2.17 and its terminology. For a fixed $x \in X(C)$, there is a term $T = \sum_{i,j} \frac{g_{i,j}}{(z - s_j)^v}$ with all $g_{i,j} \in C$ such that $\sum_{j} g_{i,j}$ is an integer $d \geq 0$ and there is a solution $v_0 + v_1 z + \cdots + v_d z^d$

of $(\frac{d}{dz} + A(x))(v_0 + v_1 z + \cdots + v_d z^d) = T(v_0 + v_1 z + \cdots + v_d z^d)$, such that $v_0 = e$ and $v_d \neq 0$. Moreover, there are only finitely many possibilities for $T$. Now we fix $T$ and consider the equation $(\frac{d}{dz} + A)(\sum_{i \geq 0} v_i z^i) = T(\sum_{i \geq 0} v_i z^i)$ with $v_0 = e$ and $v_i \in O(X) \otimes V$ for $i \geq 1$. This equation has a unique solution. The closed subset of $X$ given by $v_i(x) = 0$ for $i > d$, is denoted by $X(T)$. By assumption $X$ is the union of the finitely many sets $X(T)$. Since $X$ is irreducible, $X$ is equal to a single $X(T)$. We continue with this $T$.

Let $v_0 + v_1 z + \cdots + v_d z^d$ denote the solution corresponding to this $T$ (with again $v_0 = e$ and $v_i \in O(X) \otimes V$ for $i \geq 1$). It is, a priori, possible that $v_d$ is identically zero. Let $\ell$ be maximal such that $v_\ell$ is not identical zero. It is also possible that $v_0 + v_1 z + \cdots + v_{\ell} z^\ell$ is divisible by some $(z - s_j)$. We divide $v_0 + v_1 z + \cdots + v_{\ell} z^\ell$ by $(z - s_1)^{m_1} \cdots (z - s_r)^{m_r}$ with $m_1, \ldots, m_r \geq 0$ as large as possible (this changes the $T$ as well). The result is a section, say $v_0 + w_1 z + \cdots + w_q z^q$, of $\mathcal{M}(q \cdot [\infty])$ such that none of the expressions $w_q$ and $v_0 + w_1 s_j + \cdots + w_q s_j^q$ for $j = 1, \ldots, r$, is identical zero. Let $X'$ be the open, non-empty, subset of $X$ given by $w_q(x) \neq 0$ and the $v_0 + w_1(x) s_j + \cdots + w_q(x) s_j^q \neq 0$ for $j = 1, \ldots, r$. We claim that the section $v_0 + w_1 z + \cdots + w_q z^q$ of $\mathcal{M}(q \cdot [\infty])$ does not vanish on $X' \times \mathbb{P}^1$. For points $(x, y)$ or $(x, s_j)$ with $x$ a closed point of $X'$, this is obvious. For a point $(x, s)$ with $s \notin \{s_1, \ldots, s_r\}$ and $x \in X'(C)$, the expression $v_0 + w_1(x) z + \cdots + w_q(x) z^q$ is a solution of the differential operator $\frac{d}{dz} + A(x) - T$. Since this operator is regular at $s$, the vanishing of $v_0 + w_1(x) s + \cdots + w_q(x) s^q$ implies that $v_0 + w_1(x) z + \cdots + w_q(x) z^q$ is identical zero. This contradicts $w_q(x) \neq 0$.

In what follows, $X$ is already replaced by the non-empty open subset $X'$. 
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In the next steps, we will shrink $X$ even further. Let $\mathcal{F} = \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^1_c}$. The line bundle $\mathcal{F}$ is embedded into $\mathcal{M}(q \cdot [\infty])$ by sending the global section 1 of $\mathcal{O}_{\mathbb{P}^1_c}$ to $v_0 + w_1 z + \cdots + w_q z^q$. This induces a connection on $\mathcal{F}$ and local data for $\mathcal{F}$. Moreover, we identify $(\mathcal{O}_{\mathbb{P}^1_c})_0/(z)$ with $Cv_0$, by sending 1 to $v_0 = e$. Now we consider $\mathcal{L} := \mathcal{F}(-q \cdot [\infty]) = \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^1_c}(-q \cdot [\infty])$. The above data make $(\mathcal{L}, \nabla, Cv_0, \text{local data})$ into a family, parametrized by $X$.

The quotient $\mathcal{Q} := \mathcal{M}/\mathcal{L}$ is again a vector bundle on $\mathbb{P}^1_X$ with an induced connection and induced local data. After shrinking $X$, there exists a vector bundle $\mathcal{N}$ on $\mathbb{P}^1_c$ such that $\mathcal{Q} = \mathcal{O}_X \otimes \mathcal{N}$. A choice of an isomorphism $\lambda : \mathcal{N}_0/(z) \to V/\mathcal{E}$ induces an isomorphism $\mathcal{Q}_0/(z) \to \mathcal{O}_X \otimes (V/\mathcal{E})$. We require that this map is induced by the given isomorphism $\mathcal{M}_0/(z) \to \mathcal{O}_X \otimes V$. For every closed point $x$ of $X$, there is an induced exact sequence of connections $0 \to \mathcal{L}(x) \to \mathcal{M}(x) \to \mathcal{Q}(x) \to 0$ on $\mathbb{P}^1_c$. The action of $\text{Gal}(x)$ on $V$ induces the actions on $\mathcal{E}$ and $V/\mathcal{E}$ for the connections $\mathcal{L}(x)$ and $\mathcal{Q}(x)$. We come now to the general result.

**Proposition 3.18** Let $(\nabla, V, \{\nabla_i\})$ be a family, parametrized by a reduced, irreducible scheme $X$ of finite type over $C$. Let $W \subset V$ be a proper subspace such that $W$ is invariant under $\text{Gal}(x)$ for all $x \in X(C)$. After replacing $X$ by a non-empty open subset, there exists a family $(\mathcal{N}, \nabla^*, W, \text{local data})$ parametrized by $X$ such that:

(i) $\mathcal{N}$ is a subbundle of $\mathcal{M} = \mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1_c} \otimes V)$, invariant under $\nabla$. Moreover, $\nabla^*$, the local data of $\mathcal{N}$ and the isomorphism $\mathcal{N}_0/(z) \to \mathcal{O}_X \otimes W$ are induced by those of $\mathcal{M}$.

(ii) The sheaf $\mathcal{Q} := \mathcal{M}/\mathcal{N}$ is a vector bundle on $\mathbb{P}^1_X$, isomorphic to $\mathcal{O}_X \otimes S$ for a suitable vector bundle $S$ on $\mathbb{P}^1_c$. Moreover, $\mathcal{Q}$ can be made into a family, parametrized by $X$, with connection, local data, and isomorphism $\mathcal{Q}_0/(z) \to \mathcal{O}_X \otimes (V/W)$, induced by those of the family $\mathcal{M}$.

(iii) For every closed point $x \in X(C)$, the exact sequence

$$0 \to \mathcal{N}(x) \to \mathcal{M}(x) \to \mathcal{Q}(x) \to 0$$

of connections on $\mathbb{P}^1_c$, has the property that the action of $\text{Gal}(x)$ on $V$ induces the actions of the differential Galois groups on $W$ and $V/W$, that are produced by $\mathcal{N}(x)$ and $\mathcal{Q}(x)$.
Proof. Put \( d = \dim W \). The case \( d = 1 \) is discussed above. For the general case one considers \( L = \wedge^d W \subset \wedge^d V \) and the family \( (\wedge^d \mathcal{M}, \ldots) \) associated to \( \wedge^d V \). One finds a line bundle \( \mathcal{L} \subset \wedge^d \mathcal{M} \) (above a suitable open subset of \( X \)) with the required properties. This line bundle is decomposable since the line \( L \subset \wedge^d V \) is decomposable. Thus there exists a vector bundle \( \mathcal{N} \subset \mathcal{M} \) (above a suitable open subset of \( X \)) with \( \wedge^d \mathcal{N} = \mathcal{L} \) and \( \mathcal{N} \) has the required properties. In particular, \( \mathcal{Q} \) is a connection on \( \mathbb{P}^1_X \). It is not difficult to provide \( \mathcal{N} \) and \( \mathcal{Q} \) with the additional structure, which makes them into families, parametrized by \( X \). This proves (i) and (ii). Part (iii) follows from the explicit construction. \( \square \)

Proposition 3.18 is a sort of converse of Lemma 3.10. Indeed, let \( K \) denote the function field of \( X \). The assumption that \( W \) is invariant under all \( \operatorname{Gal}(x) \) implies that the differential Galois group \( H \subset \operatorname{GL}(K \otimes V) \) of the generic differential equation on \( \operatorname{Spec}(K) \otimes \mathbb{P}^1_C \) leaves the subspace \( K \otimes_C W \) invariant.

Proposition 5.1 and Corollary 5.3 of Hrushovski’s paper [H02] is also related to Proposition 3.18 above.

3.4.3 Constructions of linear algebra

Let \( H \) be an algebraic subgroup of \( \operatorname{GL}(V) \). In other words, \( V \) is a faithful \( H \)-module. Let \( W \) be another \( H \)-module. It is well known that \( W \) can be obtained from \( V \) by a “construction of linear algebra”. Explicitly, \( W \cong W_2/W_1 \), where \( W_1 \subset W_2 \) are \( H \)-invariant subspaces of a finite direct sum \( \oplus_i V_{m_i}^{n_i} \).

Proposition 3.19 Let a family \( (\nabla, V, \{\nabla_i\}) \), parametrized by a reduced, irreducible scheme \( X \) of finite type over \( C \), be given. Let \( H \) be an algebraic subgroup of \( \operatorname{GL}(V) \) and suppose that \( \operatorname{Gal}(x) \subset H \) for every closed point \( x \in X \). For any construction of linear algebra \( W := W_2/W_1 \), as above, there exists a family \( (\mathcal{N}, \nabla, W, \text{local data}) \), parametrized by a non-empty open subset \( U \) of \( X \) such that:

(i) For every closed point \( x \in U(C) \), the connection \( (\mathcal{N}(x), \nabla(x)) \) on \( \mathbb{P}^1_C \) is obtained by the same construction.
(ii) The action of $\text{Gal}(x)$ on $W$, induced by the construction of linear algebra, coincides with the action of the differential Galois group of the connection $\mathcal{N}(x)$ on $W$.

**Proof.** For an $H$-module of the form $\tilde{V} = \oplus V_{\mu_i}^\mu$, the construction of the new family, parametrized by $X$, is discussed in the proof of Proposition 3.16. For an $H$-submodule $W_2$ we apply Proposition 3.18 and we have to replace $X$ with an open subset of $X$. For a $H$-submodule $W_1$ of $W_2$ one applies Proposition 3.18 again. The result is a family, parametrized by an open subset of $X$, corresponding to the $H$-module $W_2/W_1$. The construction of $(\mathcal{N}, W, \ldots)$ implies at once the properties (i) and (ii). □

### 3.4.4 The set $X(U(G^0) \subset)$ is constructible

We introduce some notation. Let $H$ be a linear algebraic group over $C$ acting upon a finite dimensional vector space $W$ over $C$. For every character $\chi : H \to \mathbb{G}_m = C^*$ one defines $W_{\chi} := \{w \in W \mid hw = \chi(h)w \text{ for all } h \in H\}$. This is a subspace of $W$. Let $\chi_1, \ldots, \chi_r$ denote the distinct characters of $H$ such that $W_{\chi_i} \neq 0$. Then $\bigoplus_{i=1}^r W_{\chi_i} \subset W$ is in fact a direct sum $\bigoplus_{i=1}^r W_{\chi_i}$. This space is denoted by $\text{Ch}_H(W)$. As before, an algebraic subgroup $G \subset \text{GL}(V)$ is given. The group $U(G^0) = U(G)$ denotes the algebraic subgroup of $G$ generated by all the unipotent elements of $G$. Any character of $G^0$ is trivial on $U(G^0)$ and $G^0/U(G^0)$ is a torus. It easily follows that for any $G$-module $W$ one has $\text{Ch}_{G^0}(W) = W^{U(G^0)}$ (i.e., the set of $U(G^0)$-invariant elements $w \in W$). An essential result is the following.

**Theorem 3.20** (M. F. Singer)

There exists a faithful $G$-module $W$ such that for every subgroup $H$ of $G$ the following statements are equivalent.

1. $U(G^0) \subset H$.
2. $\text{Ch}_{G^0}(W) = \text{Ch}_{H \cap G^0}(W)$.

We note that the inclusion $\text{Ch}_{G^0}(W) \subset \text{Ch}_{H \cap G^0}(W)$ is valid for any $G$-module $W$. Moreover, for any $G$-module $W$, the implication $(1) \Rightarrow (2)$ holds. Indeed,
$U(G') \subset H$ implies that $U(G') \subset H^o \subset H \cap G^o$. One has $U(G^o) = U(H^o)$, hence

$$\text{Ch}_{H \cap G^o}(W) \subset \text{Ch}_{H^o}(W) = W^{U(H^o)} = W^{U(G^o)} = \text{Ch}_{G^o}(W).$$

For the rather involved proof of the existence of a faithful $G$-module $W$ for which the implication $(2) \Rightarrow (1)$ holds, we refer to [S93].

**Corollary 3.21** Put $m := [G : G^o]$. There exists a faithful $G$-module $W$ such that for every subgroup $H$ of $G$ the following statements are equivalent.

(i) $U(G^o) \subset H$.

(ii) For every $r \leq m^m$ and for every $H$-invariant decomposable line

$L = Cu_1 \otimes u_2 \otimes \cdots \otimes u_r \subset \text{Sym}(W, r)$, the elements $u_1, \ldots, u_r$ belong to $\text{Ch}_{G^o}(W)$.

**Proof.** $W$ will denote the $G$-module of Theorem 3.20.

(i) $\Rightarrow$ (ii). As remarked above, the implication $(1) \Rightarrow (2)$ in Theorem 3.20 holds for every $G$-module. We have

$$u_1 \otimes \cdots \otimes u_r \in \text{Ch}_{H \cap G^o}(\text{Sym}(W, r)),$$

so $u_1 \otimes \cdots \otimes u_r \in \text{Ch}_{G^o}(\text{Sym}(W, r)) = \text{Sym}(W, r)^{U(G^o)}$. Let $x_1, \ldots, x_n$ denote a basis of $W$ over $C$. The algebra $\oplus_{m \geq 0} \text{Sym}(W, m)$ is identified with $C[x_1, \ldots, x_n]$. The group $G$ acts linearly on $C[x_1, \ldots, x_n]$ and the element $u := u_1 \otimes \cdots \otimes u_r$ is a homogeneous polynomial which is a product of homogeneous linear terms. From the $U(G^o)$-invariance of $u$, the connectedness of $U(G^o)$ and the unicity of the decomposition of $u$ (up to scalars and order), one deduces that $g(u_i)$ is a $C^*$-multiple of $u_i$ for every $g \in U(G^o)$ and every $i$. We find that $u_i \in \text{Ch}_{U(G^o)}(W) = W^{U(G^o)} = \text{Ch}_{G^o}(W)$ for all $i$.

(ii) $\Rightarrow$ (i). We will show that (ii) implies condition (2) of Theorem 3.20. It suffices to show that any $H \cap G^o$-invariant line $Cu \subset W$ belongs to $\text{Ch}_{G^o}(W)$. The group $H \cap G^o$ is a subgroup of $H$ of index at most $m := [G : G^o]$. There is a normal subgroup $\tilde{H}$ of $H$ contained in $H \cap G^o$, such that $[H : \tilde{H}] \leq m^m$. Let $h_1, \ldots, h_r$ denote representatives of $H/\tilde{H}$. Then the line spanned by $h_1 u \otimes h_2 u \otimes \cdots \otimes h_r u \in \text{Sym}(W, r)$ is decomposable and invariant under $H$. By (ii), $h_i u \in \text{Ch}_{G^o}(W)$ and so $u \in \text{Ch}_{G^o}(W)$. $\square$

**Proposition 3.22** Let a family $(\nabla, V, \{\nabla_i\})$, parametrized by an irreducible, reduced $X$, be given. Let $G$ be an algebraic subgroup of $\text{GL}(V)$. Suppose that
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\[ \text{Gal}(x) \subset G \] holds for every closed point \( x \) of \( X \). There exists an open non-empty subset \( X' \) such that the set \( X'(U(G^o) \subset \) is constructible.

**Proof.** Let \( W \) be the \( G \)-module having the properties of Theorem 3.20 and Corollary 3.21. By Proposition 3.19, there corresponds to \( W \) a family \((N', \nabla, W, \ldots)\), parametrized by an open non-empty subset \( X' \) of \( X \). Again we may suppose that \( N \) is free. Consider some integer \( r \) with \( 1 \leq r \leq m^m \), where \( m := [G : G^o] \). The set \( S(r) \) of elements \( u = u_1 \otimes \cdots \otimes u_r \in \text{Sym}(W, r) \) with all \( u_i \neq 0 \), and not all \( u_i \) belonging to \( \text{Ch}_{G^o}(W) \), is constructible. By part (1) of Corollary 3.17, the set \( X'(r) \), consisting of the closed points \( x \in X'(C) \) such that \( \text{Gal}(x) \) fixes a line \( Cu \subset \text{Sym}(W, r) \) with \( u \in S(r) \), is constructible. \( X'(U(G^o) \subset \) is constructible since it is, by Corollary 3.21, the complement in \( X' \) of \( \bigcup_{1 \leq r \leq m^m} X'(r) \). \( \square \)

3.4.5 The final step, involving the Singer condition

As before, an algebraic subgroup \( G \subset GL(V) \) is given. We suppose that \( G \) satisfies the “Singer condition”. Let a family \( F := (\nabla, V, \text{local data}) \), parametrized by an irreducible, reduced \( X \), be given. We will show, by induction on the dimension of \( X \), that \( X(=G) \) is constructible.

We have shown that there exists an open non-empty \( X' \subset X \) such that \( X'(U(G^o) \subset \subset G) \) is constructible. By induction, \( \{ x \in X \setminus X' \mid \text{Gal}(x) = G \} \) is constructible. After replacing \( X \) by an irreducible component of the set \( X'(U(G^o) \subset \subset G) \), one has \( U(G^o) \subset \text{Gal}(x) \subset G \) for all \( x \in X \).

Consider a faithful \( G/U(G^o) \)-module \( W \). The family \( F \) induces a family \( G := (\nabla, \nabla, W, \text{local data}) \), parametrized by \( X \). For every \( x \in X(C) \), one has \( \text{Gal}(x) \subset G/U(G^o) \). For the family \( G \), we have to prove that \( X(= G/U(G^o)) \) is constructible. We change the notation and write \( G \) for \( G/U(G^o) \) and \( V \) for \( W \). If \( G \) is finite, then an application of Proposition 3.16 finishes the proof. If \( G \) is infinite, then \( G^o \) is a torus and \( G^o \) lies in the center of \( G \) (this is precisely the Singer condition).

We continue the proof. For a closed point \( x \) and a singular point \( s_j \) one obtains a differential module \( M(x,s_j) := C((z - s_j)) \otimes \mathcal{M}(x)|_{s_j} \) over the differential field \( C((z - s_j)) \). Let \( PVF(x, s_j) \) denote a Picard-Vessiot field
for this differential module. The formal local Galois group $\text{Gal}(x, s_j)$ is the group of the differential automorphisms of $PVF(x, s_j)/C((z - s_j))$. Let $PVF \supset C(z)$ denote the Picard-Vessiot field for the generic differential module $\mathcal{M}(x)_k$ over $C(z)$. The differential Galois group $\text{Gal}(x)$ is the group of the differential automorphisms of $PVF/C(z)$. This group is canonical embedded into $GL(V)$ by our constructions. There exists a $C(z)$-linear embedding $PVF \subset PVF(x, s_j)$. This induces an injective algebraic homomorphism $\text{Gal}(x, s_j) \rightarrow \text{Gal}(x)$. Another embedding changes this homomorphisms by conjugation (with an element in $\text{Gal}(x)$). The connected component of the identity $\text{Gal}(x, s_j)^o$ is mapped to a subgroup of $\text{Gal}(x)^o \subset G^o$, and lies therefore in the center of $G$ and $\text{Gal}(x)$. In particular, the image of $\text{Gal}(x, s_j)^o$ in $G$ does not depend on the chosen embedding $PVF \rightarrow PVF(x, s_j)$.

We note that the local connection $\mathcal{M}(x, s_j)$ is semi-simple since the formal local differential Galois group does not contain $\mathfrak{g}_n$. Indeed, by construction $U(G^o) = \{1\}$, so $\text{Gal}(x)$ does not contain a copy of $\mathfrak{g}_n$. Now there are finitely many possibilities for the equivalence class of $\mathcal{M}(x, s_j)$. It is easily seen that this equivalence class depends in a constructible way on $x$. Therefore there exists an open non-empty subset of $X$, where the equivalence classes of $\mathcal{M}(x, s_j)$ do not depend on $x$. After restricting to this open subset, all the differential modules $\mathcal{M}(x, s_j)$ are isomorphic. In particular, $PVF(x, s_j)$ and $\text{Gal}(x, s_j)$ do not depend on $x$. We will write $PVF(s_j)$ and $\text{Gal}(s_j)$ for these objects. For a fixed embedding $PVF \rightarrow PVF(s_j)$, one has a fixed image of the groups $\text{Gal}(x, s_j) = \text{Gal}(s_j) \rightarrow \text{Gal}(x)$. Moreover, the image of $\text{Gal}(x, s_j)^o$ into $\text{Gal}(x)$ does not depend on any choice and is independent of $x$.

Let $H \subset G^o$ denote the subgroup, generated by the images of all $\text{Gal}(s_j)^o$. Then $H$ does not depend on $x$ and $H$ is a connected normal subgroup of $G$. Now we take a faithful $G/H$-module $W$ and its corresponding family, parametrized by a non-empty open subset $X'$ of $X$. For notational convenience, we replace $G$ with $G/H$. For this new family, parametrized by $X'$, one has:

(i) the differential Galois groups are contained in $G$,

(ii) the formal local differential Galois groups are finite,

(iii) the singularities are regular singular,
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(iv) the group $\text{Gal}(x)$ is generated (as an algebraic group) by the finite local differential Galois groups.

We have to show that $X'(=G)$ is constructible. By [BS64] Lemme 5.11 (also known as Platonov’s Theorem), there is a finite subgroup $E \subset G$ that maps surjectively to $G/G^o$. The surjective map $\hat{G} := G^o \times E \to G$ has a finite kernel. The group $\hat{G}$ has the property: any subgroup generated by $s$ subgroups, each one of order bounded by some $D$, is finite (and in fact contained in $G^o[m] \times E$ for a suitable $m$ depending in $D$). Thus the same statement holds for $G$. It follows that all $\text{Gal}(x)$ are finite. If $G^o \neq \{1\}$, then $X'(=G) = \emptyset$. If $G^o = \{1\}$, then $G$ is finite and therefore $X'(=G)$ is constructible.

3.5 Non-constructible sets $X(=G)$

The aim of this section is to produce for any linear algebraic $G$ that does not satisfy the “Singer condition”, a family of differential equations, parametrized by some $X$, such that $X(=G)$ is not constructible. We start by investigating a rather special case namely, $G$ is a semi-direct product $G = T \rtimes E$. Here $E$ is a finite group and $T$ is a torus. Furthermore, there is given a homomorphism of groups $\psi : E \to \text{Aut}(T)$. The group structure of $G$ is then defined by the formula $ete^{-1} = \psi(e)(t)$. The induced action $\phi$ of $E$ on the character group $X(T)$ of $T$, is given by the formula $(\phi(e)(\chi))(t) = \chi(e^{-1}te)$.

**Lemma 3.23** The following properties of $G = T \rtimes E$ are equivalent.

(i) $\sum_{e \in E} \text{im}(\phi(e) - 1)$ has finite index in $X(T)$.

(ii) $\bigcap_{e \in E} \ker(\phi(e) - 1) = 0$.

(iii) The $E$-module $X(T) \otimes \mathbb{Q}$ does not contain the trivial representation.

(iv) The center of $G$ is finite.

**Proof.** The vector space $X(T) \otimes \mathbb{Q}$ is an $E$-module and can be written as a direct sum of irreducible $E$-modules $I_1, \ldots, I_r$. Consider a non-trivial irreducible representation $\rho : E \to \text{GL}(W)$ over $\mathbb{Q}$. Then the submodule $\sum_{e \in E} \text{im}(\rho(e) - 1)$ of $W$ is not zero and hence equal to $W$. Moreover,
\[ \bigcap_{e \in E} \ker(\rho(e) - 1) \] is a proper submodule of \( W \) and hence equal to \( \{0\} \). For the trivial, 1-dimensional representation \( \rho : E \to \text{GL}(\mathbb{Q}) \), one has that
\[ \sum_{e \in E} \text{im}(\rho(e) - 1) = 0 \]
and
\[ \bigcap_{e \in E} \ker(\rho(e) - 1) = \mathbb{Q}. \]
This proves the equivalence of (i),(iii) and (iii). The elements of \( T \) can be considered as group homomorphisms \( t : X(T) \to C^* \). Now \( t \) lies in the center of \( G \) if and only if \( \chi(e^{-1}te) = \chi(t) \) for every \( \chi \) and every \( e \in E \). This translates into: \( t \) is equal to 1 on the submodule \( \sum_{e \in E} \text{im}(\phi(e) - 1) \). This proves the equivalence of (i) and (iv). \( \square \)

**Lemma 3.24** As above \( G = T \rtimes E \). Suppose that \( X(T) \otimes \mathbb{Q} \) is a non-trivial irreducible \( E \)-module. Let \( H \) be an algebraic subgroup of \( G \) which maps surjectively to \( E \). Then:

(i) If \( H \neq G \), then there exists an integer \( n \geq 1 \) such that \( H \subset T[n] \rtimes E \). Here \( T[n] \) denotes the subgroup of \( T \) consisting of the elements with order dividing \( n \).

(ii) Let \( e \in E \) have order \( m > 1 \) and let \( t \in T \) be given as a homomorphism \( t : X(T) \to X(T)/\ker(\phi(e) - 1) \to C^* \). Then \( (te)^n = 1 \).

(iii) There exist integers \( N, M \geq 1 \) and subgroups \( G_n \subset T[n] \rtimes E \) for infinitely many \( n \geq 1 \) such that the following holds.

(a) The index of \( G_n \) in \( T[n] \rtimes E \) is bounded by a constant independent of \( n \).

(b) \( G \) and every \( G_n \) is generated, as an algebraic subgroup, by \( N \) elements of order \( \leq M \).

**Proof.**

(i) The subtorus \( (H \cap T)^o \) of \( T \) is invariant under the action of \( E \) on \( T \). For let \( t \in H \cap T \), then for any \( e \in E \) there exists an element \( s \in T \) such that \( se \in H \), so \( e^s t e^{-1} = set e^{-1} s e^{-1} \in H \). We find that \( H \cap T \) is invariant under the action of \( E \), so the same holds for \( (H \cap T)^o \). Let \( N \) denote the kernel of the surjective homomorphism \( X(T) \to X((H \cap T)^o) \), then \( X(T)/N \) has no torsion and \( (H \cap T)^o \) consists of the homomorphisms \( t : X(T) \to C^* \) which are 1 on \( N \). If \( N = X(T) \), then \( H \) is finite and clearly contained in \( T[n] \rtimes E \) for some \( n \geq 1 \). If \( N \neq X(T) \), then \( N = 0 \) and \( H = G \).
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(ii) One verifies that

$$(te)^m = t \cdot \psi(e)(t) \cdot \psi(e^2)(t) \cdots \psi(e^{m-1})(t).$$

For any character $\chi$ one finds

$$\chi((te)^m) = \chi(t) \cdot (\phi(e^{-1})\chi)(t) \cdots (\phi(e^{-m+1})\chi)(t).$$

Therefore the only thing we have to show is that $t$ has value 1 on the submodule $(1 + \phi(e^{-1}) + \cdots + \phi(e^{-m+1}))X(T)$ of $X(T)$. Since this submodule is contained in $\ker(\phi(e^{-1}) - 1) = \ker(\phi(e) - 1)$, one concludes that $(te)^m = 1$.

(iii) For $G$ one takes as set of generators $E$ and an element $te$, with $e \in E$ of order $m$, $t \in T$ of infinite order and $te$ of order $m$. It follows from (i) that $G$ is generated, as an algebraic subgroup, by this set. Consider an integer $n > 1$. Let $G_n$ be the subgroup of $T[n] \times E$ generated by $E$ and for every $e \in E$ a collection of products $te$, with $t \in T$, that we now describe. Let $e \in E$ have order $m > 1$. Take a $\mathbb{Z}$-basis $b_1, \ldots, b_r$ of $X(T)/\ker(\phi(e) - 1)$ and define the homomorphisms $h_1, \ldots, h_r : X(T)/\ker(\phi(e) - 1) \to C^n$ by $h_i(b_j) = 1$ if $i \neq j$ and $h_i(b_i) = \zeta_n$ for $i = 1, \ldots, r$ and with $\zeta_n$ a fixed $n$th root of unity. The $t_i e$ that we use as generators of $G_n$ are $t_i : X(T) \to X(T)/\ker(\phi(e) - 1)$.

Part (b) is clear. For the proof of part (a) we consider the obvious map $\alpha : X(T) \to M := \oplus_{e \in E} X(T)/\ker(\phi(e) - 1)$. This map is injective by Lemma 3.23. For every homomorphism $h : M \to \mu_n$, (here $\mu_n$ denotes the group of the $n$th roots of unity), the element $t = h \circ \alpha$ belongs to $G_n$. Let $N$ denote the smallest submodule of $M$ such that $\text{im} \alpha \subset N$ and $M/N$ has no torsion. Then $N$ is a direct summand of $M$, so the image of $\text{Hom}(N, \mu_n) \to \text{Hom}(X(T), \mu_n) = T[n]$ is the same as the image of $\text{Hom}(M, \mu_n) \to \text{Hom}(X(T), \mu_n) = T[n]$, and is therefore contained in $G_n$. For infinitely many of the $n$, we have $\text{Hom}(X(T), \mu_n) \subset \text{Hom}(N, \mu_n)$, and $[\text{Hom}(X(T), \mu_n) : \text{Hom}(N, \mu_n)] = [N : \text{im} \alpha]$. For these $n$ we have $[T[n] \times E : G_n] = [T[n] : G_n \cap T[n]] \leq [\text{Hom}(X(T), \mu_n) : \text{Hom}(N, \mu_n)]$, so $[T[n] \times E : G_n] \leq [N : \text{im} \alpha] < \infty$ and (a) follows. \qed

**Proposition 3.25** Suppose that $C$ is the field of the complex numbers $\mathbb{C}$. Let $G = T \times E$ and suppose that $X(T) \otimes \mathbb{Q}$ is an irreducible $E$-module. There is a moduli space $\mathcal{M}$ such that $\mathcal{M}(=G)$ is not constructible.
Proof. Let $G \subset \text{GL}(V)$ be a faithful irreducible representation. Fix a finite subset \( \{ s_1, \ldots, s_r \} \) of \( \mathbb{C}^r \) and integers \( d_i > 1 \) for \( i = 1, \ldots, r \). Let \( \pi_1 \) denote the fundamental group of \( \mathbb{P}_C^1 \setminus \{ s_1, \ldots, s_r \} \) with base point \( 0 \). Take loops \( \lambda_1, \ldots, \lambda_r \in \pi_1 \) around the \( s \) points such that \( \pi_1 \) is generated by \( \lambda_1, \ldots, \lambda_r \) and such that the only relation between these generators is \( \lambda_1 \cdots \lambda_r = 1 \). Let \( G_n, n \in I \subset \mathbb{Z} \) be subgroups of \( G \) as given by Lemma 3.24. By the previous lemma we get that for a suitable choice of \( r \) and the \( d_i \), and an infinite subset \( I_1 \subset I \), there exist homomorphisms \( \rho, \rho_n : \pi_1 \to G \subset \text{GL}(V), n \in I_1 \) with the following properties:

(a) \( \rho(\lambda_i) \) and the \( \rho_n(\lambda_i) \) have order \( d_i \) (for \( i = 1, \ldots, r \)),

(b) the image of \( \rho \) is Zariski dense in \( G \) and \( G_n = \text{im} \rho_n \) for every \( n \in I_1 \).

Let \( te \) be the element used as a topological generator of \( G \), as in the proof of Lemma 3.24. Some continuity argument shows that the eigenvalues of \( te \) and \( e \) are the same. It follows that there is an infinite set \( I_2 \subset I_1 \) such that for each \( i \) the set of eigenvalues of \( \rho_n(\lambda_i) \) and \( \rho(\lambda_i) \) are the same for all \( n \in I_2 \). The Riemann-Hilbert correspondence attaches to each \( \rho_n, n \in I_2 \) a differential module \( M_n \cong \mathbb{C}(z) \otimes V \) over \( \mathbb{C}(z) \) (unique up to conjugation, see [PS03], Theorem 6.15). For each \( M_n \) and each \( i \), there is a unique lattice \( A_{n,i} \subset \mathbb{C}((z - s_i)) \otimes M_n \), with the following property. \( A_{n,i} \) has a basis, on which the differential is given by \( \frac{d}{dz(z - s_i)} + \frac{A_{n,i}}{z - s_i} \), where \( A_{n,i} \) is a diagonal matrix with diagonal entries in \( [0, 1] \cap \mathbb{Q} \). We can take \( A_{n,i} \) independent of \( n \). By [PS03] Lemma 6.18, these data define a unique connection \( (M_n, \nabla) \) with generic differential module \( M_n \). Now \( M_n \) is in general not free, but has the form \( O(a_1) \oplus \cdots \oplus O(a_v) \) with \( a_1 \geq \cdots \geq a_v \) and \( v := \text{dim} V \). The sum \( a_1 + \cdots + a_v \) is fixed since the local exponents of \( \Lambda^* M_n \) are given. Since \( \rho_n \) is irreducible the defect of \( M_n \) is uniformly bounded (see [PS03], Proposition 6.21). It follows that there is an infinite subset \( I_3 \subset I_2 \) such that \( M_n \) is of type \( a_1 \geq \cdots \geq a_v \) for all \( n \in I_3 \). The embedding of \( V \) in \( M_n \) and the regularity of \( M_n \) at the point \( z = 0 \) yield a canonical isomorphism \( \mathbb{C}[z]_{(z)} \otimes V \to (M_n)_0 \). One defines now a moduli problem by fixing the type of the vector bundle \( \mathcal{M} \) (namely \( a_1 \geq \cdots \geq a_v \)), an identification \( \mathbb{C}[z]_{(z)} \to \mathcal{M}_0 \) and the above local data. There is a universal family, parametrized by a variety \( \mathcal{M} \). Then \( \mathcal{M}(= G_n) \) is not empty for \( n \in I_3 \). We remove from \( \mathcal{M}(\subset \mathcal{M}) \) the union of the finitely many closed subsets \( \mathcal{M}(\subset T \times E') \) with \( E' \) a proper subgroup of \( E \). For notational convenience we call the result again \( \mathcal{M}(\subset \mathcal{M}) \). The set
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$\mathcal{M}(= G)$ is the complement in $\mathcal{M}(\subset G)$ of the sets $Z_n := \mathcal{M}(\subset T[n] \times E)$ for $n \geq 1$. It suffices now to show that $\bigcup_{n \geq 1} Z_n$ is not constructible. Indeed, $\mathcal{M}(= G)$ is the complement in the closed set $\mathcal{M}(\subset G)$ of the non-constructible set $\bigcup_{n \geq 1} Z_n$.

By construction, $\{Z_n\}$ is an increasing sequence of closed sets, i.e., $Z_n \subseteq Z_{n+1}$ and $Z_n \neq \bigcup_{i \in \mathbb{N}} Z_i \quad \forall \ n \in \mathbb{N}$. Suppose that this union is equal to $\bigcup_{i=1}^{d} O_i \cap F_i$ with open sets $O_i$ and closed sets $F_i$. For some $i$ the sets $Z_n \cap (O_i \cap F_i)$ again form an increasing sequence of closed subsets. After replacing $O_i \cap F_i$ by a suitable irreducible component, say $Y$, we have an increasing sequence of closed subsets $Y_n = Z_n \cap Y$ with union $Y$ and such that each $Y_n \neq Y$. This is not possible because the field $\mathbb{C}$ is uncountable.

Remarks 3.26

(1) The moduli space $m$ occurring in the proof of Proposition 3.25 is in general not the one studied in detail in Chapter 2, since the vector bundle $\mathcal{M}$ is not free. Suppose that one of the local data $\frac{d}{d(z-n)} + \frac{A_n}{z-n}$ is such that the eigenvalues of $A_i$ have multiplicity 1, then one can change each $\mathcal{M}_n$ (with $n \in I$) into a free vector bundle by shifting the eigenvalues of $A_i$ over integers. There are only finitely many ways to do this. Thus for some infinite subset $I' \subset I$ one single change of $A_i$ will make all $\mathcal{M}_n$ with $n \in I'$ into a free vector bundle. Now one can define the moduli space $\mathcal{M}$ by a free vector bundle $\mathcal{M}$ with $H^0(\mathbb{P}_C, \mathcal{M})$ identified with $V$ and with the prescribed local data.

(2) The proof of Proposition 3.25 extends to the case where $C$ is any algebraically closed field, not algebraic over $\mathbb{Q}$. Indeed, it suffices to consider a field $C$ of finite transcendence degree $\geq 1$. This field is embedded into $\mathbb{C}$. The moduli space $\mathcal{M}$ of the proof descends to $C$, i.e., $\mathcal{M} = \mathcal{M}_C \otimes_C \mathbb{C}$ for a suitable space $\mathcal{M}_C$.

The group $G$ is given as an algebraic subgroup of $GL(V)$ where $V$ is a vector space over $C$. One easily verifies that $\mathcal{M}(\subset G \otimes_C \mathbb{C}) = \mathcal{M}_C(\subset G) \otimes_C \mathbb{C}$. The same statement is valid for the groups $G_n$. It follows that $\mathcal{M}_C(= G)$ is not constructible.

We now give the proof of the general result, omitting some of the more obvious details.
Theorem 3.27 Let $C$ be the field of the complex numbers $\mathbb{C}$. Suppose that the linear algebraic group $G$ does not satisfy the Singer condition. Then there is a moduli space $\mathbb{M}$ such that $\mathbb{M}(= G)$ is not constructible.

Proof. As we will show, it suffices to prove this theorem for a linear algebraic group $G'$ for which there exists a surjective morphism $G' \to G$ with finite kernel. By [BS64] Lemma 5.11, there exists a finite subgroup $E$ of $G$ such that $E \to G/G^o$ is surjective. Thus we may replace $G$ with $G^o \times E$. The group $G^o/U(G^o)$ is a torus.

Lemma 3.28 (We use the above notations) There is a torus $T \subset G^o$, invariant under conjugation with the elements of $E$, such that $T \to G^o/U(G^o)$ is surjective and has a finite kernel.

Proof. First we will assume $G^o$ to be reductive. Then by [Sp98] Corollary 8.1.6 $(G^o, G^o)$ is semi-simple and $G^o = (G^o, G^o) \cdot R(G^o)$, where $R(G^o)$ denotes the radical of $G^o$. By [Sp98] Proposition 7.3.1, $R(G^o)$ is a central torus of $G^o$ and $R(G^o) \cap (G^o, G^o)$ is finite. Furthermore, by [Sp98] Theorem 8.1.5, we have $(G^o, G^o) \subset U(G^o)$. We have a surjective map $R(G^o) \to G^o/(G^o, G^o)$, so $G^o/(G^o, G^o)$ is a torus, and we find $(G^o, G^o) = U(G^o)$. The subgroup $R(G^o) \subset G^o$ is a characteristic subgroup, so in particular $eR(G^o)e^{-1} = R(G^o)$ for all $e \in E$. We find that we can take $T = R(G^o)$.

We now consider the general case. We define $T$ to be a maximal torus in $R(G^o)$. We have $R(G^o) = T \times R_u(G^o)$, where $R_u(G^o)$ is the unipotent radical of $G^o$. The image of $R(G^o)$ under the map $\pi : G^o \to G^o/R(G^o)$ is the radical of $G^o/R_u(G^o)$, and clearly $\pi(R(G^o)) = \pi(T)$. We find that $\pi$ defines an isomorphism of $T$ with the radical of $G^o/R_u(G^o)$, so the canonical map $T \to G^o/U(G^o)$ is surjective and has a finite kernel. The only thing left to show is that $T$ is invariant under conjugation with the elements of $E$. For $e \in E$, we have that $eTe^{-1}$ is again a maximal torus in $R(G^o)$, so we can write $eTe^{-1} = rTr^{-1}$ for some $r \in R(G^o)$. Because $R(G^o) = T \times R_u(G^o)$, we can take $r \in R_u(G^o)$. Let $N := \{u \in R_u(G^o) | uTu^{-1} = T\}$, then $N$ is a normal subgroup of $R_u(G^o)$ and we find a map $c : E \to R_u(G^o)/N, e \mapsto r$. Let $e \in E$, then $e$ is semi-simple, so $c(e)$ is semisimple, but also unipotent, because $c(e) \in R_u(G^o)/N$. Therefore $c(e) = 1$, $\forall e \in E$, so indeed $T$ is invariant under conjugation with the elements of $E$. \[\square\]
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Thus we may replace $G^*$ with $U(G^*) \rtimes T$. After replacing $T$ with a torus $T'$ such that $T' \to T$ is surjective and has a finite kernel, one can write $T$ as a product of two tori $T_1$ and $T_2$, both invariant under conjugation by $E$ and such that the group $T_2 \rtimes E$ satisfies the assumptions of Lemma 3.24. To be precise, let $X(T)$ be the character group of $T$, which has a structure of $E$-module. Then we can write $X(T) \otimes \mathbb{Q}$ as a direct sum of irreducible $E$-modules, say $X(T) \otimes \mathbb{Q} = D_1 \oplus \cdots \oplus D_r$. Since $G$ does not satisfy the Singer condition, we may assume that $D_1$ is a non-trivial $E$-module. Let $X_2$ be the projection of $X(T)$ on $D_1$, and $X_1$ the projection of $X(T)$ on $D_2 \oplus \cdots \oplus D_r$. Then $X(T) \subseteq X_1 \oplus X_2$ has finite index. Now let $T_i$ be a torus with $X(T_i) = X_i$, $i = 1, 2$, then $T' = T_1 \times T_2$, has the desired properties. The result after these changes is a group $G'$ of the form

$$(U(G^*) \rtimes T_1) \rtimes (T_2 \rtimes E)$$

which maps surjectively to $G$ and has a finite kernel. We will construct a moduli space $\mathcal{M}$ such that $\mathcal{M}(=G)$ is not constructible.

One takes a finite subset $\{b_1, \ldots, b_l, s_1, \ldots, s_r\}$ in $\mathbb{C}^*$. The fundamental group $\pi_1$ of the complement of this set in $\mathbb{P}^1 \mathbb{C}$, with base point 0, is given by generators $\mu_1, \ldots, \mu_l, \lambda_1, \ldots, \lambda_r$ according to loops around these points. The only relation is $\mu_1 \cdots \mu_l \lambda_1 \cdots \lambda_r = 1$. We will consider homomorphisms $\rho : \pi_1 \to G'$ by assigning images for these $t + s$ generators. For notational convenience we will ignore the relation between the generators of $\pi_1$. The trick which allows us to do so is the following. One doubles the finite set by adding new points $s_1^*, \ldots, s_l^*, b_1^*, \ldots, b_l^*$. The fundamental group has now generators $\mu_1, \ldots, \mu_l, \lambda_1, \ldots, \lambda_r, \lambda_1^*, \ldots, \lambda_r^*, \mu_1^*, \ldots, \mu_l^*$. The only relation is their product being 1. Suppose that we want to assign elements $g_1, \ldots, g_l, b_1, \ldots, h_r \in G'$ to $\mu_1, \ldots, \lambda_r$. Then for the larger fundamental group, we complete this by assigning $h_1^{-1}, \ldots, h_l^{-1}, g_1^{-1}, \ldots, g_l^{-1}$ to the generators $\lambda_1^*, \ldots, \mu_l^*$. The homomorphisms $\rho_n : \pi_1 \to G'$ that interest us are given by:

(a) $\rho_n(\mu_1), \ldots, \rho_n(\mu_{l-1}) \in U(G^*)$; these elements are unipotent, $\neq 1$ and they generate $U(G^*)$ as an algebraic group. Moreover, these elements will not depend on $n$.

(b) $\rho_n(\mu_l) \in T_1$ which generates $T_1$ as an algebraic group. Moreover, this element will not depend on $n$. 

(c) $\rho_n(\lambda_1), \ldots, \rho_n(\lambda_r) \in T_2 \times E$ are chosen as in the proof of Proposition 3.25.

As above this is completed by assigning values to $\mu_1^i, \ldots, \lambda_r^i$. The homomorphism $\rho_n : \pi_1 \to G$ are obtained by composing $\rho_n^i$ with $G' \to G$. We take an irreducible faithful $G$-module $V$. Riemann-Hilbert (see [PS03] Theorem 6.15) produces a differential module $M_n = \mathbb{C}(\pi) \otimes V$ with singularities in $\{b_1, \ldots, s_1, \ldots, a_1^i, \ldots, b_r^i, \ldots, b_r^i\}$. The local monodromies at the points $b_1, \ldots, b_r$ are fixed and we choose local connections for these singular points. For the local connections at the regular singular points $s_1, \ldots, s_r$ we make a choice which fits infinitely many of the $\rho_n$. The local data at the other points $a_1^i, \ldots, b_r^i$ are just the negatives of the corresponding points in $\{b_1, \ldots, s_r\}$.

As in the proof of Proposition 3.25, there exists an infinite subset $I$ of $\mathbb{N}$, such that the corresponding vector bundles $M_n$ have the same type. This defines the moduli problem and the moduli family, parametrized by some space $\mathcal{M}$. According to Proposition 3.22, $\mathcal{M}(G) \subset \mathbb{C}(G)$ is constructible. Let $H$ denote the image of the group $U(G) \times T_1$ in $G$. Then it can be seen that $\mathcal{M}(H \subset \subset G)$ is also constructible. The final part of the proof of Proposition 3.25 applies here as well and the result is that $\mathcal{M}(= G)$ is not constructible.

\(\Box\)

**Remarks 3.29** Another formulation of the Singer condition.

(1) The constructions in Lemma 3.23, Lemma 3.24, Proposition 3.25 and Theorem 3.27 lead to the following observation.

A linear algebraic group $G$ does not satisfy the Singer-condition if and only if it has a factor group $H$ of dim $\geq 1$, with the following property: There exist integers $N, M, I > 1$ such that every algebraic subgroup $K \subset H$ which is mapped surjectively to $H/H^o$ contains an algebraic subgroup of index $\leq I$ which is, as algebraic group, generated by $N$ elements of order $\leq M$.

(2) Theorem 3.27 remains valid for an algebraically closed field $C$ that is not algebraic over $\mathbb{Q}$ (See Remarks 3.26).

(3) For Theorem 3.15 to hold, it is essential to consider families of differential equations on $\mathbb{P}^1$. For example on an elliptic curve $E$ over $\mathbb{C}$, one can construct a family of differential equations parametrized by some $X$, such that $X(= \mathbb{C}^*)$ is not constructible (see [S93] p.384). If this family is pushed
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down to $\mathbb{P}_C^l$, then after a shift one obtains the Lamé family we considered in Example 3.8.