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Balancing and model reduction for discrete-time nonlinear systems based on Hankel singular value analysis

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Abstract—This paper is concerned with balanced realization and model reduction for discrete-time nonlinear systems. Singular perturbation type balanced truncation method is proposed. In this procedure, the Hankel singular values and the related controllability and observability properties are preserved, which is a natural generalization of both the linear discrete-time case and the nonlinear continuous-time case.

I. INTRODUCTION

In linear control systems theory, balanced realization and model reduction theory plays an important role in both theoretical and practical research fields [11]. Motivated by this, its nonlinear extension was investigated by many authors [9], [5], [8], [4]. The authors have provided a new balanced realization method based on singular value analysis of the Hankel operator of the nonlinear plant [1], [2] as a precise discrete-time counterpart of our continuous-time case. In those former results, balancing and model reduction method for continuous-time nonlinear systems was obtained, although its discrete-time version was not investigated.

Balanced realization for discrete-time nonlinear systems were also investigated by some authors [10], [6], [3]. However, though there is a strong similarity to the continuous-time case, those results are not immediately obtained from the continuous-time results. In particular, model reduction theory based on balancing for discrete-time nonlinear systems was not obtained so far.

In this paper, we provide a balancing and model reduction method for discrete-time nonlinear systems. This method is a natural nonlinear generalization of the linear case as well as a discrete-time counterpart of our continuous-time case result. We prove that there exists a balanced realization for nonlinear discrete-time systems which is quite similar to the continuous-time case and that a model reduction method based on this realization and a singular perturbation based truncation approach derives a reduced order model which preserves several important properties of the original system such as controllability, observability and the gain property.

II. PROBLEM SETTING AND PRELIMINARIES

Consider an $\ell_2$-stable discrete-time nonlinear system

\begin{align}
\Sigma: \quad & x(t+1) = f(x(t), u(t)) \\
& y(t) = h(x(t), u(t))
\end{align}  \tag{1}

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$. Its controllability operator $\mathcal{C} : \ell_2^m(\mathbb{Z}_+) \to \mathbb{R}^n$ and observability operator $\mathcal{O} : \mathbb{R}^n \to \ell_2^r(\mathbb{Z}_+)$ are defined by

\begin{align}
x^0 = \mathcal{C}(u) : \quad & x(t-1) = f(x(t), u(t)) \quad x(\infty) = 0 \\
x^0 = x(0) \\
y = \mathcal{O}(x^0) : \quad & x(t+1) = f(x(t), 0) \quad x(0) = x^0
\end{align}

The Hankel operator is given by their composition

$\mathcal{H} = \mathcal{O} \circ \mathcal{C}$.

The corresponding controllability and observability functions are defined by

\begin{align}
L_c(x) &= \frac{1}{2} \| \mathcal{C}^\dagger(x) \|_{\ell_2^r}^2 \tag{2} \\
L_o(x) &= \frac{1}{2} \| \mathcal{O}(x) \|_{\ell_2^n}^2 \tag{3}
\end{align}

where $\mathcal{C}^\dagger$ is the norm-minimizing pseudo-inverse of $\mathcal{C}$, that is,

$\mathcal{C}^\dagger(x) = \arg \inf_{\mathcal{C}(\xi) = x} \| u \|_{\ell_2}$.

Balanced realization investigated in this paper (also balanced realization for continuous-time systems in [1], [2]) is closely related to the solution of singular value analysis of the Hankel operator $\mathcal{H}$ as

$$(d\mathcal{H}(u))^\ast \circ \mathcal{H}(u) = \lambda u, \quad \lambda \in \mathbb{R}.$$

Solutions of this equation are important because they characterize critical points of $\| \mathcal{H}(u)\|/\|u\|$, hence the gain maximizing input $\arg \sup_{u \in \ell_2^n} (\| \mathcal{H}(u)\|/\|u\|)$ is also contained in them.

In the authors’ former result, the following theorem was proved.

Theorem 1: [3] Suppose that $\mathcal{C}$, $\mathcal{C}^\dagger$ and $\mathcal{O}$ are differentiable, and that there exist $\lambda \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ satisfying

$$\frac{\partial L_o(\xi)}{\partial \xi} = \lambda \frac{\partial L_o(\xi)}{\partial \xi}.$$

Then $v \in \ell_2^n(\mathbb{Z}_+)$ defined by

$$v := \mathcal{C}^\dagger(\xi)$$
satisfies the equation for singular value analysis of $\mathcal{H} \vphantom{(\mathcal{H}(v))}{(\mathcal{H}(v))}$ 
\begin{equation}
(\mathcal{H}(v)) = \lambda v.
\end{equation}

Suppose moreover that the Jacobian linearization of $\Sigma$ has non-zero and distinct Hankel singular values. Then there exist $n$ solutions curves $\xi = \xi_i(s) \in \mathbb{R}^n$, $s \in \mathbb{R}$ satisfying $\xi_i(0) = 0$ for Equation (4) in a neighborhood of the origin.

Here we call the solution $v$ of Equation (5) a singular vector of $\mathcal{H}$, and the corresponding input-output ratio
\[ \sigma = \frac{\|\mathcal{H}(v)\|}{\|v\|} \]
a singular value of $\mathcal{H}$, respectively. Singular value functions and singular vector functions corresponding to $\xi_i(s)$ are defined as follows for convenience.
\begin{align}
\nu_i(s) & := C^\dagger(\xi_i(s)) \\
\sigma_i(s) & := \frac{\|\mathcal{H}(\nu_i(s))\|}{\|\nu_i(s)\|}
\end{align}
The curves in the state-space $\xi_i(s)$ play the role of the coordinate axes of the balanced realization. Balanced realization and the corresponding model reduction method in the continuous-time case was derived based on them. See [11], [2] for the detail.

### III. MAIN RESULTS

#### A. Observability and controllability functions

As a preparation for the model reduction of discrete-time systems, we need to characterize the observability and controllability functions $L_o(x)$ and $L_c(x)$ by algebraic equations which are similar to the Hamilton Jacobi equations in the continuous-time case.

**Lemma 1:** Suppose that $x = 0$ of the system
\[ x(t + 1) = f(x(t), 0) \]
is asymptotically stable. Then a smooth observability function $L_o(x)$ in (3) exists if and only if
\[ L_o(f(x, 0)) - L_o(x) + \frac{1}{2} h(x, 0)^T h(x, 0) = 0, \quad L_o(0) = 0 \]  
has a smooth solution $L_o(x)$.

**Proof:** Sufficiency is proved first. Suppose that the observability function $L_o(x)$ exists. Then the definition of the observability function (3) implies that
\[ L_o(x(0)) = \frac{1}{2} \sum_{t=0}^{\infty} h(x(t), 0)^T h(x(t), 0) \]
\[ = \frac{1}{2} \sum_{t=1}^{\infty} h(x(t), 0)^T h(x(t), 0) + \frac{1}{2} h(x(0), 0)^T h(x(0), 0) \]
\[ = L_o(x(1)) + \frac{1}{2} h(x(0), 0)^T h(x(0), 0) \]
\[ = L_o(f(x(0), 0)) + \frac{1}{2} h(x(0), 0)^T h(x(0), 0). \]
This equation has to hold for an arbitrary initial state $x(0)$, that is, it satisfies the equation (8) since $L_o(0) = 0$. This proves sufficiency.

Next, necessity is proved. Suppose that the equation (8) has a smooth solution $L_o(x)$. The equation (8) implies that
\[ \tilde{L}_o(x) = \tilde{L}_o(F(x)) + \frac{1}{2} h(x, 0)^T h(x, 0) \]
\[ + \frac{1}{2} h(x, 0)^T h(x, 0) + \frac{1}{2} h(F(x), 0)^T h(F(x), 0) \]
\[ = \cdots \]
\[ = \lim_{k \to \infty} \left( \tilde{L}_o(F^k(x)) + \frac{1}{2} \sum_{t=0}^{k} h(F^t(x), 0)^T h(F^t(x), 0) \right) \]
\[ = \lim_{k \to \infty} \tilde{L}_o(F^k(x)) + L_o(x) \]
\[ = L_o(x) \]
where $F(x) := f(x, 0)$. The last equation holds because the system $x(t + 1) = F(x(t))$ is asymptotically stable and because $L_o(0) = 0$. This completes the proof.

This result is a natural nonlinear generalization of the linear case result. In the linear case, the dynamics (1) reduces to
\[ \Sigma : \quad \begin{cases} x(t + 1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \]
with appropriate matrices $A$, $B$, $C$ and $D$. Here the observability function is in a quadratic form
\[ L_o(x) = \frac{1}{2} x^T G_o x. \]
The algebraic equation (8) reduces down to
\[ A^T G_o A - G_o + C^T C = 0 \]
which is the Lyapunov equation for the observability Gramian in the linear case.

A similar result for the controllability function is obtained as follows. Let us consider an optimal control problem minimizing a cost function
\[ \min_{\varepsilon \in \mathcal{E}(\mathcal{E}^+) \cap 0, \ v(0) = x(0)} \sum_{t=0}^{\infty} \|u(t)\|^2 \]
for the dynamics of $\mathcal{C}$
\[ x(t + 1) = f^{-1}(x(t), u(t)) \]
where $\varepsilon^{-1}$ denotes the inverse of $f(x, u)$ with respect to $x$, that is,
\[ f(f^{-1}(x, u), u) = x \]
holds. Let us denote the input $u$ achieving the minimization in (9) by $u = u^*(x(t))$. Then the dynamics of $\mathcal{C}^\dagger : x^0 \mapsto \nu$ becomes
\[ \mathcal{C}^\dagger : \quad \begin{cases} x(t + 1) = f^{-1}(x(t), u^*(x(t))) & x(0) = x^0 \\ \nu(t) = u^*(x(t)) \end{cases} \]
Lemma 2: Suppose that $x = 0$ of the feedback system

$$x(t + 1) = f^{-1}(x(t), u^*(x(t)))$$

is asymptotically stable. Then a smooth controllability function $L_c(x)$ in (2) exists if and only if

$$L_c(f^{-1}(x, u^*(x))) - L_c(x) + \frac{1}{2}u^*(x)^T u^*(x), \quad L_c(0) = 0$$

has a smooth solution $L_c(x)$.

Proof: This lemma can be proved as a corollary of Lemma 1 by identifying Hamilton-Jacobi equations.

These results are natural generalizations of the continuous-time case results where the equations (8) and (10) are Hamilton-Jacobi equations.

B. Balanced realization

As in the continuous-time case [2], we can prove the existence of balanced realization for discrete-time nonlinear systems.

Theorem 2: Consider the state-space system $\Sigma$ in (1) and suppose that its Jacobian linearization has non-zero and distinct Hankel singular values. Then, in a neighborhood of the origin, there exists a coordinate transformation converting $\Sigma$ into a system whose controllability and observability functions are described by

$$L_c(x) = \frac{1}{2} \sum_{i=1}^{n} \frac{x_i^2}{\sigma_i(x_i)}$$

$$L_o(x) = \frac{1}{2} \sum_{i=1}^{n} x_i \sigma_i(x_i)$$

with the singular value functions $\sigma_i$'s defined in (7). In particular, if the above coordinate transformation is defined globally, then

$$\sup_{u \in \mathcal{U}(2, \varepsilon)} \frac{\|\mathcal{H}(u)\|}{\|u\|} = \max_{i \in \mathbb{R}} \sup \sigma_i(s).$$

The proof follows along the same lines as the proof of Theorem 5 in [2], and it is omitted for the reason of space. This realization is a natural nonlinear generalization of the linear case, because the balanced realization in the linear case has the controllability and observability functions

$$L_c(x) = \frac{1}{2} x^T G_c^{-1} x, \quad L_o(x) = \frac{1}{2} x^T G_o x$$

with the controllability and observability Grammians $G_c$ and $G_o$ which are balanced as

$$G_c = G_o = \text{diag}(\sigma_1, \ldots, \sigma_n)$$

with the Hankel singular values of the system. In Theorem 2, we have its nonlinear counterpart

$$L_c(x) = \frac{1}{2} x^T G_c(x)^{-1} x, \quad L_o(x) = \frac{1}{2} x^T G_o(x) x$$

with

$$G_c(x) = G_o(x) = \text{diag}(\sigma_1(x), \ldots, \sigma_n(x))$$

and divide the state-space according to $k$ as

$$x = (x^a, x^b) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$$

$$f(x, u) = \begin{pmatrix} f^a(x^a, x^b, u) \\ f^b(x^a, x^b, u) \end{pmatrix} \in \mathbb{R}^k \times \mathbb{R}^{n-k}.$$
the system $\Sigma$ is asymptotically stable. Then we obtain explicit forms
\begin{equation}
\Sigma^a : \begin{cases} 
  x^a(t+1) &= f^a(x^a(t), u^a(t)) \\
  y^a(t) &= h^a(x^a(t), u^a(t)) 
\end{cases} 
\tag{15}
\end{equation}
\begin{equation}
\Sigma^b : \begin{cases} 
  x^b(t+1) &= f^b(x^b(t), u^b(t)) \\
  y^b(t) &= h^b(x^b(t), u^b(t)) 
\end{cases} 
\tag{16}
\end{equation}

with
\begin{align*}
  f^a(x^a(t), u^a(t)) &= f^a(x^a(t), \hat{f}^a(x^a(t), u^a(t)), u^a(t)) \\
  h^a(x^a(t), u^a(t)) &= h(x^a(t), f^b(x^a(t), u^a(t)), u^a(t)) \\
  f^b(x^b(t), u^b(t)) &= f^b(\hat{f}^b(x^b(t), u^b(t)), x^b(t), u^b(t)) \\
  h^b(x^b(t), u^b(t)) &= h(f^b(x^b(t), u^b(t)), x^b(t), u^b(t))
\end{align*}
by substituting the equations (12) and (14) for $\Sigma$. For those reduced order systems, we can prove the following properties.

**Theorem 3:** Consider the system $\Sigma$ in (1) and the truncated systems $\Sigma^a$ and $\Sigma^b$ in (15) and (16). Then, in a neighborhood of the origin, $\Sigma^a$ and $\Sigma^b$ are balanced in the sense of Theorem 2 and
\begin{align*}
  \sigma^a_i(x^a_k) &= \sigma_i(x^a_k) \quad i \in \{1, \ldots, k\} \\
  \sigma^b_i(x^b_k) &= \sigma_{k+i}(x^b_k) \quad i \in \{1, \ldots, n-k\}
\end{align*}
hold with $\sigma^a_i$'s and $\sigma^b_i$'s the singular value functions of the systems $\Sigma^a$ and $\Sigma^b$, respectively. In particular, if those functions are defined globally, then
\[
  \sup_{u \in L^2_2(\mathbb{R})} \left\| \mathcal{H}(u) \right\| = \sup_{s \in \mathbb{R}} \sigma^a_i(s).
\]

**Proof:** Suppose that the system $\Sigma$ in (1) is balanced in the sense of Theorem 2. Then it implies that $L_o(x)$ can be divided into two parts
\[
  L_o(x) = L^a_o(x^a) + L^b_o(x^b) \tag{17}
\]
where
\begin{align*}
  L^a_o(x^a) &:= \frac{1}{2} \sum_{i=1}^{k} x^2_i \sigma_i(x_i) \\
  L^b_o(x^b) &:= \frac{1}{2} \sum_{i=k+1}^{n} x^2_i \sigma_i(x_i).
\end{align*}

On the other hand, the equations (11) and (13) imply that
\begin{align}
  f^a(\bar{f}^a(x^b, u), x^b, u) &= f^a(x^b, u) \tag{18} \\
  f^b(x^a, \bar{f}^b(x^a, u), u) &= f^b(x^a, u). \tag{19}
\end{align}

Let us substitute (14) for (8). Then we obtain
\[
  0 = \left[ L_o(f(x,0)) - L_o(x) + \frac{1}{2} h(x,0)^T h(x,0) \right]_{x^b = \tilde{f}^b(x^a, u)} \\
  = L_o(f(x, \tilde{f}^b(x^a,0)), 0) - L_o(x^a, \tilde{f}^b(x^a,0)) \\
  + \frac{1}{2} h(x, \tilde{f}^b(x^a,0),0)^T h(x, \tilde{f}^b(x^a,0),0) \\
  = (L^a_o(f^a(x^a, \tilde{f}^b(x^a,0)), 0) + L^b_o(f^b(x^a, \tilde{f}^b(x^a,0),0))) \\
  - \left( L^a_o(x^a) + L^b_o(\tilde{f}^b(x^a,0),0) \right) \\
  + \frac{1}{2} h(x, \tilde{f}^b(x^a,0),0)^T h(x, \tilde{f}^b(x^a,0),0) \\
  = L^a_o(\tilde{f}^a(x^a,0)) - L^a_o(x^a) + \frac{1}{2} h(x, \tilde{f}^b(x^a,0)^T h(x, \tilde{f}^b(x^a,0),0).
\]
Here the third equation follows from (17), and the last equation follows from (18) and (19). Then Lemma 1 implies that $L^a_o(x^a)$ is the observability function of the system $\Sigma^a$. Further, it can be easily proved that $L^b_o(x^b)$ is the observability function of $\Sigma^b$ by substituting (12).

In a similar way, as in the proof of Lemma 2, by identifying $C^1$ with $O$, we can prove that the controllability functions $L^a_c(x^a)$ and $L^b_c(x^b)$ of the systems $\Sigma^a$ and $\Sigma^b$ are given by
\begin{align*}
  L^a_c(x^a) &:= \frac{1}{2} \sum_{i=1}^{k} x^2_i \sigma_i(x_i) \\
  L^b_c(x^b) &:= \frac{1}{2} \sum_{i=k+1}^{n} x^2_i \sigma_i(x_i)
\end{align*}
which prove the former part of the theorem. The latter part follows immediately. (See [21]) This completes the proof.

This theorem reveals several properties of the proposed model reduction method:
- This model reduction derives balanced reduced order models.
- Singular value functions are preserved and, in particular, the gain of the related Hankel operator (which is called Hankel norm) is preserved.
- Since singular value functions are preserved, some properties related to controllability and observability of the original system is preserved.

This is both a natural nonlinear generalization of the linear case result [7] and a natural discrete-time counterpart of the continuous-time nonlinear systems case [1], though that was based on balanced truncation, where here we use a singular perturbation model reduction procedure so that we preserve the structure.

**IV. Conclusion**

This paper was devoted to balanced realizations and model reduction for discrete-time nonlinear dynamical systems based on Hankel singular value analysis. Firstly, we proved the existence of a balanced realization similar to continuous-time case result. Secondly, a balanced truncation method
based on a singular perturbation approach was proposed. In this method, several important properties of the original system such as controllability, observability and the gain property are preserved.

V. REFERENCES


