Balancing and model reduction for discrete-time nonlinear systems based on Hankel singular value analysis
Fujimoto, Kenji; Scherpen, Jacquelien M.A.

Published in:
Proceedings of the 16th MTNS Conference

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2004

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

Copyright
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

Take-down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): http://www.rug.nl/research/portal. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Download date: 07-09-2020
Balancing and model reduction for discrete-time nonlinear systems 

based on Hankel singular value analysis

Kenji Fujimoto\textsuperscript{a} and Jacquelien M. A. Scherpen\textsuperscript{b}

\textsuperscript{a}Dept. of Mechanical Science and Engineering
Graduate School of Engineering
Nagoya University
Furo-cho, Chikusa-ku, Nagoya 464-8603, Japan
fujimoto@nuem.nagoya-u.ac.jp

\textsuperscript{b}Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
J.M.A.Scherpen@dcsc.tudelft.nl

Abstract—This paper is concerned with balanced realization and model reduction for discrete-time nonlinear systems. Singular perturbation type balanced truncation method is proposed. In this procedure, the Hankel singular values and the related controllability and observability properties are preserved, which is a natural generalization of both the linear discrete-time case and the nonlinear continuous-time case.

I. INTRODUCTION

In linear control systems theory, balanced realization and model reduction theory plays an important role in both theoretical and practical research fields [11]. Motivated by this, its nonlinear extension was investigated by many authors [9], [5], [8], [4]. The authors have provided a new balanced realization method based on singular value analysis of the Hankel operator of the nonlinear plant [1], [2] as a precise controllability and observability properties are preserved, which is a natural generalization of both the linear discrete-time case and the nonlinear continuous-time case.

Balanced realization for discrete-time nonlinear systems were also investigated by some authors [10], [6], [3]. However, though there is a strong similarity to the continuous-time case, those results are not immediately obtained from the continuous-time results. In particular, model reduction theory based on balancing for continuous-time nonlinear systems was obtained, although its discrete-time version was not investigated.

In this paper, we provide a balancing and model reduction method for discrete-time nonlinear systems. This method is a natural nonlinear generalization of the linear case as well as a discrete-time counterpart of our continuous-time case result. We prove that there exists a balanced realization for nonlinear discrete-time systems which is quite similar to the continuous-time case and that a model reduction method based on this realization and a singular perturbation based truncation approach derives a reduced order model which preserves several important properties of the original system such as controllability, observability and the gain property.

II. PROBLEM SETTING AND PRELIMINARIES

Consider an $\ell_2$-stable discrete-time nonlinear system

\[ \Sigma : \begin{cases} x(t+1) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{cases} \quad (1) \]

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$. It’s controllability operator $C : \ell_2^m(\mathbb{Z}_+) \rightarrow \mathbb{R}^n$ and observability operator $O : \mathbb{R}^n \rightarrow \ell_p^2(\mathbb{Z}_+)$ are defined by

\[ x^0 = C(u) : \begin{cases} x(t-1) &= f(x(t), u(t)) \quad x(\infty) = 0 \\ x(0) &= 0 \\ y(t) &= h(x(t), 0) \end{cases} \]

\[ y = O(x^0) : \begin{cases} x(t+1) &= f(x(t), 0) \quad x(0) = x^0 \end{cases} \]

The Hankel operator is given by their composition $H = O \circ C$.

The corresponding controllability and observability functions are defined by

\[ L_x(x) = \frac{1}{2} \| C^\dagger(x) \|_{\ell_2}^2 \quad (2) \]

\[ L_y(x) = \frac{1}{2} \| O(x) \|_{\ell_2}^2 \quad (3) \]

where $C^\dagger$ is the norm-minimizing pseudo-inverse of $C$, that is,

\[ C^\dagger(x) = \arg \inf_{C^\dagger(x) \in x} \| u \|_{\ell_2}. \]

Balanced realization investigated in this paper (also balanced realization for continuous-time systems in [1], [2]) is closely related to the solution of singular value analysis of the Hankel operator $H$ as

\[ (dH(u))^* \circ H(u) = \lambda u, \quad \lambda \in \mathbb{R}. \]

Solutions of this equation are important because they characterize critical points of $\| H(u) \| / \| u \|$, hence the gain maximizing input $\arg \sup_{u \in \ell_2} (\| H(u) \| / \| u \|)$ is also contained in them.

In the authors’ former result, the following theorem was proved.

Theorem 1: [3] Suppose that $C$, $C^\dagger$ and $O$ are differentiable, and that there exist $\lambda \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ satisfying

\[ \frac{\partial L_y(\xi)}{\partial \xi} = \lambda \frac{\partial L_x(\xi)}{\partial \xi}. \quad (4) \]

Then $v \in \ell_2^m(\mathbb{Z}_+)$ defined by

\[ v := C^\dagger(\xi) \]
satisfies the equation for singular value analysis of \( \mathcal{H} \)
\[
(\alpha \mathcal{H}(v))^* \circ \mathcal{H}(v) = \lambda \ v.
\] (5)

Suppose moreover that the Jacobian linearization of \( \Sigma \) has
non-zero and distinct Hankel singular values. Then there exist
\( n \) solutions curves \( \xi = \xi_i(s) \in \mathbb{R}^n, \ s \in \mathbb{R} \) satisfying \( \xi_i(0) = 0 \)
for Equation (4) in a neighborhood of the origin.

Here we call the solution \( v \) of Equation (5) a singular vector of \( \mathcal{H} \), and the corresponding input-output ratio
\[
\sigma = \frac{\|\mathcal{H}(v)\|}{\|v\|}
\] a singular value of \( \mathcal{H} \), respectively. Singular value functions and singular vector functions
are defined as follows for convenience.
\[
\nu_i(s) := \mathcal{C}^!(\xi_i(s)) \quad (6)
\]
\[
\sigma_i(s) := \frac{\|\mathcal{H}(\nu_i(s))\|}{\|\nu_i(s)\|} \quad (7)
\]
The curves in the state-space \( \xi_i(s) \) play the role of the coordinate axes of the balanced realization. Balanced realization and the corresponding model reduction method in the
continuous-time case was derived based on them. See [11], [2] for the detail.

III. MAIN RESULTS

A. Observability and controllability functions

As a preparation for the model reduction of discrete-
time systems, we need to characterize the observability
and controllability functions \( L_0(x) \) and \( L_c(x) \)
by algebraic equations which are similar to the Hamilton Jacobi equations in
the continuous-time case.

**Lemma 1:** Suppose that \( x = 0 \) of the system
\[
x(t + 1) = f(x(t), 0)
\]
is asymptotically stable. Then a smooth observability function \( L_0(x) \) in (3) exists if and only if
\[
L_0(f(x, 0)) - L_0(x) + \frac{1}{2} h(x, 0)^T h(x, 0) = 0, \quad L_0(0) = 0
\] (8)
has a smooth solution \( L_0(x) \).

**Proof:** Sufficiency is proved first. Suppose that the
observability function \( L_0(x) \) exists. Then the definition of the
observability function (3) implies that
\[
L_0(x(0)) = \frac{1}{2} \sum_{t=0}^{\infty} h(x(t), 0)^T h(x(t), 0)
\]
\[
= \frac{1}{2} \sum_{t=1}^{\infty} h(x(t), 0)^T h(x(t), 0)
\]
\[
+ \frac{1}{2} h(x(0), 0)^T h(x(0), 0)
\]
\[
= L_0(x(1)) + \frac{1}{2} h(x(0), 0)^T h(x(0), 0)
\]
\[
= L_0(f(x(0), 0)) + \frac{1}{2} h(x(0), 0)^T h(x(0), 0).
\]

This equation has to hold for an arbitrary initial state \( x(0) \),
that is, it satisfies the equation (8) since \( L_0(0) = 0 \). This proves sufficiency.

Next, necessity is proved. Suppose that the equation (8)
has a smooth solution \( L_0(x) \). The equation (8) implies that
\[
\bar{L}_0(x) = L_0(F(x)) + \frac{1}{2} h(x, 0)^T h(x, 0)
\]
\[
= \bar{L}_0(F(x))
\]
\[
+ \frac{1}{2} h(x, 0)^T h(x, 0) + \frac{1}{2} h(F(x), 0)^T h(F(x), 0)
\]
\[
= \ldots
\]
\[
= \lim_{k \to \infty} \left( L_0(F^k(x)) + \sum_{i=0}^{k} h(F^i(x), 0)^T h(F^i(x), 0) \right)
\]
\[
= \lim_{k \to \infty} \bar{L}_0(F^k(x)) + L_0(x)
\]
\[
= L_0(x)
\]
where \( F(x) := f(x, 0) \). The last equation holds because the system \( x(t + 1) = F(x(t)) \) is asymptotically stable and
because \( L_0(0) = 0 \). This completes the proof.

This result is a natural nonlinear generalization of the
linear case result. In the linear case, the dynamics (1) reduces to
\[
\Sigma : \begin{cases} x(t + 1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}
\]
with appropriate matrices \( A, B, C \) and \( D \). Here the observability
function is in a quadratic form
\[
L_0(x) = \frac{1}{2} x^T G_0 x.
\]
The algebraic equation (8) reduces down to
\[
A^T G_0 A - G_0 + C^T C = 0
\]
which is the Lyapunov equation for the observability Gram-
ian in the linear case.

A similar result for the controllability function is obtained
as follows. Let us consider an optimal control problem
minimizing a cost function
\[
\min \sum_{t=0}^{\infty} \|u(t)\|^2
\] (9)
for the dynamics of \( \mathcal{C} \)
\[
x(t + 1) = f^{-1}(x(t), u(t))
\]
where \( f^{-1} \) denotes the inverse of \( f(x, u) \) with respect to \( x \),
that is,
\[
f(f^{-1}(x, u), u) = x
\]
holds. Let us denote the input \( u \) achieving the minimization in (9) by \( u = u^*(x) \). Then the dynamics of \( \mathcal{C}^! : x^0 \mapsto v \)
becomes
\[
\mathcal{C}^! : \begin{cases} x(t + 1) = f^{-1}(x(t), u^*(x(t))) \\ v(t) = u^*(x(t)) \end{cases} \quad x(0) = x^0
\]
Lemma 2: Suppose that \( x = 0 \) of the feedback system
\[
x(t + 1) = f^{-1}(x(t), u^*(x(t)))
\]
is asymptotically stable. Then a smooth controllability function
\( L_c(x) \) in (2) exists if and only if
\[
L_c(f^{-1}(x, u^*(x))) - L_c(x) + \frac{1}{2} u^*(x)^T u^*(x), \quad L_c(0) = 0
\]
has a smooth solution \( L_c(x) \).

**Proof:** This lemma can be proved as a corollary of Lemma 1 by identifying \( C^1 \) with \( O \).

These results are natural generalizations of the continuous-time case results where the equations (8) and (10) are Hamilton-Jacobi equations.

**B. Balanced realization**

As in the continuous-time case [2], we can prove the existence of balanced realization for discrete-time nonlinear systems.

**Theorem 2:** Consider the state-space system \( \Sigma \) in (1) and suppose that its Jacobian linearization has non-zero and distinct Hankel singular values. Then, in a neighborhood of the origin, there exists a coordinate transformation converting \( \Sigma \) into a system whose controllability and observability functions are described by

\[
L_c(x) = \frac{1}{2} \sum_{i=1}^{n} \frac{x_i^2}{\sigma_i(x_i)}
\]

\[
L_o(x) = \frac{1}{2} \sum_{i=1}^{n} x_i^2 \sigma_i(x_i)
\]

with the singular value functions \( \sigma_i \)'s defined in (7). In particular, if the above coordinate transformation is defined globally, then

\[
\sup_{u \in \mathbb{E}_2(\Sigma_c)} \frac{||H(u)||}{||u||} = \max_{\sigma \in \mathbb{R}} \sup_{x} \sigma_i(s).
\]

The proof follows along the same lines as the proof of Theorem 5 in [2], and it is omitted for the reason of space. This realization is a natural nonlinear generalization of the linear case, because the balanced realization in the linear case has the controllability and observability functions

\[
L_c(x) = \frac{1}{2} x^T G_c^{-1} x, \quad L_o(x) = \frac{1}{2} x^T G_o x
\]

with the controllability and observability Grammians \( G_c \) and \( G_o \) which are balanced as

\[
G_c = G_o = \text{diag}(\sigma_1, \ldots, \sigma_n)
\]

with the Hankel singular values of the system. In Theorem 2, we have its nonlinear counterpart

\[
L_c(x) = \frac{1}{2} x^T G_c(x)^{-1} x, \quad L_o(x) = \frac{1}{2} x^T G_o(x)x
\]

with

\[
G_c(x) = G_o(x) = \text{diag}(\sigma_1(x), \ldots, \sigma_n(x))
\]

with the singular value functions \( \sigma_i(\cdot) \)'s of the Hankel operator \( \mathcal{H} \).

**C. Model reduction**

This subsection gives a model reduction method based on the balanced realization given in Theorem 2 with a singular perturbation type balanced truncation technique.

Consider the system \( \Sigma \) in (1) and suppose that the system is balanced in the sense of Theorem 2. Suppose moreover that the singular value functions satisfy

\[
\max_{\pm \sigma} \sigma_i(\pm s) > \max_{\pm \sigma} \sigma_{i+1}(\pm s).
\]

Namely, the coordinate axis \( x_i \) plays a more important role than \( x_j \) in the input-output mapping. Moreover we assume that

\[
\max_{\pm \sigma} \sigma_k(\pm s) > \max_{\pm \sigma} \sigma_{k+1}(\pm s)
\]

holds for a certain \( k \), and divide the state-space according to \( k \) as

\[
x = (x^a, x^b) \in \mathbb{R}^k \times \mathbb{R}^{n-k}
\]

\[
f(x, u) = \left( f^a(x^a, x^b, u), f^b(x^a, x^b, u) \right) \in \mathbb{R}^k \times \mathbb{R}^{n-k}.
\]

Then, accordingly, we obtain two reduced order systems by a singular perturbation based truncation method

\[
\Sigma^a : \begin{cases} x^a(t+1) = f^a(x^a(t), x^b(t), u^a(t)) \\ x^b(t) = f^b(x^a(t), x^b(t), u^a(t)) \\ y^a(t) = h(x^a(t), x^b(t), u^a(t)) \\ y^b(t) = h(x^a(t), x^b(t), u^b(t)) \end{cases}
\]

\[
\Sigma^b : \begin{cases} x^a(t+1) = f^a(x^a(t), x^b(t), u^a(t)) \\ x^b(t+1) = f^b(x^a(t), x^b(t), u^b(t)) \\ y^a(t) = h(x^a(t), x^b(t), u^a(t)) \\ y^b(t) = h(x^a(t), x^b(t), u^b(t)) \end{cases}
\]

Here we suppose that the equation

\[
x^a = f^a(x^a, x^b, u)
\]

has a unique solution

\[
x^a = f^a(x^b, u),
\]

and that the equation

\[
x^b = f^b(x^a, x^b, u)
\]

has a unique solution

\[
x^b = f^b(x^a, u).
\]

Note that these equations always have solutions at least in a neighborhood of the origin if the Jacobian linearization of
the system $\Sigma$ is asymptotically stable. Then we obtain explicit forms

$$
\Sigma^a : \begin{cases}
x^a(t+1) &= \tilde{f}^a(x^a(t), u^a(t)) \\
y^a(t) &= h^a(x^a(t), u^a(t))
\end{cases}
$$

(15)

$$
\Sigma^b : \begin{cases}
x^b(t+1) &= \tilde{f}^b(x^b(t), u^b(t)) \\
y^b(t) &= h^b(x^b(t), u^b(t))
\end{cases}
$$

(16)

with

$$
\tilde{f}^a(x^a(t), u^a(t)) := f^a(x^a(t), \tilde{p}^b(x^a(t), u^a(t)), u^a(t))
$$

$$
h^a(x^a(t), u^a(t)) := h(x^a(t), \tilde{f}^b(x^a(t), u^a(t)), u^a(t))
$$

$$
\tilde{f}^b(x^b(t), u^b(t)) := f^b(\tilde{f}^a(x^b(t), u^b(t)), x^b(t), u^b(t))
$$

$$
h^b(x^b(t), u^b(t)) := h(f^a(x^b(t), u^b(t)), x^b(t), u^b(t))
$$

by substituting the equations (12) and (14) for $\Sigma$. For those reduced order systems, we can prove the following properties.

Theorem 3: Consider the system $\Sigma$ in (1) and the truncated systems $\Sigma^a$ and $\Sigma^b$ in (15) and (16). Then, in a neighborhood of the origin, $\Sigma^a$ and $\Sigma^b$ are balanced in the sense of Theorem 2 and

$$
\sigma_i^a(x_i^a) = \sigma_i(x_i^a) \quad i \in \{1, \ldots, k\}
$$

$$
\sigma_i^b(x_i^b) = \sigma_{k+i}(x_i^b) \quad i \in \{1, \ldots, n-k\}
$$

hold with $\sigma_i^a$’s and $\sigma_i^b$’s the singular value functions of the systems $\Sigma^a$ and $\Sigma^b$, respectively. In particular, if those functions are defined globally, then

$$
\sup_{u \in \mathcal{U}^a_{\theta}(\mathcal{Z}_+)} ||H(u)|| = \sup_{s \in \mathbb{R}} \sigma_i^a(s).
$$

Proof: Suppose that the system $\Sigma$ in (1) is balanced in the sense of Theorem 2. Then it implies that $L_o(x)$ can be divided into two parts

$$
L_o(x) = L_o^a(x^a) + L_o^b(x^b)
$$

(17)

where

$$
L_o^a(x^a) := \frac{1}{2} \sum_{i=1}^{k} x_i^a \sigma_i(x_i)
$$

$$
L_o^b(x^b) := \frac{1}{2} \sum_{i=k+1}^{n} x_i^b \sigma_i(x_i).
$$

On the other hand, the equations (11) and (13) imply that

$$
f^a(\tilde{f}^a(x^b, u), x^b, u) = \tilde{f}^a(x^a, u)
$$

(18)

$$
f^b(x^a, \tilde{f}^b(x^a, u), u) = \tilde{f}^b(x^a, u)
$$

(19)

Let us substitute (14) for (8). Then we obtain

$$
0 = \left[ L_o(f(x, 0)) - L_o(x) + \frac{1}{2} h(x, 0)^T h(x, 0) \right]_{x^b = f^b(x^a, u)}
$$

$$
= L_o(f(x^a, \tilde{f}^b(x^a, 0), 0)) - L_o(x^a, \tilde{f}^b(x^a, 0))
$$

$$
+ \frac{1}{2} h(x^a, \tilde{f}^b(x^a, 0), 0)^T h(x^a, \tilde{f}^b(x^a, 0), 0)
$$

$$
= \left( L_o^a(f^a(x^a, \tilde{f}^b(x^a, 0), 0)) + L_o^b(f^b(x^a, \tilde{f}^b(x^a, 0), 0)) \right)
$$

$$
- \left( L_o^a(x^a) + L_o^b(\tilde{f}^b(x^a, 0)) \right)
$$

$$
+ \frac{1}{2} h(x^a, \tilde{f}^b(x^a, 0), 0)^T h(x^a, \tilde{f}^b(x^a, 0), 0)
$$

$$
= L_o^a(\tilde{f}^a(x^a, 0)) - L_o^a(x^a) + \frac{1}{2} h(x^a, 0)^T h(x^a, 0).
$$

Here the third equation follows from (17), and the last equation follows from (18) and (19). Then Lemma 1 implies that $L_o^b(x^b)$ is the observability function of the system $\Sigma^a$. Further, it can be easily proved that $L_o^b(x^b)$ is the observability function of $\Sigma^b$ by substituting (12).

In a similar way, as in the proof of Lemma 2, by identifying $\mathcal{C}^i$ with $O$, we can prove that the controllability functions $L_c^a(x^a)$ and $L_c^b(x^b)$ of the systems $\Sigma^a$ and $\Sigma^b$ are given by

$$
L_c^a(x^a) := \frac{1}{2} \sum_{i=1}^{k} x_i^a \sigma_i(x_i)
$$

$$
L_c^b(x^b) := \frac{1}{2} \sum_{i=k+1}^{n} x_i^b \sigma_i(x_i)
$$

which prove the former part of the theorem. The latter part follows immediately. (See [2].) This completes the proof.

This theorem reveals several properties of the proposed model reduction method:

- This model reduction derives balanced reduced order models.
- Singular value functions are preserved and, in particular, the gain of the related Hankel operator (which is called Hankel norm) is preserved.
- Since singular value functions are preserved, some properties related to controllability and observability of the original system is preserved.

This is both a natural nonlinear generalization of the linear case result [7] and a natural discrete-time counterpart of the continuous-time nonlinear systems case [1], though that was based on balanced truncation, where here we use a singular perturbation model reduction procedure so that we preserve the structure.

IV. CONCLUSION

This paper was devoted to balanced realizations and model reduction for discrete-time nonlinear dynamical systems based on Hankel singular value analysis. Firstly, we proved the existence of a balanced realization similar to continuous-time case result. Secondly, a balanced truncation method...
based on a singular perturbation approach was proposed. In this method, several important properties of the original system such as controllability, observability and the gain property are preserved.

V. REFERENCES